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A Note on Recurrent Points

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Abstract: We show that some definitions of recurrent points are equivalent in the setting of general topological spaces.

Keywords: Topological space; continuous map; discrete dynamical system; recurrent point; ω-limit set; T_1-space; Poincaré Recurrence Theorem

This paper is dedicated to the memory of Professor José Sousa Ramos.

1 Introduction

When dealing with a discrete dynamical system (X, f), that is, a general topological space X and a continuous function $f : X \to X$ (in short, $f \in C(X,X)$), the main goal is to analyze the asymptotic behaviour of the *orbits* for any $x \in X$

$$\mathrm{Orb}_f(x) := \{x, f(x), f^2(x), \ldots, f^n(x), \ldots\},$$

or equivalently to study how the orbits of the system behave when n goes to infinity. Here, we define the *(positive) iterates* of f as $f^1 = f, f^n = f \circ f^{n-1}$ $(n \geq 2)$ and $f^0 = \mathrm{Id}|_X$ (the *identity map on* X).

This objective is easily achieved in some particular cases. For instance, let $f : [0,1] \to [0,1]$ be an interval map for which $f(x) < x$ if $x \in (0,1]$ and $f(0) = 0$: in this situation we obtain that 0 is a *global attractor* since $\lim_{n\to\infty} f^n(x) = 0$ for any point x. But, in general, the description turns into a more difficult (and intriguing) behaviour of the orbits, for example, let us mention the paradigmatic logistic map $f(x) = 4x(1-x)$, in spite of its certainly simple aspect, its dynamics conceals a fascinating series of different motions for its orbits, from the simple accumulation towards periodic points to complex dynamics involving attracting Cantor sets or even dense orbits (see [18] and [7] to learn more details of this logistic map).

The reader has surely noticed that this long–term study is strongly related with the analysis of different sets of return points, those points whose orbits intersect in the large all their prescribed open neighbourhoods. Among them, the simplest case corresponds to periodic orbits. Recall that $x \in X$ is a *periodic point* if $f^m(x) = x$ for some positive integer m (if $m = 1$ we say that x is a *fixed point*, and in general m is the *order* or *period* of x under f when in addition $f^j(x) \neq x$ for $0 < j < m$). This idea of returning to itself can be generalized by the notion of *recurrence* which means the intersection of the orbit $\mathrm{Orb}_f(x)$ with any of the open neighbourhoods U of x. To learn something more about the different notions of return of points, the reader is encouraged to consult [22, §1.3], where it is possible to find, in the setting of interval maps, several approaches to the idea of return by the notions of *periodicity, almost periodicity in the sense of Bohr, uniformly recurrence, recurrence, no-wandering points, chain recurrence*, etcetera. It is also studied the relation between these notions and the complexity or simplicity of the dynamics of the map [22, §4].

We concentrate our attention on the topological notion of recurrence. The purpose of this note is to present three equivalent notions of recurrence, to give a little historical account of this concept and to remark the topological character of our approach.

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2 Three notions of recurrence

In [14] the definition of recurrent point for (X, f) is given as follows (we will consider that f is continuous, though in [14] f was assumed to be a homeomorphism):

Definition 1(Gottschalk, [14]). *Given a discrete dynamical system (X, f), we say that $x \in X$ is a* recurrent point *of f whenever for any open neighbourhood $\mathscr{U} = \mathscr{U}(x)$ of x there exists a positive integer $N = N(x, \mathscr{U})$ such that $f^N(x) \in \mathscr{U}(x)$. The set of recurrent points will be denoted by* $\mathrm{Rec}(f)$.

Soon after, Erdös and Stone gave in [10] simpler proofs of some results of [14] about properties, among others, of recurrent points. There the restrictive hypothesis of being f a homeomorphism is replaced simply by the continuity of f. For instance, it is proved that the sets of recurrent points of f and any iterate f^n are the same, now in the setting of continuous maps and arbitrary topological spaces.

However, in [10] the authors gave the following definition of recurrent point:

Definition 2(Erdös-Stone, [10]). *We say that $x \in X$ is a* recurrent point *of the discrete dynamical system (X, f) whenever for any open neighbourhood $\mathscr{W} = \mathscr{W}(x)$ of x there exists an infinite set of positive integers n such that $f^n(x) \in \mathscr{W}$. We write $\widetilde{\mathrm{Rec}}(f)$ to denote the set of such points.*

Moreover, they comment: “This definition is equivalent to Gottschalk's if X is a T_1-space”. Let us justify this assertion. Remember that X is a T_1-topological space if given two different points x_1, x_2 in X there exist neighbourhoods $\mathscr{U}_1$ and $\mathscr{U}_2$ of x_1, x_2, respectively, such that $x_i \notin \mathscr{U}_j$ for $i \neq j$, $i, j \in \{1, 2\}$. Notice that the T_1-definition implies that each singleton $\{x\}$ is closed. Observe that the inclusion $\widetilde{\mathrm{Rec}}(f) \subseteq \mathrm{Rec}(f)$ is immediate. Moreover, if X is T_1 and $x \in \mathrm{Rec}(f)$, given a neighbourhood $\mathscr{U} = \mathscr{U}(x)$, let n_1 be the first positive integer such that $f^{n_1}(x) \in \mathscr{U}$. If $f^{n_1}(x) = x$ the periodicity of x yields an infinite set of values, namely $n_k = kn_1$, for which the definition of $\widetilde{\mathrm{Rec}}(f)$ is satisfied. If $f^{n_1}(x) \neq x$, then $\mathscr{U} \setminus \{f^{n_1}(x)\}$ is an open set containing x, so the definition of $\mathrm{Rec}(f)$ gets the existence of a new positive integer $n_2 > n_1$ such that $f^{n_2}(x) \in \mathscr{U} \setminus \{f^{n_1}(x)\} \subset \mathscr{U}$ and with $f^j(x) \notin \mathscr{U}$ if $n_1 < j < n_2$. Reasoning in an analogous way, we deduce the existence of an infinite set of positive integers $(n_j)_j$ holding Definition 2. Hence, $\mathrm{Rec}(f) \subseteq \widetilde{\mathrm{Rec}}(f)$, which ends the assertion about the equivalence between Definitions 1 and Definition 2 for T_1-spaces.

The above comment was the leitmotif for investigating the relationship between the two definitions in the general case of an arbitrary topological space, without additional conditions on separation axioms over the space. As far as we know this equivalence has no been treated in the literature and we try to fulfill this (small) lack.

On the other hand, some authors define a recurrent point in terms of ω-limit sets (in this case the space X is assumed to be compact metric): x is *recurrent* if and only if $x \in \omega_f(x)$, where $\omega_f(x)$ is meant the set of accumulation points of the orbit of x under f. Nevertheless, following a parallel development to that described in [5], it is possible to extend this definition to the general setting of arbitrary topological spaces as follows.

Given a general discrete dynamical system (X, f) and a point $x \in X$, we define the *ω-limit set of x under f* by

$$\omega_f(x) := \bigcap_{m \geq 0} \overline{\bigcup_{k \geq m} \{f^k(x)\}},$$

where $\overline{A}$ is the *closure* of a set $A \subseteq X$, that is, the smallest closed set containing A.

We are in a position to state the third definition of recurrent point.

Definition 3.*Let (X, f) be a discrete dynamical system. We say that $x \in X$ is a* recurrent point *if $x \in \omega_f(x)$. We denote by $\mathrm{Rec}_\omega(f)$ the set of recurrent points according to this definition of recurrence.*

Notice that $\omega_f(x)$ is always a closed and f-invariant set (that is, $f(\omega_f(x)) \subseteq \omega_f(x)$). It can be an empty set, consider for instance the system $(\mathbb{R}, f)$ with $f(x) = x + 1$. However, if X is compact we can ensure that $\omega_f(x) \neq \emptyset$ according to the well-known topological result establishing that a compact space verifies the *finite intersection property*, namely if we have a family of closed sets in which any finite subfamily has non-empty intersection then the family itself has also non-empty intersection (see for more details [9, §XI]).

3 The definitions are equivalent

In this section we will prove that the three above definitions of recurrence are equivalent.

Proposition 1.*Let X be a topological space and let $f : X \to X$ be continuous. We have*

$$\mathrm{Rec}_\omega(f) = \mathrm{Rec}(f) = \widetilde{\mathrm{Rec}}(f).$$

*Proof.*First, we prove the equality

$$\mathrm{Rec}(f) = \widetilde{\mathrm{Rec}}(f).$$

In this case, it suffices to prove that $\mathrm{Rec}(f) \subseteq \widetilde{\mathrm{Rec}}(f)$, since the other inclusion is immediate. So let $x \in \mathrm{Rec}(f)$, we will show that x holds Definition 2. Let $\mathscr{U}$ be a neighbourhood of x, then by Definition 1, there exists a positive integer N_1 such that $f^{N_1}(x) \in \mathscr{U}$. Since f^{N_1} is a continuous map, given a neighbourhood $\mathscr{W}_1$ of the iterate

$f^{N_1}(x)$ we get a neighbourhood $\mathscr{U}_1$ of x holding $f^{N_1}(\mathscr{U}_1) \subseteq \mathscr{W}_1 \cap \mathscr{U} =: \widetilde{\mathscr{W}_1}$. Then, $f^{N_1}(\mathscr{U}_1 \cap \mathscr{U}) \subseteq f^{N_1}(\mathscr{U}_1) \subseteq \widetilde{\mathscr{W}_1}$ (notice that $x \in \mathscr{U}_1 \cap \mathscr{U} =: \widetilde{\mathscr{U}_1} \neq \emptyset$). On the other hand, by applying Definition 1 to $\widetilde{\mathscr{U}_1}$ we obtain a positive integer N_2 (it could be $N_1 = N_2$) such that $f^{N_2}(x) \in \widetilde{\mathscr{U}_1}$, and consequently $f^{N_1+N_2}(x) = f^{N_1}(f^{N_2}(x)) \in f^{N_1}(\widetilde{\mathscr{U}_1}) \subseteq f^{N_1}(\mathscr{U}_1) \subseteq \widetilde{\mathscr{W}_1} \subseteq \mathscr{U}$.

In the same manner we can see that it is possible to find a sequence of neighbourhoods $\mathscr{U} \supseteq \widetilde{\mathscr{U}_1} \supseteq \widetilde{\mathscr{U}_2} \supseteq \widetilde{\mathscr{U}_3} \supseteq \ldots$, and a sequence of positive integers $(N_k)_k$ so that $f^{N_1+\ldots+N_k}(x) \in \mathscr{U}$. It follows that x satisfies the statement of Definition 2, therefore $x \in \widetilde{\mathrm{Rec}}(f)$.

Secondly, we prove that $\mathrm{Rec}_\omega(f) = \mathrm{Rec}(f)$. If $x \in \mathrm{Rec}_\omega(f)$, by using the notion of closure of a set, it is a simple matter to see that any neighbourhood $\mathscr{U}$ of x intersects to $\overline{\{f^j(x), f^{j+1}(x), \ldots\}}$ for all $j \geq 1$, and consequently there exists a positive integer m such that $f^m(x) \in \mathscr{U}$, proving that $x \in \mathrm{Rec}(f)$.

On the other hand, if $x \in \mathrm{Rec}(f) = \widetilde{\mathrm{Rec}}(f)$, by Definition 2 and the notion of ω-limit set we have $x \in \overline{\{f^j(x), f^{j+1}(x), \ldots\}}$ for all $j \geq 1$. Hence, $x \in \omega_f(x)$, or equivalently $x \in \mathrm{Rec}_\omega(f)$.

Remark. It is well-known that in the case of metric spaces (X,d) we can characterize the set of recurrent points under sequences of iterates: x is recurrent if and only if there exists an increasing sequence of positive integers $(n_k)_k$ such that $d(f^{n_k}(x), x)$ tends to zero when k goes to infinity. This follows directly from the equivalence of recurrence via Definition 3 dealing with ω-limit sets. A precise proof can be consulted in [5].

Remark. Let (X,τ) be a topological space. If $f \in C(X,X)$ and $x \in \mathrm{Rec}(f)$, then $\omega_f(x) = \overline{\mathrm{Orb}_f(x)}$.

The inclusion $\omega_f(x) \subseteq \overline{\mathrm{Orb}_f(x)}$ is always true by the definition of ω-limit set. If, in addition, x is a recurrent point, Proposition 1 yields $x \in \omega_f(x)$ and the f-invariance of $\omega_f(x)$ gives $\mathrm{Orb}_f(x) \subseteq \omega_f(x)$. Finally, $\overline{\mathrm{Orb}_f(x)} \subseteq \omega_f(x)$ since $\omega_f(x)$ is closed.

We finish this section by emphasizing that it would be interesting to analyze the topological properties of a system from the more general point of view of arbitrary topological spaces, without additional assumptions on the space or with the minimal amount of topological hypothesis. In this sense, as an illustrative example let us give the following properties on the characterizations of minimal sets and on the link between minimal sets and uniformly recurrent points. Before we refresh two well-known definitions in the setting of discrete dynamics.

Definition 4. *Given a discrete dynamical system (X,f), we say that $M \subseteq X$ is a* minimal set *if it is nonempty, closed and f-invariant and no proper subset of M possesses these properties of closure and f-invariance.*

Following [5], we say that $x \in X$ is uniformly recurrent *whenever for any neighbourhood $\mathscr{U} = \mathscr{U}(x)$ of x there exists a positive integer $N = N(x,\mathscr{U})$ such that if $f^m(x) \in \mathscr{U}, m \geq 0$, then $f^{m+k}(x) \in \mathscr{U}$ for some k with $0 < k \leq N$. The set of uniformly recurrent points of f is denoted by* $\mathrm{UR}(f)$.

It is straightforward to check the following result.

Proposition 2. *Let (X,f) a discrete dynamical system and let M be a nonempty set. The following properties are equivalent:*

– *M is a minimal set.*
– *$\mathrm{Orb}_f(x) = M$ for all $x \in M$.*
– *$\omega_f(x) = M$ for all $x \in M$.*

As a consequence if M is minimal then $M \subseteq \mathrm{Rec}(f)$.

The proof of the following result is inspired by [5, §V, Proposition 5]. In this reference it is proved the same property in the setting of compact metric spaces.

Proposition 3. *Let (X,f) be a discrete dynamical system. Suppose that X is a T_1-space. If M is minimal and $x \in M$, then x is uniformly recurrent.*

Proof. First, notice that according to Proposition 2 x is necessarily a recurrent point. To obtain a contradiction, suppose that x is not uniformly recurrent. Then there exist an open neighbourhood U of x and an increasing sequence of positive integers $(n_j)_{j\geq 1}$ such that $f^{n_j}(x) \in U$ but $f^s(x) \notin U$ if $s \in \{n_j+1, n_j+2, \ldots, n_j+j\}$. Define $R := \overline{\{f^{n_1}(x), f^{n_2}(x), \ldots, f^{n_j}(x), \ldots\}}$. Since M is minimal and $R \subseteq U$ we have $\emptyset \neq R \subseteq \overline{U} \cap M$. Let $z \in R \subseteq M$, the minimality of M together with Proposition 2 implies the existence of $m > 0$ such that $f^m(z) \in U$. Now, the continuity of f^m yields the existence of a neighbourhood V of z with $f^m(V) \subseteq U$. By the definition of R as a closure, for V there is an index n_j such that $f^{n_j}(x) \in V$. Taking into account that X is a T_1-space, it is not restrictive to assume that $m < j$ (otherwise, we take $\widetilde{V} = V \setminus \{f^{n_1}(x), \ldots, f^{n_j}(x)\}$, again an open set since each point in a T_1-space is closed, see [9], and we apply the same reasoning as above to $\widetilde{V}$ instead of V). In this case, $f^{n_j}(x) \in V$ and also $f^{m+n_j}(x) \in U$, with $m < j$, a contradiction. This ends the proof.

We next give a partial converse of Proposition 3. Following [9, p.141] a topological space (X,τ) is *regular* if it is a Hausdorff space holding that each $z \in X$ and closed set Y not containing z have disjoint neighbourhoods, that is, if Y is closed and $z \notin Y$ then there is a neighbourhood V of z and an open $W \supseteq Y$ such that $U \cap W = \emptyset$.

Proposition 4. *Let (X,τ) be a regular topological space and let (X,f) be a discrete dynamical system. If a point $x \in X$ is uniformly recurrent, then $\overline{\mathrm{Orb}_f(x)}$ is a minimal set of X.*

*Proof.*First at all, realize that $\overline{\mathrm{Orb}_f(x)}$ is clearly non-empty and closed, moreover it is invariant due to the continuity of f: $f(\overline{\mathrm{Orb}_f(x)}) \subseteq \overline{f(\mathrm{Orb}_f(x))} = \overline{\mathrm{Orb}_f(f(x))} \subseteq \overline{\mathrm{Orb}_f(x)}$.

We prove that $\overline{\mathrm{Orb}_f(x)}$ is minimal. Arguing by contradiction, assume that there exists a closed f-invariant set A such that $\emptyset \subsetneq A \subset \overline{\mathrm{Orb}_f(x)}$. In this case, note that $x \notin A$, otherwise $\overline{\mathrm{Orb}_f(x)} \subseteq A$. Since X is regular, there exist a neighbourhood W of x and an open set $V \supseteq A$ such that $V \cap W = \emptyset$.

For the open W, according to the definition of uniformly recurrent point, there exists a positive integer κ such that

$$\{f^j(x), f^{j+1}(x), \ldots, f^{j+\kappa}(x)\} \cap W \neq \emptyset, \text{ for all } j \geq 0. \quad (1)$$

On the other hand, let y be an arbitrary point in A. Then $\{y, f(y), \ldots, f^\kappa(y)\} \cap W = \emptyset$ since A is invariant. To each $j = 0, 1, \ldots, \kappa$, we associate an open neighbourhood $Z_j \subseteq V$ of $f^j(y)$, so $Z_j \cap W = \emptyset$. By continuity of f^j, we can find an open neighbourhood $V_j \subseteq V$ of y such that $f^j(V_j) \subseteq Z_j$, $j = 0, 1, \ldots, \kappa$. Put $\widetilde{V} = \bigcap_{s=1}^{\kappa} V_s$ (observe that $y \in \widetilde{V} \neq \emptyset$). In this case, for any $j \in \{0, 1, \ldots, \kappa\}$

$$f^j(\widetilde{V}) = f^j\left(\bigcap_{s=1}^{\kappa} V_s\right) \subseteq \bigcap_{s=1}^{\kappa} f^j(V_s) \subseteq f^j(V_j) \subseteq Z_j,$$

and therefore $f^j(\widetilde{V}) \cap W = \emptyset$. Finally, taking into account that $y \in \overline{\mathrm{Orb}_f(x)}$, there exists a certain iterate $f^q(x) \in \widetilde{V}$. Consequently, $f^{j+q}(x) \notin W$ for $j = 0, 1, \ldots, \kappa$, which contradicts (1).

*Remark.*Notice that any metric space is regular, so we have extended [5, Proposition 5, Chapter V] to a larger class of topological spaces. It is an open question to know whether the above result works or not in the more general setting of T_2-spaces. In the case of compact Hausdorff spaces, the property remains true: here the space is regular (see [9, Theorem 1.2, Chapter XI]), therefore it is regular and the property applies.

*Remark.*It is worth of mentioning that the above property is already established in [16], but no proof (or mention to the proof) is given in the survey paper. Consequently, for the sake of completeness we have included the proof in our paper.

4 Historical remarks

The dynamical notion of recurrence has its origin in the work of the French mathematician Henri Poincaré. In his studies about Celestial Mechanics he tried to find qualitative properties of all the solutions of certain systems of differential equations rather than to obtain analytic expressions of some solutions. Applying this new point of view, in [20, §8][1] appeared the original version of his celebrated Poincaré Recurrence Theorem in this form:

Théorème I. *Supposons que le point P reste à distance finie, et que le volume $\int dx_1 dx_2 dx_3$ soit un invariant intégral; si l'on considère une région r_0 quelconque, quelque petite que soit cette région, il y aura des trajectoires qui la traverseront une infinité de fois.*

This theorem also appears in [21, §26, Stabilité à la Poisson], the usual reference to cite Poincaré Recurrence Theorem. Roughly speaking, it establishes that given a system of differential equations whose phase space is a bounded domain V inside the n-dimensional Euclidean space $\mathbb{R}^n$, and we suppose that the system preserves the flow volume (respect with an appropriate measure) then we can choose initial conditions such that their solutions intersect indefinitely any region $r_0 \subseteq V$.

Following [12], the Poincaré's memory [20] can be consider as one of the origins of both the disciplines we now call Topological Dynamics and Ergodic Theory.[2] On one hand, the study of the stability of periodic solutions is made under discretizing the system by the Poincaré's map and starts the topological approach to the dynamics, and on the other hand the probabilistic study of the recurrence will be give rise a few decades later to the ergodic theory:

"Nous conviendrons de dire que la probabilité pour que la position initiale du point mobile P appartienne à une certaine région r_0 est à la probabilité pour que cette position initiale appartienne à une autre région r'_0 dans le même rapport que le volume de r_o au volume de r'_0. Les probabilités ainsi définies, je me propose d'établir que la probabilité pour qu'une trajectoire issue d'un point de r_0 ne traverse pas cette région plus de k fois est nulle, quelque grand que soit k et quelque petite que soit la région r_0. C'est là ce que j'entends quand je dis que les trajectoires qui ne traversent r_0 qu'un nombre fini de fois sont exceptionnelles"([20, §8]).

Concerning to this ergodic approach, a modern version[3] whose proof can be consulted in [24, p. 26], establishes[4]

[1] This famous memory won the King Oscar II's Prize competition. More information about the history of this competition and the analysis of the error in the Poincaré's memory can be consulted in [1].

[2] As J. de Vries explains in [8, p. 1], we can say that Topological Dynamics is *the study of transformation groups with respect to those topological properties whose prototype occurred in classical dynamics*. Rephrasing [15, p. iii] and replacing topological by measure-theoretic we have the definition of Ergodic Theory. As a curiosity the word "ergodic" is an amalgamation of the Greek words "ergon" (work) and "odos" (path) ([24, p. 2]).

[3] By using the modern measure theory developed by Lebesgue, Carathéodory was the first mathematician to state and prove a modern version of Poincaré's Recurrence Theorem, see [6].

[4] If $(X, \mathscr{B}, m)$ is a probability space, a measure-preserving transformation $T : X \to X$ is a measurable map (that is,

Theorem 1(Poincaré Recurrence Theorem). *Let $T:X\to X$ be a measure-preserving transformation of a probability space $(X,\mathscr{B},m)$. Let $E\in\mathscr{B}$ with $m(E)>0$. Then almost all points of E return infinitely often to E under positive iteration by T (i.e, there exists $F\subset E$ with $m(F)=m(E)$ such that for every $x\in F$ there is a sequence $n_1<n_2<\ldots$ of natural numbers with $T^{n_j}(x)\in F$ for each j).*

As a consequence, we can affirm that if we impose some conditions on finiteness of a measure and on preservation of that measure, we deduce that almost every point x is recurrent (except for a set of zero measure), we obtain qualitative information of the asymptotics of the orbits instead of knowing a precise analytic expression for these orbits.

For more information about Poincaré Recurrence Theorem, we recommend the reading of [13], [12] and [2].

The French mathematician uses the terminology *stability in the sense of Poisson* to denote our actual notion of recurrence. Other authors use a different terminology to appoint the concept of recurrence and other related notions of return. To have information on this different terminology, the reader can consult the comparative table appearing in [8, p. 150]. In this table we find a list of different books on topological dynamics ([15], [19], [13], [24], [8], ...) and the used terminology in each of them. As a curiosity, in some cases, as interval maps, this range allows us to detect chaos[5] whenever the set of recurrent and uniformly recurrent points are not equal([5, §VI, Corollary 8]). In [22, Theorem 4.19] there is a list of 45 equivalent statements to characterize simple interval maps, those having no periodic points of period not a power of two.

The next milestone to be mentioned is the well-known Birkhoff Recurrence Theorem which can be considered as the discrete counterpart of Poincaré's Recurrence Theorem. [6] It establishes that if (X,d) is a compact metric space and $T:X\to X$ is a homeomorphism, then $\mathrm{Rec}(f)\neq\emptyset$, that is, there exists a point $x\in X$ and a subsequence $(n_j)_j$ of increasing positive integers such that $d(T^{n_j}(x),x)\to 0$. In fact, the result remains true if T is continuous. [7] It is worth point out that Birkhoff uses the word recurrent point (in the frame of a continuous flow in a compact space) in the sense of uniformly recurrent point ([4, p. 199]): "*In order that a point group generated by the motion P_t be recurrent, it is necessary and sufficient that for any positive quantity ε, however small, there exists a positive quantity T so large that any arc P_tP_{t+T} of the curve of motion has points within distance ε for every point of the curve of motion*".

Actually, we can say more: if X is a compact topological space and $f\in C(X,X)$, then $\mathrm{Rec}(f)\neq\emptyset$ since the compactness implies the existence of a minimal set $M\subseteq X$ (the proof is based on Zorn's Lemma) and Proposition 2 applies.

5 Conclusion

In this note we have presented three equivalent definitions of recurrent points in the context of discrete dynamical systems. From Section 4 we have seen that in its origin this notion was established in the context of continuous systems moving into bounded regions of the n-dimensional Euclidean space. As a direct consequence, when discrete recurrence is taking into account the most part of the authors agreed with considering compact metric spaces. Nevertheless, it is immediate to translate the notion to arbitrary topological spaces. Similar studies can be done when analyzing other topological properties of the systems (uniformly recurrence, minimality, transitivity, etcetera). So the line of researching we stress here proposes to analyze the topological properties of a system from the more basic set of assumptions, without separation axioms to be possible. Finally, another conclusion we extract from this note is the helpful of the set of returning points in order to discuss the simplicity or complexity of a discrete system, in particular to interval maps. Let us mention that the analysis of complexity in nonlinear dynamics is a subject extensively treated by J. Sousa Ramos and collaborators.

$T^{-1}(\mathscr{B})\subseteq\mathscr{B})$) such that $m(T^{-1}(A))=m(A)$ for all the elements A of the σ-algebra $\mathscr{B}$.

[5] In the sense that $f\in C([0,1],[0,1])$ has a periodic point whose period is not a power of 2 (or equivalently, that the topological entropy $h(f)$ is positive – see [24] for a general definition of topological entropy for compact metric spaces and [5, §VIII, Prop. 34] for a proof of this equivalence).

[6] In fact, by [17] it is known that if $T:X\to X$ is a continuous transformation of a compact metric space X into itself, then there exists a probability measure μ in the probabilistic space $(X,\mathscr{B}(X),\mu)$, where $\mathscr{B}(X)$ is the Borel σ-algebra, such that T preserves the measure; consequently, by applying Poincaré Recurrence Theorem we have that T has recurrent points.

[7] It is interesting to mention the so-called Birkhoff Multiple Recurrence Theorem, due to Furstenberg and Weiss [11], which ensures the existence of common recurrent points (multiply recurrent points) for a finite number of continuous maps $T_1,\ldots,T_N$ from a compact metric space X into itself such that $T_i\circ T_j=T_j\circ T_i$ with $i,j\in\{1,\ldots,N\}, i\neq j$. As a curiosity, this result of topological dynamics allows us to find a new proof of the celebrated van der Waerden's Theorem [23]: For any partition of the natural numbers into a finite number of classes $\mathbb{N}=A_1\cup A_2\cup\ldots\cup A_N$, at least one element A_j of the partition contains arithmetic progressions of arbitrary length. Even more, in [11] (see also the monograph [13]) it is highlighted the surprising relationship between the fields of Topological Dynamics and Combinatorial Number Theory.

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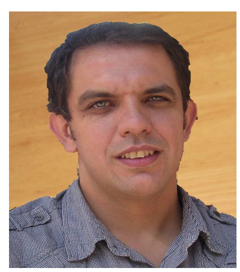

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