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Structure of the Global Attractor for Weak Solutions of a Reaction-Diffusion Equation

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Abstract: In this paper we study the structure of the global attractor for a multivalued semiflow generated by weak solutions of a reaction-diffusion equation in which uniqueness of the Cauchy problem is not guaranteed, improving the results of a previous paper. Under suitable assumptions, we prove that the global attractor can be characterized using either the unstable manifold of the set of stationary points or the stable one but considering in this last case only solutions in the set of bounded complete trajectories.

Keywords: reaction-diffusion equations, set-valued dynamical system, global attractor, unstable manifolds

This paper is dedicated to the memory of Professor José Sousa Ramos.

1 Introduction

The problem of studying the structure of global attractors for infinite-dimensional dynamical systems is amazing. In the particular case of reaction-diffusion equations beautiful results in this direction have been proved (see e.g. [6], [7], [11], [20], [21]).

The first step in such problems is to establish that the global attractor is the unstable manifold of the set of stationary points. In the single-valued case, when for example the nonlinear term is a polynomial or its derivative satisfies some assumptions, this is well known [3], [4], [23]. The problem is more complicated when uniqueness of the Cauchy problem is not guaranteed. In such a case, a multivalued semiflow has to be defined, rather than a semigroup, and different types of solutions can be considered. Some results in this direction have been obtained for differential inclusions and reaction-diffusion equations (see [2], [10], [13]). We observe that new interesting situations can appear in such equations (see [2]).

In this paper we will study the structure of the global attractor of a reaction-diffusion equation in which the nonlinear term satisfy suitable growth and dissipative conditions, but there is no condition ensuring uniqueness of the Cauchy problem (like e.g. a monotonicity assumption). Such equation generates in the general case a multivalued semiflow having a global compact attractor (see [12], [15], [25]), which is the union of all bounded complete trajectories of the semiflow. In our previous paper [13] three different semiflows are considered, depending on the regularity of the solutions: weak, regular or strong ones. In the case of the semiflows generated by either regular or strong solutions, it is proved that the global attractor is the unstable manifold of the set of stationary points and the stable one but considering in this last case only solutions in the set of bounded complete trajectories. However, in the case of weak solutions such result was not obtained, but a weaker one stating that the attractor is the closure of the stable set restricted to the set of bounded complete trajectories. Now, we improve the theorem of [13] for weak solutions and obtain, under additional assumptions on the parameters of the problem, the same result as for regular and strong solutions.

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2 Setting of the problem

In a bounded domain $\Omega \subset \mathbb{R}^3$ with sufficiently smooth boundary $\partial\Omega$ we consider the problem

$$\begin{cases} u_t - \Delta u + f(u) = h, \quad x \in \Omega,\ t > 0, \\ u|_{\partial\Omega} = 0, \\ u(0) = u_0, \end{cases} \tag{1}$$

where

$$\begin{array}{c} h \in L^2(\Omega), \\ f \in C(\mathbb{R}), \\ |f(u)| \le C_1(1 + |u|^{p-1}), \quad \forall u \in \mathbb{R}, \\ f(u)u \ge \alpha u^p - C_2, \quad \forall u \in \mathbb{R}, \end{array} \tag{2}$$

with $2 \le p \le 4$, $C_1, C_2, \alpha > 0$.

We denote by A the operator $-\Delta$ with Dirichlet boundary conditions, so that $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. As usual, denote the eigenvalues and the eigenfunctions of A by λ_i, e_i, $i = 1, 2...$

Denote $F(u) = \int_0^u f(s)ds$. From (2) we have that $\liminf\limits_{|u|\to\infty} \frac{f(u)}{u} = \infty$, and for some $D_1, D_2, \delta > 0$,

$$|F(u)| \le D_1(1 + |u|^p),\ F(u) \ge \delta |u|^p - D_2, \tag{3}$$

for all $u \in \mathbb{R}$.

In what follows we denote $H = L^2(\Omega)$, $V = H_0^1(\Omega)$, and $\|\cdot\|$, $(\cdot,\cdot)$ will be the norm and the scalar product in $L^2(\Omega)$. We denote by $\|\cdot\|_X$ the norm in the abstract Banach space X, whereas $(\cdot,\cdot)_Y$ will be the scalar product in the abstract Hilbert space Y. Also, $P(X)$ will be the set of all non-empty subsets of X.

On the other hand, we define the usual spaces

$$V^{2\alpha} = D(A^\alpha) = \{u \in H : \sum_{i=1}^{\infty} \lambda_i^{2\alpha} |(u, e_i)|^2 < \infty\},$$

where $\alpha \ge 0$. We recall the following well known result, which is a particular case of [22, Lemma 37.8] for our operator $A = -\Delta$ in a three-dimensional domain.

Lemma 1. *$D(A^\alpha) \subset W^{k,q'}(\Omega)$ whenever $q' \ge 2$ and k is an integer such that*

$$k - \frac{3}{q'} < 2\alpha - \frac{3}{2}.$$

Also, it is well known that $V^s \subset H^s(\Omega)$ for all $s \ge 0$ (see [24, Chapter IV] or [19]).

A function

$$u \in L^2_{loc}(0, +\infty; V) \bigcap L^p_{loc}(0, +\infty; L^p(\Omega))$$

is called a weak solution of (1) on $(0, +\infty)$ if for all $T > 0$, $v \in V$, $\eta \in C_0^\infty(0, T)$,

$$\int_0^T (u, v)\eta_t dt = \int_0^T ((u, v)_V + (f(u), v) - (h, v))\, \eta dt.$$

It is well known [1, Theorem 2] or [5, p.284] that for any $u_0 \in H$ there exists at least one weak solution of (1) with $u(0) = u_0$ (and it may be non unique) and that any weak solution of (1) belongs to $C([0, +\infty); H)$. Moreover, the function $t \mapsto \|u(t)\|^2$ is absolutely continuous and

$$\begin{array}{c} \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + \|u(t)\|_V^2 \\ +(f(u(t)), u(t)) - (h, u(t)) = 0 \text{ a.e.} \end{array} \tag{4}$$

We define

$$\begin{array}{c} K^+ = \{u(\cdot) : u(\cdot) \text{ is a weak solution of (1)}\}, \\ G : \mathbb{R}^+ \times H \to P(H), \\ G(t, u_0) = \{u(t) : u(\cdot) \in K^+,\ u(0) = u_0\}. \end{array} \tag{5}$$

Definition 1. *Let X be a complete metric space with metric ρ. A multivalued map $G : \mathbb{R}^+ \times X \to P(X)$ is a multivalued semiflow (m-semiflow) if:*

1. $G(0, u_0) = u_0$, $\forall u_0 \in X$;
2. $G(t + s, u_0) \subset G(t, G(s, u_0))$, $\forall\, t, s \ge 0$, $\forall u_0 \in X$.

It is called strict if

$$G(t + s, u_0) = G(t, G(s, u_0)),$$

for all $t, s \ge 0$, $u_0 \in X$

Definition 2. *A set $\Theta \subset X$ is called a global attractor of G, if:*

1. *$\Theta \subset G(t, \Theta)$, $\forall t \ge 0$ (i.e. it is negatively semi-invariant);*
2. *For any bounded set $B \subset X$,*

$$\text{dist}_X(G(t, B), \Theta) \to 0,\ \text{as } t \to +\infty, \tag{6}$$

where

$$\text{dist}_X(A, B) = \sup_{x\in A} \inf_{y\in B} \rho(x, y);$$

3. *It is minimal, that is, for any closed set C satisfying (6) it holds $\Theta \subset C$.*

The global attractor is called invariant if $\Theta = G(t, \Theta)$, $\forall t \ge 0$.

The map G defined by (5) is a strict multivalued semiflow which possesses a global compact invariant connected attractor [12], [15], [16]. Our aim is to give a characterization of the attractor. First we shall define complete trajectories for problem (1).

Definition 3. *A map $\gamma : \mathbb{R} \to H$ is called a complete trajectory of K^+ if*

$$\gamma(\cdot + h)|_{[0,+\infty)} \in K^+,\ \forall h \in \mathbb{R},$$

that is, if $\gamma|_{[\tau,+\infty)}$ is a weak solution of (1) on $(\tau, +\infty)$, $\forall \tau \in \mathbb{R}$. We denote by $\mathbb{F}$ the set of all complete trajectories of K^+.

Let $\mathbb{K}$ be the set of all bounded (in the H norm) complete trajectories. It is shown in [13] that the global attractor of G is the union of all bounded complete trajectories, and also that $\gamma(\cdot)$ is a complete trajectory of K^+ if and only if the map $t \mapsto \gamma(t)$ is continuous and

$$\gamma(t+s) \in G(t, \gamma(s)), \ \forall t \geq 0, s \in \mathbb{R}.$$

We recall that the global attractor Θ is called stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$G(t, O_\delta(\Theta)) \subset O_\varepsilon(\Theta), \forall\, t \geq 0.$$

In [14, Theorem 3.18] the global attractor of the m-semiflow G is proved to be stable.

Summarizing all these results we have the following theorem.

Theorem 1. *Under conditions (2) the m-semiflow (5) has a global compact invariant attractor $\Theta \subset H$ which is connected, stable and*

$$\Theta = \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}\} = \bigcup_{t \in \mathbb{R}} \{\gamma(t) : \gamma(\cdot) \in \mathbb{K}\}. \qquad (7)$$

Let $\mathfrak{R}$ be the set of all stationary points of (1), i.e., the points $u \in V$ such that

$$-\Delta u + f(u) = h \text{ in } H^{-1}(\Omega). \qquad (8)$$

It is proved in [13, Lemmas 12, 14 and Theorem 13] that $\mathfrak{R} \neq \varnothing$ and the following properties are equivalent:

1. $u_0 \in \mathfrak{R}$;
2. $u_0 \in G(t, u_0)$ for all $t \geq 0$;
3. The function $u(t) \equiv u_0$ belongs to K^+.

We define now the sets:

$$M^-(\mathfrak{R}) = \left\{ \begin{array}{c} z : \exists \gamma(\cdot) \in \mathbb{K}, \ \gamma(0) = z, \\ \text{dist}_H(\gamma(t), \mathfrak{R}) \to 0, \ t \to +\infty \end{array} \right\},$$
$$M^+(\mathfrak{R}) = \left\{ \begin{array}{c} z : \exists \gamma(\cdot) \in \mathbb{F}, \ \gamma(0) = z, \\ \text{dist}_H(\gamma(t), \mathfrak{R}) \to 0, \ t \to -\infty \end{array} \right\}. \qquad (9)$$

$M^+(\mathfrak{R})$ is the unstable set of $\mathfrak{R}$. $M^-(\mathfrak{R})$ is the stable set of $\mathfrak{R}$ but considering only bounded complete trajectories.

It is known [3], [4, p.106], [23] that under additional assumptions ensuring that G is a single-valued semigroup, the set Θ is bounded in $H^2(\Omega) \bigcap H_0^1(\Omega)$ and

$$\Theta = M^+(\mathfrak{R}).$$

Moreover, in [4, p.106] it is proved that

$$\Theta = M^+(\mathfrak{R}) = M^-(\mathfrak{R}). \qquad (10)$$

We observe that an equivalent definition of the set $M^+(\mathfrak{R})$ is the following

$$M^+(\mathfrak{R}) = \left\{ \begin{array}{c} z : \exists \gamma(\cdot) \in \mathbb{K}, \ \gamma(0) = z, \\ \text{dist}_H(\gamma(t), \mathfrak{R}) \to 0, \ t \to -\infty \end{array} \right\},$$

as for every complete trajectory $\gamma(\cdot) \in \mathbb{F}$ as in (9) we have that the set $\cup_{t \in \mathbb{R}} \gamma(t)$ is bounded, so that the inclusion $\gamma(\cdot) \in \mathbb{K}$ follows.

The aim of our paper is to obtain (10) for K^+ under suitable assumptions.

3 Previous results

If we consider the multivalued semiflow generated by regular or strong solutions of (2), then formula (10) is shown to be true in [13]. We shall recall these theorems in this section. We note that, although in [13] these results are stated for $p = 4$, the proofs work for the more general case where $2 \leq p \leq 4$.

3.1 Regular solutions

The function

$$u \in L^2_{loc}(0, +\infty; V) \bigcap L^p_{loc}(0, +\infty; L^p(\Omega))$$

is called a regular solution of (1) on $(0, +\infty)$ if for all $T > 0$, $v \in V$ and $\eta \in C_0^\infty(0, T)$ we have

$$\int_0^T (u, v)\eta_t dt = \int_0^T ((u, v)_V + (f(u), v) - (h, v))\, \eta dt, \quad (11)$$

and

$$u \in L^\infty(\varepsilon, T; V), \qquad (12)$$
$$u_t \in L^2(\varepsilon, T; H), \ \forall\, 0 < \varepsilon < T. \qquad (13)$$

Any regular solution u satisfies

$$u \in L^2(\varepsilon, T; D(A)). \qquad (14)$$

Also, for any $u_0 \in H$ there exists at least one regular solution $u(\cdot)$ such that $u(0) = u_0$.

Let

$$K_r^+ = \{u(\cdot) : u \text{ is a regular solution of (1)}\}.$$

We define now the map $G_r : \mathbb{R}^+ \times H \to P(H)$ by

$$G_r(t, u_0) = \{u(t) : u \in K_r^+ \text{ and } u(0) = u_0\},$$

which is a multivalued semiflow. G_r possesses a global compact attractor Θ_r. Moreover, for any set B bounded in H we have

$$dist_V(G_r(t, B), \Theta_r) \to 0 \text{ as } t \to +\infty, \qquad (15)$$

and also that Θ_r is compact in V.

We can characterize the attractor as the union of all bounded complete trajectories. The map $\gamma : \mathbb{R} \to L^2(\Omega)$ is called a complete trajectory of K_r^+ if

$$\gamma(\cdot + h)\mid_{[0,+\infty)} \in K_r^+ \text{ for any } h \in \mathbb{R}.$$

The set of all complete trajectories of K_r^+ will be denoted by $\mathbb{F}_r$. Let $\mathbb{K}_r$ be the set of all complete trajectories which are bounded in H, and let $\mathbb{K}_r^1$ be the set of all complete trajectories which are bounded in V. These sets are proved to coincide, that is, $\mathbb{K}_r = \mathbb{K}_r^1$. Moreover, Θ_r is the union of all points lying in a bounded complete trajectory, that is,

$$\begin{aligned}\Theta_r &= \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_r\} \\ &= \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_r^1\} \\ &= \cup_{t\in\mathbb{R}}\{\gamma(t) : \gamma(\cdot) \in \mathbb{K}_r\} \\ &= \cup_{t\in\mathbb{R}}\{\gamma(t) : \gamma(\cdot) \in \mathbb{K}_r^1\}.\end{aligned} \tag{16}$$

As in the case of weak solutions, the following properties are equivalent:

1. $u_0 \in \mathfrak{R}$;
2. $u_0 \in G_r(t, u_0)$ for all $t \geq 0$;
3. The function $u(t) \equiv u_0$ belongs to K_r^+.

It is clear that if the map $\gamma : \mathbb{R} \to H$ is a complete trajectory of K_r^+, then

$$\gamma(t+s) \in G_r(t, \gamma(s)) \text{ for all } s \in \mathbb{R} \text{ and } t \geq 0. \tag{17}$$

Conversely, the map $\gamma : \mathbb{R} \to H$ is a complete trajectory of K_r^+ if and only if $\gamma \in L^\infty_{loc}(\mathbb{R}; V)$, $\gamma_t \in L^2_{loc}(\mathbb{R}; H)$ and (17) holds.

We define the sets

$$M_r^-(\mathfrak{R}) = \left\{ \begin{array}{c} z : \exists\gamma(\cdot) \in \mathbb{K}_r,\ \gamma(0) = z, \\ \mathrm{dist}_H(\gamma(t), \mathfrak{R}) \to 0,\ t \to +\infty \end{array} \right\},$$
$$M_r^+(\mathfrak{R}) = \left\{ \begin{array}{c} z : \exists\gamma(\cdot) \in \mathbb{F}_r,\ \gamma(0) = z, \\ \mathrm{dist}_H(\gamma(t), \mathfrak{R}) \to 0,\ t \to -\infty \end{array} \right\}.$$

As in the case of weak solutions, in the definition of $M_r^+(\mathfrak{R})$ we can replace $\mathbb{F}_r$ by $\mathbb{K}_r$, since every γ as given in the definition of $M_r^+(\mathfrak{R})$ belongs to $\mathbb{K}_r$.

Theorem 2. *Under conditions (2) it holds*

$$\Theta_r = M_r^+(\mathfrak{R}) = M_r^-(\mathfrak{R}). \tag{18}$$

Moreover,

$$\begin{aligned} M_r^-(\mathfrak{R}) &= \left\{ \begin{array}{c} z : \exists\gamma(\cdot) \in \mathbb{K}_r,\ \gamma(0) = z, \\ \mathrm{dist}_V(\gamma(t), \mathfrak{R}) \to 0,\ t \to +\infty \end{array} \right\}, \\ M_r^+(\mathfrak{R}) &= \left\{ \begin{array}{c} z : \exists\gamma(\cdot) \in \mathbb{F}_r,\ \gamma(0) = z, \\ \mathrm{dist}_V(\gamma(t), \mathfrak{R}) \to 0,\ t \to -\infty \end{array} \right\}. \end{aligned} \tag{19}$$

3.2 Strong solutions

In this section we shall define a semiflow in the phase space V. For this aim we introduce now a stronger concept of solution for (1).

The function

$$u \in L^2_{loc}(0, +\infty; V) \bigcap L^p_{loc}(0, +\infty; L^p(\Omega))$$

is called a strong solution of (1) on $(0, +\infty)$ if for all $T > 0$, $v \in V$ and $\eta \in C_0^\infty(0, T)$ we have that (11) holds and

$$\begin{aligned} &u \in L^\infty(0, T; V), \\ &u_t \in L^2(0, T; H),\ \forall\, T > 0. \end{aligned}$$

Any strong solution u satisfies

$$u \in L^2(0, T; D(A)) \cap C([0, +\infty); V).$$

For any $u_0 \in V$ there exists at least one strong solution $u(\cdot)$ such that $u(0) = u_0$.

Let

$$K_s^+ = \{u(\cdot) : u \text{ is a strong solution of (1)}\}.$$

We define now the map $G_s : \mathbb{R}^+ \times V \to P(V)$ by

$$G_s(t, u_0) = \{u(t) : u \in K_s^+ \text{ and } u(0) = u_0\},$$

which is a strict multivalued semiflow. G_s possesses a global compact attractor Θ_s in the phase space V. Moreover, $\Theta_s = \Theta_r$, that is, the regular and strong attractors coincide.

The map $\gamma : \mathbb{R} \to V$ is called a complete trajectory of G_s if

$$\gamma(\cdot + h)\mid_{[0,+\infty)} \in K_s^+ \text{ for any } h \in \mathbb{R}.$$

The set of all complete trajectories of K_s^+ will be denoted by $\mathbb{F}_s$. Let $\mathbb{K}_s$ be the set of all complete trajectories which are bounded in V. It holds $\mathbb{K}_s = \mathbb{K}_r^1 = \mathbb{K}_r$. That is, the sets of regular and strong complete bounded solutions are the same.

As before, we can characterize the attractor Θ_s as the union of all points lying in a bounded complete trajectory:

$$\Theta_s = \{\gamma(0) : \gamma(\cdot) \in \mathbb{K}_s\} = \cup_{t\in\mathbb{R}}\{\gamma(t) : \gamma(\cdot) \in \mathbb{K}_s\}.$$

Also, the following properties are equivalent:

1. $u_0 \in \mathfrak{R}$;
2. $u_0 \in G_s(t, u_0)$ for all $t \geq 0$;
3. The function $u(t) \equiv u_0$ belongs to K_s^+.

As in the case of weak solutions the map $\gamma : \mathbb{R} \to V$ is a complete trajectory of K_s^+ if and only if $\gamma(\cdot)$ is continuous and

$$\gamma(t+s) \in G_s(t, \gamma(s)) \text{ for all } s \in \mathbb{R} \text{ and } t \geq 0.$$

Finally, we have the following result.

Theorem 3. *Under conditions (2) it holds*

$$\Theta_s = M_s^+(\mathfrak{R}) = M_s^-(\mathfrak{R}),$$

where

$$M_s^-(\mathfrak{R}) = \left\{ \begin{array}{c} z : \exists\gamma(\cdot) \in \mathbb{K}_s,\ \gamma(0) = z, \\ \mathrm{dist}_V(\gamma(t), \mathfrak{R}) \to 0,\ t \to +\infty \end{array} \right\},$$
$$M_s^+(\mathfrak{R}) = \left\{ \begin{array}{c} z : \exists\gamma(\cdot) \in \mathbb{F}_s,\ \gamma(0) = z, \\ \mathrm{dist}_V(\gamma(t), \mathfrak{R}) \to 0,\ t \to -\infty \end{array} \right\}.$$

Remark. In the case of trajectory attractors some results in regular spaces have been obtained also in [8].

4 Structure of the global attractor

We shall prove that formula (10) is true for weak solutions if we assume additionally that either $h \in L^{\infty}(\Omega)$ or $p \leq 3$.

4.1 Regularity of the attractor in V

We shall prove that the global attractor is compact in V under the additional assumption $h \in L^{\infty}(\Omega)$. Using this result we will obtain formula (10).

First, we recall the following result about the boundedness of the global attractor in the space $L^{\infty}(\Omega)$.

Lemma 2. *[13, Lemma 20] Under conditions (2) and $h \in L^{\infty}(\Omega)$ the set Θ is bounded in $L^{\infty}(\Omega)$.*

Using this lemma we can prove now the compactness of the attractor in V.

Theorem 4. *Under conditions (2) and $h \in L^{\infty}(\Omega)$ the global attractor Θ is compact in V.*

Moreover, any weak solution $u(\cdot)$ with $u_0 \in \Theta$ is a strong solution. Hence, $G(t,u_0) = G_s(t,u_0)$ for all $u_0 \in \Theta$.

Proof. Let us consider an arbitrary bounded complete trajectory $u(\cdot) \in \mathbb{K}$. Due to the definition of weak solution $u(t) \in V$ for a.a $t \in \mathbb{R}$. We take such $\tau \in \mathbb{R}$ that $u(\tau) \in V$ and consider the following Cauchy problem

$$\begin{cases} v_t = \Delta v - \overline{f}(t,x) + h(x), \quad x \in \Omega,\ t > \tau, \\ v|_{\partial\Omega} = 0, \\ v|_{t=\tau} = u(\tau), \end{cases} \tag{20}$$

where $\overline{f}(t,x) = f(u(t,x))$. Since $u(t) \in \Theta$, for any $t \in \mathbb{R}$, and Θ is bounded in $L^{\infty}(\Omega)$ by Lemma 2, we have that $\overline{f} \in L^{\infty}(\mathbb{R} \times \Omega)$. Thus for the linear problem (20) from well-known results one can deduce that $v \in C([\tau,T];V) \cap L^2(0,T;D(A))$ for all $T > \tau$. From uniqueness of the solution of the Cauchy problem (20) $v \equiv u$ on $[\tau,+\infty)$ and, therefore, $u(t) \in V$ for all $t \geq \tau$. It means that $u(t) \in V$ for all $t \in \mathbb{R}$ and from formula (7), $\Theta \subset V$.

It follows that the restriction of $u(\cdot) \in \mathbb{K}$ to any interval $[\tau,+\infty)$ is a strong solution. Hence, any weak solution $u(\cdot)$ with $u_0 \in \Theta$ is a strong solution and $G(t,u_0) = G_s(t,u_0)$ for all $u_0 \in \Theta$.

From the energy inequality (see [15])

$$\|u(t)\|^2 + \int_s^t \|u(\tau)\|_V^2 d\tau + 2\alpha \int_s^t \|u(\tau)\|_{L^p(\Omega)}^p d\tau$$
$$\leq \|u(s)\|^2 + K\left(1 + \|h\|^2\right)(t-s)$$

and the boundedness of Θ in the space H we deduce that there exists $C_1 > 0$ such that

$$\int_t^{t+1} \|u(s)\|_V^2 ds \leq C_1(1+\|h\|^2),\ \forall\, t \in \mathbb{R}.$$

So, for arbitrary $t \in \mathbb{R}$ we find $\tau \in [t,t+1]$ such that $\|u(\tau)\|_V^2 \leq C_1(1+\|h\|^2)$. For problem (20) it is standard to obtain the inequality

$$\|v(t)\|_V^2 \leq e^{-\delta(t-\tau)}\|v(\tau)\|_V^2 + C_2,\ \forall\, t \geq \tau,$$

where $\delta,\ C_2 > 0$. Thus

$$\|u(t)\|_V^2 \leq C_1(1+\|h\|^2) + C_2,\ \forall\, t \in \mathbb{R}.$$

Hence, the global attractor is bounded in V.

Finally, let us prove that Θ is compact in V. Consider an arbitrary sequence $\{z_n\} \subset \Theta$. Since Θ is bounded in V, we can assume that $z_n \to z$ weakly in V. Then $\|z\|_V \leq \liminf \|z_n\|_V$. It remains to prove that $z_n \to z$ strongly in V. In view of (7) there exist $u_n \in \mathbb{K}$ such that $z_n = u_n(0)$. Since $\overline{f}_n(t,x) = f(u_n(t,x))$ is bounded in $L^{\infty}((-1,0)\times\Omega)$ and $\|u_n(t)\|_V \leq C_3$ for all n and $t \in [-1,0]$, we obtain

$$\left\|\frac{du_n}{dt}\right\|^2 + \frac{d}{dt}\|u_n\|_V^2 \leq C_4 \tag{21}$$

and

$$\int_{-1}^{0} \left\|\frac{du_n}{dt}\right\|^2 ds \leq C_5,$$
$$\int_{-1}^{0} \|\Delta u_n\|^2 ds \leq C_6, \text{ for all } n.$$

Hence, in a standard way using the Ascoli-Arzelà theorem we have that up to a subsequence

$$\begin{aligned} &u_n \to u \text{ in } C([-1,0],H), \\ &u_n \to u \text{ weakly in } L^2\left(-1,0;H^2(\Omega)\right), \\ &\frac{du_n}{dt} \to \frac{du}{dt} \text{ weakly in } L^2(-1,0;H). \end{aligned} \tag{22}$$

Then a standard argument gives

$$u_n(t_n) \to u(t) \text{ weakly in } V \text{ if } t_n \to t \in [-1,0].$$

On the other hand, the previous estimates imply by the Compactness Theorem [18] that

$$u_n \to u \text{ strongly in } L^2(-1,0;V). \tag{23}$$

Also, it is standard to show that $u(\cdot)$ is a weak solution to (1) with $u(0) = z$.

In view of (21) the functions $J_n(t) = \|u_n(t)\|_V^2 + C_4 t$, $J(t) = \|u(t)\|_V^2 + C_4 t$ are continuous and non-increasing in $[-1,0]$. Moreover, (23) implies that $J_n(t) \to J(t)$ for a.a. $t \in (-1,0)$. Take $-1 < t_m < 0$ such that $t_m \to 0$ and $J_n(t_m) \to J(t_m)$ for all m. Then

$$\begin{aligned} &J_n(0) - J(0) \\ &\leq J_n(t_m) - J(0) \\ &\leq |J_n(t_m) - J(t_m)| + |J(t_m) - J(0)|, \end{aligned}$$

so that for any $\varepsilon > 0$ there exist $m(\varepsilon)$ and $N(m)$ such that $J_n(0) - J(0) \leq \varepsilon$ if $n \geq N$. Then $\lim\ \sup J(t_n) \leq \lim\ \sup J(0)$, so that

$$\lim\ \sup \|u^n(0)\|_V^2 \leq \|u(0)\|_V^2 .$$

As $\|z\|_V \leq \liminf\ \|z_n\|_V$, we obtain

$$\|z_n\|_V \to \|z\|_V ,$$

so that $z_n \to z$ strongly in V.

Further, we can prove the main result about the structure of the global attractor for weak solutions.

Theorem 5. *Under conditions (2) and $h \in L^\infty(\Omega)$ equality (10) holds.*

Proof. First, we prove that $\Theta = \Theta_s$. In view of Theorem 4, $G(t,u_0) = G_s(t,u_0)$ for any $u_0 \in \Theta$. Also, Θ is compact in V. Hence, for any $\varepsilon > 0$ there exists $T(\varepsilon)$ such that

$$\Theta = G(t,\Theta) = G_s(t,\Theta) \subset O_\varepsilon(\Theta_s) \text{ for } t \geq T.$$

Thus, $\Theta \subset \Theta_s$. Since the converse inclusion is obvious, we obtain $\Theta = \Theta_s$.

Now, by Theorem 3 we have

$$\Theta = \Theta_s = M_s^+(\mathfrak{R}) = M_s^-(\mathfrak{R}),$$

where

$$M_s^-(\mathfrak{R}) = \left\{ \begin{array}{c} z : \exists \gamma(\cdot) \in \mathbb{K}_s,\ \gamma(0) = z, \\ \mathrm{dist}_V(\gamma(t),\mathfrak{R}) \to 0,\ t \to +\infty \end{array} \right\},$$
$$M_s^+(\mathfrak{R}) = \left\{ \begin{array}{c} z : \exists \gamma(\cdot) \in \mathbb{F}_s,\ \gamma(0) = z, \\ \mathrm{dist}_V(\gamma(t),\mathfrak{R}) \to 0,\ t \to -\infty \end{array} \right\}.$$

Since $M_s^+(\mathfrak{R}) \subset M^+(\mathfrak{R})$, we have $\Theta \subset M^+(\mathfrak{R})$. But $M^+(\mathfrak{R}) \subset \Theta$ follows from (7), so that $\Theta = M^+(\mathfrak{R})$ follows. In the same way we obtain $\Theta = M^-(\mathfrak{R})$.

4.2 Regularity of weak solutions

In this section we will prove that with an additional assumption on p every weak solution is in fact a regular solution. From this fact we shall obtain formula (10).

Lemma 3. *Assume that $2 \leq p \leq 3$ in condition (2). Then any weak solution $u(\cdot)$ satisfies*

$$u \in C([\varepsilon,T];V) \cap L^2(\varepsilon,T;D(A)),$$
$$u_t \in L^2(\varepsilon,T;H),$$

for all $\varepsilon > 0$, i.e., it is a regular solution.

Proof. From

$$\int_\Omega |f(u(t,x))|^{\frac{p}{p-1}} dx \leq C_1 + C_2 \int_\Omega |u(t,x)|^p dx$$

we obtain that

$$\|f(u(t))\|^2_{L^{\frac{p}{p-1}}(\Omega)} \leq C_3 + C_4 \|u(t)\|^{2p-2}_{L^p(\Omega)}.$$

Using the Sobolev embedding $H^r(\Omega) \subset L^p(\Omega)$ if $r = \left(\frac{3}{2} - \frac{3}{p}\right) \leq \frac{1}{2}$ (as $p \leq 3$) and the Gagliardo-Nirenberg inequality

$$\|v\|_{H^r(\Omega)} \leq C_5 \|v\|_{H^{\frac{1}{2}}(\Omega)} \leq C_6 \|v\|^{\frac{1}{2}} \|v\|^{\frac{1}{2}}_{H^1(\Omega)},$$

we have

$$\begin{aligned} &\|f(u(t))\|^2_{L^{\frac{p}{p-1}}(\Omega)} \\ &\leq C_3 + C_7 \|u(t)\|^{p-1} \|u(t)\|^{p-1}_{H^1(\Omega)} \\ &\leq C_8 + C_9 \|u(t)\|^2 \|u(t)\|^2_{H^1(\Omega)}. \end{aligned}$$

Thus,

$$\begin{aligned} &\|f(u)\|_{L^2\left(0,T;L^{\frac{p}{p-1}}(\Omega)\right)} \\ &\leq C_{10}\left(1 + \|u\|_{C([0,T];H)} \|u\|_{L^2(0,T;H^1(\Omega))}\right). \end{aligned}$$

Set $d(t,x) = f(u(t,x))$ for $(t,x) \in (0,T) \times \Omega$. Then

$$\begin{aligned} &d \in L^2\left(0,T;L^{\frac{p}{p-1}}(\Omega)\right) \subset L^2\left(0,T;H^{-r}(\Omega)\right) \\ &\subset L^2\left(0,T;V^{-r}\right) \subset L^2\left(0,T;V^{r-1}\right). \end{aligned}$$

We consider the problem

$$\begin{cases} v_t - \Delta v = -d(t,x) + h(x), & x \in \Omega,\ t > 0, \\ v|_{\partial\Omega} = 0, \\ v(\tau) = u(\tau). \end{cases}$$

We note that $u(\tau) \in V \subset V^r$ for a.a. $\tau > 0$. For such τ in view of [22, p.163, Th. 42.12] there exists a unique weak solution $v(\cdot)$ such that $v \in C([\tau,T];V^r) \cap L^2\left(\tau,T;V^{r+1}\right)$. Hence, $u \in C([\varepsilon,T];V^r) \cap L^2\left(\varepsilon,T;V^{r+1}\right)$ for all $\varepsilon > 0$.

We shall prove that $f(u(\cdot)) \in L^2(\varepsilon,T;H)$. As this is obvious if $p = 2$, we consider that $2 < p \leq 3$. We note that $V^r \subset H^r(\Omega) \subset L^p(\Omega)$. Also, by Lemma 1 with $\alpha = \frac{r+1}{2}$, $r = 3\left(\frac{1}{2} - \frac{1}{p}\right)$, $k = 1$ we obtain that $V^{r+1} \subset W^{1,q'}(\Omega)$ for any $q' < p$. On the other hand, by the Sobolev embedding theorems we have $W^{1,q'}(\Omega) \subset L^q(\Omega)$, for $1 \leq q < \frac{3p}{3-p}$ if $2 < p < 3$ ($q < +\infty$ if $p = 3$). Thus, the inequality $p(p-1) < \frac{3p}{3-p}$, for all $2 < p < 3$, implies that

$$u \in C([\varepsilon,T];L^p(\Omega)) \cap L^2\left(\varepsilon,T;L^{p(p-1)}(\Omega)\right).$$

By (2) we have

$$\begin{aligned}
&\|f(u(t))\|^2 \\
&= \int_\Omega |f(u(t,x))|^2 dx \\
&\le C_{11} + C_{12} \int_\Omega |u(t,x)|^{2(p-1)} dx \\
&\le C_{13} + C_{14} \|u(t)\|_{L^p(\Omega)}^{p-1} \|u(t)\|_{L^{p(p-1)}(\Omega)}^{p-1}.
\end{aligned}$$

Therefore, $f(u(\cdot)) \in L^2(\varepsilon, T; H)$. Then standard results imply that

$$u \in C([\varepsilon, T]; V) \cap L^2(\varepsilon, T; D(A))$$

and $u_t \in L^2(\varepsilon, T; H)$.

Remark. For reaction-diffusion inclusions similar results about the regularity of weak solutions has been obtained in [9], [17].

Now, we can prove the result about the structure of the global attractor for weak solutions.

Theorem 6. *Assume that* $2 \le p \le 3$ *in condition (2). Then equality (10) holds.*

Proof. First, we prove that $\Theta = \Theta_r$. In view of Lemma 3, $G(t, u_0) = G_r(t, u_0)$ for any $u_0 \in H$. Hence, for any $\varepsilon > 0$ there exists $T(\varepsilon)$ such that

$$\Theta = G(t, \Theta) = G_r(t, \Theta) \subset O_\varepsilon(\Theta_r) \text{ for } t \ge T.$$

Thus, $\Theta \subset \Theta_r$. Since the converse inclusion is obvious, we obtain $\Theta = \Theta_r$.

Now, by Theorem 2 we have

$$\Theta = \Theta_r = M_r^+(\mathfrak{R}) = M_r^-(\mathfrak{R}),$$

where

$$M_r^-(\mathfrak{R}) = \left\{ \begin{array}{c} z : \exists \gamma(\cdot) \in \mathbb{K}_r,\ \gamma(0) = z, \\ \text{dist}_V(\gamma(t), \mathfrak{R}) \to 0,\ t \to +\infty \end{array} \right\},$$
$$M_r^+(\mathfrak{R}) = \left\{ \begin{array}{c} z : \exists \gamma(\cdot) \in \mathbb{F}_r,\ \gamma(0) = z, \\ \text{dist}_V(\gamma(t), \mathfrak{R}) \to 0,\ t \to -\infty \end{array} \right\}.$$

Since $M_r^+(\mathfrak{R}) \subset M^+(\mathfrak{R})$, we have $\Theta \subset M^+(\mathfrak{R})$. But $M^+(\mathfrak{R}) \subset \Theta$ follows from (7), so that $\Theta = M^+(\mathfrak{R})$ follows. In the same way we obtain $\Theta = M^-(\mathfrak{R})$.

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