Applied Mathematics & Information Sciences Letters
An International Journal

http://dx.doi.org/10.18576/amisl/040203

Characterizations of Hemirings by $(\in, \in \vee q_k)^*$-Intuitionistic Fuzzy k-Ideals

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Received: 26 May 2015, Revised: 17 Sep. 2015, Accepted: 30 Sep. 2015
Published online: 1 May 2016

Abstract: The aim of this paper is to introduce the notion of $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideal, which is the generalization of $(\in, \in \vee q)^*$-intuitionistic fuzzy k-ideal, and characterize hemirings by the properties of $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideals. And also characterized k-regular hemiring by the properties of $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideals.

Keywords: $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left k-ideal, $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right k-ideal

1 Introduction

A non-empty set R together with two associative binary operations, addition "$+$" and multiplication "$.$ " such that "$.$ "distributes over "$+$" from both sides, is called a Semiring. Semirings which are regarded as a generalization of rings, was first introduced by Vandiver. By a hemiring, we mean a semiring with a zero and with a commutative addition.

Ideals of hemirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals. Many results in rings apparently have no analogues in hemirings using only ideals. Henriksen [9] defined a more restricted class of ideals in semirings, which is called the class of k-ideals, with the property that if the semiring R is a ring then a complex in R is a k-ideal if and only if it is a ring ideal.

The fundamental concept of a fuzzy set, introduced by Zadeh [16], was applied by many researchers to generalize some of the basic concepts of algebra. In [13] Azirel Rosenfeld used the idea of fuzzy set to introduce the notions of fuzzy subgroups. In [2] Ahsan et el. initiated the study of fuzzy semirings. The fuzzy algebraic structures play an important role in mathematics with wide applications in theoretical physics, computer sciences, control engineering, information sciences, coding theory and topological spaces [7,15].

The general properties of fuzzy k-ideals of semirings were described in [5,6,8,11,17].

In this paper we define the concept of $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideals of hemiring and obtain some result concerning it, and characterize hemiregular hemirings with the help of $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideals.

2 preliminaries

Recall that a semiring is an algebraic system $(R, +, .)$ consisting of a non empty set R together with two binary operations "$+$" and "$.$ " , which are called addition and multiplication, respectively such that $(R, +)$ and $(R, .)$ are semigroups linked by the following distributive laws:

$a(b+c) = ab+ac$ and
$(a+b)c = ac+bc$

for all $a,b,c \in R$.

By a zero of a semiring $(R, +, .)$, we mean an element $0 \in R$ such that $0. x = x. 0 = 0$ and $0+x = x+0 = x$ for all $x \in R$. A semiring with a zero such that $(R,+)$ is a commutative semigroup is called a hemiring. A non-empty subset A of a hemiring R is called a sub hemiring of R if it contains 0 and closed with respect to addition and multiplication of R. A non-empty subset $I$of a hemiring R is called a left (right) ideal of R if I is closed under addition and $RI \subseteq I (IR \subseteq I)$. Furthermore I is called an ideal if it is both a left ideal and right ideal of R. A left (right) ideal of R is called a left

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(right) k-ideal of R if $a,b \in I$, $x \in R$, $x+a=b$ implies $x \in I$.

Let R be a hemiring and $A \subseteq R$. Then k-closure $\overline{A}$of A is defined by $\overline{A} = \{x \in R : x+a_1 = a_2$ for some $a_1, a_2 \in A\}$. If A is a left ideal of R, then $\overline{A}$ is the smallest left k-ideal of R containing A.

Let X be a non-empty fixed set. An intuitionistic fuzzy subset A of X is an object having the form

$$A = \{\langle x, \mu_A(x),\ \lambda_A(x)\ : x \in X\rangle\}$$

where the functions $\mu_A : X \longrightarrow [0,1]$ and $\lambda_A : X \longrightarrow [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of nonmembership (namely $\lambda_A(x)$) of each element of $x \in X$ to A, respectively, and $0 \leq \mu_A(x) + \lambda_A(x) \leq 1$ for all $x \in X$. For the sake of simplicity, we use the symbol $A = (\mu_A, \lambda_A)$ for the intuitionistic fuzzy subset (briefly, IFS) $A = \{\langle x, \mu_A(x),\ \lambda_A(x)\ : x \in X\rangle\}$. If $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ are intuitionistic fuzzy subsets of X, then

(1) $A \subseteq B \Longleftrightarrow \mu_A(x) \leq \mu_B(x)$ and $\lambda_A(x) \geq \lambda_B(x)$ for all $x \in X$.

(2) $A = B \Longleftrightarrow A \subseteq B$ and $B \subseteq A$.

(3) Complement of A is denoted and defined by $A' = (\lambda_A, \mu_A)$.

If $\{A_i : i \in I\}$ is a family of intuitionistic fuzzy subset of X, then by the union and intersection of this family we mean an intuitionistic fuzzy subsets

(4) $\cup_{i\in I} A_i = (\vee_{i\in I}\mu_{A_i}, \wedge_{i\in I}\lambda_{A_i})$

(5) $\cap_{i\in I} A_i = (\wedge_{i\in I}\mu_{A_i}, \vee_{i\in I}\lambda_{A_i})$.

Let a be a point in a non-empty set X. If $\alpha \in (0,1]$ and $\beta \in [0,1)$ are two real numbers such that $0 \leq \alpha + \beta \leq 1$ then IFS

$$a(\alpha, \beta) = \langle x, a_\alpha, 1 - a_{1-\beta}\rangle$$

is called an intuitionistic fuzzy point(IFP) in X, where α and β are the degree of membership and nonmembership of $a(\alpha,\ \beta)$ respectively and $a \in X$ is the support of $a(\alpha,\ \beta)$.

Let $a(\alpha,\ \beta)$ be an IFP in X, and $A = (\mu_A, \lambda_A)$ is an IFS in X. Then $a(\alpha,\ \beta)$ is said to belong to A, written $a(\alpha,\ \beta) \in A$, if $\mu_A(a) \geq \alpha$ and $\lambda_A(a) \leq \beta$ and quasi-coincident with A, written $a(\alpha,\ \beta)qA$, if $\mu_A(a) + \alpha > 1$, and $\lambda_A + \beta < 1$. $a(\alpha,\ \beta) \in \vee qA$, means that $a(\alpha,\ \beta) \in A$ or $a(\alpha,\ \beta)qA$ and $a(\alpha,\ \beta) \in \wedge qA$, means that $a(\alpha,\ \beta) \in A$ and $a(\alpha,\ \beta)qA$ and $a(\alpha,\ \beta)\overline{\in \vee q}A$, means that $a(\alpha,\ \beta) \in \vee qA$ doesn't hold.

Let $x(t,\ s)$ be an IFP in X, and $A = (\mu_A, \lambda_A)$ be an IFS in R, Then for all $x, y \in R$ and $t \in (0,1]$, $s \in [0,1)$, we define the following:

(i) $x(t,s)q_kA$ if $\mu_A(x) + t + k > 1$ and $\lambda_A(x) + s + k < 1$.

(ii) $x(t,s) \in \vee q_kA$ if $x(t,s) \in A$ or $x(t,s)q_kA$.

(iii) $x(t,s) \in \wedge q_kA$ if $x(t,s) \in A$ and $x(t,s)q_kA$.

(iv) $x(t,s)\overline{\in \vee q_k}A$ means that $x(t,s) \in \vee q_kA$ doesn't hold.

where $k \in [0,1)$.

2.1 Definition [10]

An IFS $A = (\mu_A, \lambda_A)$ of a hemiring R is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy ideal of R, if $\forall\ x, y \in R$ and $t_1, t_2 \in (0,1]$, $s_1, s_2 \in [0,1)$

$(1a)$ $x(t_1,s_1), y(t_2,s_2) \in A$
$\Rightarrow (x+y)(\min(t_1,t_2), \max(s_1,s_2)) \in \vee q_kA$

$(2a)$ $x(t_1,s_1) \in A$, $y \in R$
$\Rightarrow (yx)(t_1,s_1) \in \vee q_kA$

$(2'a)$ $x(t_1,s_1) \in A$, $y \in R$
$\Rightarrow (xy)(t_1,s_1) \in \vee q_kA$.

2.2 Theorem [10]

Let $A = (\mu_A, \lambda_A)$ be an intuitionistic fuzzy subset of a hemiring R. Then $(1a) \Longrightarrow (1b)$, $(2a) \Longrightarrow (2b)$, $(2'a) \Longrightarrow (2'b)$ where $\forall\ x, y \in R$ and $k \in [0,1)$,

$(1b)$ $\mu_A(x+y) \geq \min\left\{\begin{array}{c}\mu_A(x), \mu_A(y)\\ , \frac{1-k}{2}\end{array}\right\}$ and $\lambda_A(x+y) \leq \max\left\{\lambda_A(x), \lambda_A(y), \frac{1-k}{2}\right\}$

$(2b)$ $\mu_A(yx) \geq \min\left\{\mu_A(x), \frac{1-k}{2}\right\}$ and $\lambda_A(yx) \leq \max\left\{\lambda_A(x), \frac{1-k}{2}\right\}$.

$(2'b)$ $\mu_A(xy) \geq \min\left\{\mu_A(x), \frac{1-k}{2}\right\}$ and $\lambda_A(xy) \leq \max\left\{\lambda_A(x), \frac{1-k}{2}\right\}$.

Converse of the above result is not true in general.(See example 3.5).

2.3 Definition

Let $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ be intuitionistic fuzzy subsets of a hemiring R. Then the k- product of A and B is denoted and defined by $A \odot B = \langle \mu_A \odot \mu_B,\ \lambda_A \odot \lambda_B\rangle$, where

$$(\mu_A \odot \mu_B)(x) = \left\{\begin{array}{l} \bigvee_{x+\sum_{i=1}^m a_ib_i = \sum_{j=1}^n a'_jb'_j} \left\{\begin{array}{c}\left(\wedge_{i=1}^m \mu_A(a_i)\right) \wedge \\ \left(\wedge_{i=1}^m \mu_B(b_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(a'_j)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_B(b'_j)\right)\end{array}\right\} \\ 0 \text{ if } x \text{ cannot be expressed as} \\ \quad x + \sum_{i=1}^m a_ib_i = \sum_{j=1}^n a'_jb'_j \end{array}\right.$$

$$(\lambda_A \odot \lambda_B)(x) = \left\{\begin{array}{l} \bigwedge_{x+\sum_{i=1}^m a_ib_i = \sum_{j=1}^n a'_jb'_j} \left\{\begin{array}{c}\left(\vee_{i=1}^m \lambda_A(a_i)\right) \vee \\ \left(\vee_{i=1}^m \lambda_B(b_i)\right) \vee \\ \left(\vee_{j=1}^n \lambda_A(a'_j)\right) \vee \\ \left(\vee_{j=1}^n \lambda_B(b'_j)\right)\end{array}\right\} \\ 1 \text{ if } x \text{ cannot be expressed as} \\ \quad x + \sum_{i=1}^m a_ib_i = \sum_{j=1}^n a'_jb'_j \end{array}\right.$$

2.4 Definition

If $S \subseteq R$, then the intuitionistic characteristic function of S is defined by $C_S = (\chi_S, \chi_S^c)$

$$C_S = \begin{cases} (1,0) & if\ x \in S \\ (0,1) & if\ x \notin S \end{cases}$$

In particular, we let $\overline{1} = (\chi_R, \chi_R^c)$ be the intuitionistic fuzzy set in R.

2.5 Lemma

Let R be a hemiring and $P, Q \subseteq R$. Then we have
(1) $P \subseteq Q$ if and only if
$C_P = (\chi_P, \chi_P^c) \subseteq (\chi_Q, \chi_Q^c) = C_Q$.
(2) $C_P \cap C_Q = C_{P \cap Q}$.
(3) $C_P \odot C_Q = C_{\overline{PQ}}$.

*Proof.*This proof is analogous to ([14] Lemma 2.9).

3 $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideals in hemiring

3.1 Definition

An IFS $A = (\mu_A, \lambda_A)$ of a hemiring R is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy left k-ideal of R, if $\forall\ x, y, z \in R$ and $t_1, t_2 \in (0,1]$, $s_1, s_2 \in [0,1)$
$(1a)$ $x(t_1, s_1)$, $y(t_2, s_2) \in A$
$\Rightarrow (x+y)(\min(t_1,t_2), \max(s_1,s_2)) \in \vee q_k A$
$(2a)$ $x(t_1, s_1) \in A, y \in R$
$\Rightarrow (yx)(t_1, s_1) \in \vee q_k A$
$(3a)$ $x + y = z$, $y(t_1, s_1)$, $z(t_2, s_2) \in A$
$\Longrightarrow (x)(\min(t_1,t_2), \max(s_1,s_2)) \in \vee q_k A$.
$(\in, \in \vee q_k)$-intuitionistic fuzzy right k-ideal are defined similarly. An intuitionistic fuzzy set is called an $(\in, \in \vee q_k)$-intuitionistic fuzzy k-ideal of R if it is both an $(\in, \in \vee q_k)$-intuitionistic fuzzy left k-ideal and an $(\in, \in \vee q_k)$-intuitionistic fuzzy right k-ideal of R.

3.2 Theorem

Let A be an intuitionistic fuzzy subset of a hemiring R. Then $(3a) \Longrightarrow (3b)$, where
$(3b)$ $\mu_A(x) \geq \min\{\mu_A(y), \mu_A(z), \frac{1-k}{2}\}$ and $\lambda_A(x) \leq \max\{\lambda_A(y), \lambda_A(z), \frac{1-k}{2}\}$.

Proof.$(3a) \Rightarrow (3b)$
Let A be an intuitionistic fuzzy subset of a hemiring R, and $(3a)$ holds. Suppose that $(3b)$ doesn't hold. Then there exist $x, y, z \in R$ such that
$\mu_A(x) < \min\{\mu_A(y), \mu_A(z), \frac{1-k}{2}\}$ or $\lambda_A(x) > \max\{\lambda_A(y), \lambda_A(z), \frac{1-k}{2}\}$. So there exit three possible cases.
(i) $\mu_A(x) < \min\{\mu_A(y), \mu_A(z), \frac{1-k}{2}\}$ and $\lambda_A(x) \leq \max\{\lambda_A(y), \lambda_A(z), \frac{1-k}{2}\}$
(ii) $\mu_A(x) \geq \min\{\mu_A(y), \mu_A(z), \frac{1-k}{2}\}$ and $\lambda_A(x) > \max\{\lambda_A(y), \lambda_A(z), \frac{1-k}{2}\}$
(iii) $\mu_A(x) < \min\{\mu_A(y), \mu_A(z), \frac{1-k}{2}\}$ and $\lambda_A(x) > \max\{\lambda_A(y), \lambda_A(z), \frac{1-k}{2}\}$.
For the first case, there exist $t \in (0,1]$ such that
$\mu_A(x) < t < \min\{\mu_A(a), \mu_A(b), \frac{1-k}{2}\}$. Now choose $s = 1 - t$. Then clearly $y(t,s) \in A$ and $z(t,s) \in A$ but $(x)(t,s)\overline{\in \vee q_k} A$. Which is a contradiction. Second case is similar to this case.
Now consider case (iii)
$\mu_A(x) < \min\{\mu_A(y), \mu_A(z), \frac{1-k}{2}\}$ and $\lambda_A(x) > \max\{\lambda_A(y), \lambda_A(z), \frac{1-k}{2}\}$. Then there exist $t \in (0,1]$ and $s \in [0,1)$, such that
$\mu_A(x) < t \leq \min\{\mu_A(y), \mu_A(z), \frac{1-k}{2}\}$ and $\lambda_A(x) > s \geq \max\{\lambda_A(y), \lambda_A(z), \frac{1-k}{2}\}$
$\Longrightarrow y(t,s) \in A$ and $z(t,s) \in A$ but $(x)(t,s)\overline{\in \vee q_k} A$. Which is again a contradiction. So our supposition is wrong. Hence $(3b)$ holds.

3.3 Definition

Let $A = (\mu_A, \lambda_A)$ be an IFS of a hemiring R. Then A is called an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left k-ideal of R if it satisfies the conditions $(1b)$, $(2b)$ and $(3b)$.

3.4 Remark

Every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left k-ideal $A = (\mu_A, \lambda_A)$ of R need not be an $(\in, \in \vee q_k)$-intuitionistic fuzzy left k-ideal of R.

3.5 Example

Let $\mathbb{N}$ be the set of all non negative integers and $A = \langle \mu_A, \lambda_A \rangle$ be an IFS of $\mathbb{N}$ defined as follows

$$\mu_A(x) = \begin{cases} 0.4 & \text{if } x \in \langle 4 \rangle \\ 0.3 & \text{if } x \in \langle 2 \rangle - \langle 4 \rangle \\ 0 & \text{otherwise} \end{cases}$$

$$\lambda_A(x) = \begin{cases} 0.4 & \text{if } x \in \langle 4 \rangle \\ 0.3 & \text{if } x \in \langle 2 \rangle - \langle 4 \rangle \\ 0 & \text{otherwise} \end{cases}$$

For all $x, y, z \in \mathbb{N}$
$(1b)$ $\mu_A(x+y) \geq \min\{\mu_A(x), \mu_A(y), 0.4\}$ *and* $\lambda_A(x+y) \leq \max\{\lambda_A(x), \lambda_A(y), 0.4\}$
$(2b)$ $\mu_A(xy) \geq \min\{\mu_A(y), 0.4\}$ *and* $\lambda_A(xy) \leq \max\{\lambda_A(y), 0.4\}$
$(3b)$ $x + y = z$

$\Longrightarrow \mu_A(x) \geq \min\{\mu_A(y), \mu_A(z), 0.4\}$ and $\lambda_A(x) \leq \max\{\lambda_A(y), \lambda_A(z), 0.4\}$

Thus $A = (\mu_A, \lambda_A)$ is an $(\in, \in \vee q_{0.2})^*$-intuitionistic fuzzy k-ideal of $\mathbb{N}$. But

$2(0.25,\ 0.35),\ 2(0.25,\ 0.35) \in A$

$\Longrightarrow (2.2)(0.25,\ 0.35)\overline{\in \vee q_{0.2}}A$. Thus $A = (\mu_A, \lambda_A)$ is not an $(\in, \in \vee q_{0.2})$-intuitionistic fuzzy k-ideal of $\mathbb{N}$.

3.6 Definition

Let A and B be intuitionistic fuzzy subsets of a hemiring R. Then the intuitionistic fuzzy subsets A_k, $A \cap_k B$, and $A \odot_k B$ are defined as following:

$A \cap k = \left\langle \mu_A \wedge \frac{1-k}{2},\ \lambda_A \vee \frac{1-k}{2} \right\rangle = A_k$

$$A \cap_k B = \left\langle (\mu_A \wedge \mu_B) \wedge \frac{1-k}{2},\ (\lambda_A \vee \lambda_B) \vee \frac{1-k}{2} \right\rangle = (A \cap B)_k$$

$$A \odot_k B = \left\langle (\mu_A \odot \mu_B) \wedge \frac{1-k}{2},\ (\lambda_A \odot \lambda_B) \vee \frac{1-k}{2} \right\rangle = (A \odot B)_k$$

3.7 Theorem

If A and B are $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideals of R, then $A \odot_k B$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideal of R and $A \odot_k B \subseteq A \cap_k B$.

Proof. Let A and B be $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideals of R and $x, y \in R$. Then $(\mu_A \odot_k \mu_B)(x) =$

$$\bigvee_{x+\sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j} \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \mu_A(a_i)\right) \wedge \\ \left(\wedge_{i=1}^m \mu_B(b_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(a'_j)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_B(b'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

and $(\mu_A \odot_k \mu_B)(y) =$

$$\bigvee_{y+\sum_{k=1}^p c_k d_k = \sum_{l=1}^q c'_l d'_l} \left\{ \begin{array}{c} \left(\wedge_{k=1}^p \mu_A(c_k)\right) \wedge \\ \left(\wedge_{k=1}^p \mu_B(d_k)\right) \wedge \\ \left(\wedge_{l=1}^q \mu_A(c'_l)\right) \wedge \\ \left(\wedge_{l=1}^q \mu_B(d'_l)\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

Thus

$(\mu_A \odot_k \mu_B)(x) \wedge (\mu_A \odot_k \mu_B)(y) \wedge \frac{1-k}{2} =$

$$\left[\bigvee_{x+\sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j} \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \mu_A(a_i)\right) \wedge \\ \left(\wedge_{i=1}^m \mu_B(b_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(a'_j)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_B(b'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2} \right] \wedge \left[\bigvee_{y+\sum_{k=1}^p c_k d_k = \sum_{l=1}^q c'_l d'_l} \left\{ \begin{array}{c} \left(\wedge_{k=1}^p \mu_A(c_k)\right) \wedge \\ \left(\wedge_{k=1}^p \mu_B(d_k)\right) \wedge \\ \left(\wedge_{l=1}^q \mu_A(c'_l)\right) \wedge \\ \left(\wedge_{l=1}^q \mu_B(d'_l)\right) \end{array} \right\} \wedge \frac{1-k}{2} \right] \wedge \frac{1-k}{2}$$

$$= \left[\bigvee_{\substack{x+\sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j \\ y+\sum_{k=1}^p c_k d_k = \sum_{l=1}^q c'_l d'_l}} \left[\begin{array}{c} \left(\wedge_{i=1}^m \mu_A(a_i)\right) \wedge \\ \left(\wedge_{i=1}^m \mu_B(b_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(a'_j)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_B(b'_j)\right) \wedge \\ \left(\wedge_{k=1}^p \mu_A(c_k)\right) \wedge \\ \left(\wedge_{k=1}^p \mu_B(d_k)\right) \wedge \\ \left(\wedge_{l=1}^q \mu_A(c'_l)\right) \wedge \\ \left(\wedge_{l=1}^q \mu_B(d'_l)\right) \end{array} \right] \wedge \frac{1-k}{2} \right] \wedge \frac{1-k}{2}$$

$$\leq \left\{ \begin{array}{c} \bigvee_{x+y+\sum_{s=1}^u e_s f_s = \sum_{t=1}^v e'_t f'_t} \left[\begin{array}{c} \left(\wedge_{s=1}^u \mu_A(e_s)\right) \wedge \\ \left(\wedge_{s=1}^u \mu_B(f_s)\right) \wedge \\ \left(\wedge_{t=1}^v \mu_A(e'_t)\right) \wedge \\ \left(\wedge_{t=1}^v \mu_B(f'_t)\right) \end{array} \right] \\ \wedge \frac{1-k}{2} \end{array} \right.$$

$= (\mu_A \odot_k \mu_B)(x+y)$.

Now $(\lambda_A \odot_k \lambda_B)(x) =$

$$\bigwedge_{x+\sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j} \left\{ \begin{array}{c} \left(\vee_{i=1}^m \lambda_A(a_i)\right) \vee \\ \left(\vee_{i=1}^m \lambda_B(b_i)\right) \vee \\ \left(\vee_{j=1}^n \lambda_A(a'_j)\right) \vee \\ \left(\vee_{j=1}^n \lambda_B(b'_j)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

and $(\lambda_A \odot_k \lambda_B)(y) =$

$$\bigwedge_{y+\sum_{k=1}^p c_k d_k = \sum_{l=1}^q c'_l d'_l} \left\{ \begin{array}{c} \left(\vee_{k=1}^p \lambda_A(c_k)\right) \vee \\ \left(\vee_{k=1}^p \lambda_B(d_k)\right) \\ \vee \left(\vee_{l=1}^q \lambda_A(c'_l)\right) \vee \\ \left(\vee_{l=1}^q \lambda_B(d'_l)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

Thus
$(\lambda_A \odot_k \lambda_B)(x) \vee (\lambda_A \odot_k \lambda_B)(y) \vee \frac{1-k}{2} =$

$$\left[\bigwedge_{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j}\left\{\begin{array}{c}\left(\vee_{i=1}^m\lambda_A(a_i)\right)\vee\\ \left(\vee_{i=1}^m\lambda_B(b_i)\right)\vee\\ \left(\vee_{j=1}^n\lambda_A(a'_j)\right)\vee\\ \left(\vee_{j=1}^n\lambda_B(b'_j)\right)\end{array}\right\}\vee\frac{1-k}{2}\right]\vee$$

$$\left[\bigwedge_{y+\sum_{k=1}^p c_kd_k=\sum_{l=1}^q c'_ld'_l}\left\{\begin{array}{c}\left(\vee_{k=1}^p\lambda_A(c_k)\right)\vee\\ \left(\vee_{k=1}^p\lambda_B(d_k)\right)\vee\\ \left(\vee_{l=1}^q\lambda_A(c'_l)\right)\vee\\ \left(\vee_{l=1}^q\lambda_B(d'_l)\right)\end{array}\right\}\vee\frac{1-k}{2}\right]\vee$$

$$\frac{1-k}{2}$$

$$=\left[\bigwedge_{\substack{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j\\ y+\sum_{k=1}^p c_kd_k=\sum_{l=1}^q c'_ld'_l}}\left[\begin{array}{c}\left(\vee_{i=1}^m\lambda_A(a_i)\right)\vee\\ \left(\vee_{i=1}^m\lambda_B(b_i)\right)\vee\\ \left(\vee_{j=1}^n\lambda_A(a'_j)\right)\vee\\ \left(\vee_{j=1}^n\lambda_B(b'_j)\right)\vee\\ \left(\vee_{k=1}^p\lambda_A(c_k)\right)\vee\\ \left(\vee_{k=1}^p\lambda_B(d_k)\right)\vee\\ \left(\vee_{l=1}^q\lambda_A(c'_l)\right)\vee\\ \left(\vee_{l=1}^q\lambda_B(d'_l)\right)\end{array}\right]\vee\frac{1-k}{2}\right]\vee\frac{1-k}{2}$$

$$\geq\left\{\bigwedge_{x+y+\sum_{s=1}^u e_sf_s=\sum_{t=1}^v e'_tf'_t}\left\{\begin{array}{c}\left(\vee_{s=1}^u\lambda_A(e_s)\right)\vee\\ \left(\vee_{s=1}^u\lambda_B(f_s)\right)\vee\\ \left(\vee_{t=1}^v\lambda_A(e'_t)\right)\vee\\ \left(\vee_{t=1}^v\lambda_B(f'_t)\right)\end{array}\right\}\vee\frac{1-k}{2}\right\}$$

$= (\lambda_A \odot_k \lambda_B)(x+y)$.
Similarly,
$(\mu_A \odot_k \mu_B)(x) \wedge \frac{1-k}{2} =$

$$\left[\bigvee_{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j}\left\{\begin{array}{c}\left(\wedge_{i=1}^m\mu_A(a_i)\right)\wedge\\ \left(\wedge_{i=1}^m\mu_B(b_i)\right)\wedge\\ \left(\wedge_{j=1}^n\mu_A(a'_j)\right)\wedge\\ \left(\wedge_{j=1}^n\mu_B(b'_j)\right)\end{array}\right\}\wedge\frac{1-k}{2}\right]\wedge$$

$$\frac{1-k}{2}$$

$$=\bigvee_{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j}\left\{\begin{array}{c}\left(\wedge_{i=1}^m\mu_A(a_i)\right)\wedge\\ \left(\wedge_{i=1}^m\left(\mu_B(b_i)\wedge\frac{1-k}{2}\right)\right)\wedge\\ \left(\wedge_{j=1}^n\mu_A(a'_j)\right)\wedge\\ \left(\wedge_{j=1}^n\left(\mu_B(b'_j)\wedge\frac{1-k}{2}\right)\right)\end{array}\right\}\wedge\frac{1-k}{2}$$

$$\leq\left\{\begin{array}{c}\bigvee_{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j}\left\{\begin{array}{c}\left(\wedge_{i=1}^m\mu_A(a_i)\right)\wedge\\ \left(\wedge_{i=1}^m\mu_B(b_ir)\right)\wedge\\ \left(\wedge_{j=1}^n\mu_A(a'_j)\right)\wedge\\ \left(\wedge_{j=1}^n\mu_B(b'_jr)\right)\end{array}\right\}\\ \wedge\frac{1-k}{2}\end{array}\right\}$$

$$\leq\left\{\begin{array}{c}\bigvee_{xr+\sum_{k=1}^p g_kh_k=\sum_{t=1}^q g'_th'_t}\left\{\begin{array}{c}\left(\wedge_{k=1}^p\mu_A(g_k)\right)\wedge\\ \left(\wedge_{k=1}^p\mu_B(h_k)\right)\wedge\\ \left(\wedge_{t=1}^q\mu_A(g'_t)\right)\wedge\\ \left(\wedge_{t=1}^q\mu_B(h'_t)\right)\end{array}\right\}\\ \wedge\frac{1-k}{2}\end{array}\right\}$$

$= (\mu_A \odot_k \mu_B)(xr)$.
And
$(\lambda_A \odot_k \lambda_B)(x) \vee \frac{1-k}{2} =$

$$\left[\bigwedge_{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j}\left\{\begin{array}{c}\left(\vee_{i=1}^m\lambda_A(a_i)\right)\vee\\ \left(\vee_{i=1}^m\lambda_B(b_i)\right)\vee\\ \left(\vee_{j=1}^n\lambda_A(a'_j)\right)\vee\\ \left(\vee_{j=1}^n\lambda_B(b'_j)\right)\end{array}\right\}\vee\frac{1-k}{2}\right]$$

$$\vee\frac{1-k}{2}$$

$$=\bigwedge_{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j}\left\{\begin{array}{c}\left(\vee_{i=1}^m\lambda_A(a_i)\right)\vee\\ \left(\vee_{i=1}^m\left(\lambda_B(b_i)\vee\frac{1-k}{2}\right)\right)\vee\\ \left(\vee_{j=1}^n\lambda_A(a'_j)\right)\vee\\ \left(\vee_{j=1}^n\left(\lambda_B(b'_j)\vee\frac{1-k}{2}\right)\right)\end{array}\right\}$$

$$\vee\frac{1-k}{2}$$

$$\geq\bigwedge_{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j}\left\{\begin{array}{c}\left(\vee_{i=1}^m\lambda_A(a_i)\right)\vee\\ \left(\vee_{i=1}^m\lambda_B(b_ir)\right)\vee\\ \left(\vee_{j=1}^n\lambda_A(a'_j)\right)\vee\\ \left(\vee_{j=1}^n\lambda_B(b'_jr)\right)\end{array}\right\}\vee\frac{1-k}{2}$$

$$\geq\bigwedge_{xr+\sum_{k=1}^p g_kh_k=\sum_{t=1}^q g'_th'_t}\left\{\begin{array}{c}\left(\vee_{k=1}^p\lambda_A(g_k)\right)\vee\\ \left(\vee_{k=1}^p\lambda_B(h_k)\right)\vee\\ \left(\vee_{t=1}^q\lambda_A(g'_t)\right)\vee\\ \left(\vee_{t=1}^q\lambda_B(h'_t)\right)\end{array}\right\}\vee\frac{1-k}{2}$$

$= (\lambda_A \odot_k \lambda_B)(xr)$.
Analogously we can prove that
$(\mu_A \odot_k \mu_B)(x) \wedge \frac{1-k}{2} \leq (\mu_A \odot_k \mu_B)(rx)$ and
$(\lambda_A \odot_k \lambda_B)(x) \vee \frac{1-k}{2} \geq (\lambda_A \odot_k \lambda_B)(rx)$ for all $x, y \in R$.
This means that $A \odot_k B$ is an $(\in, \in \vee q_k)^*$-intuitionistic

fuzzy ideal of R. Now to prove the condition $(3b)$. Let $x+a=b$, observe that $a+\sum_{i=1}^{m}a_ib_i=\sum_{j=1}^{n}a'_jb'_j$ and

$b+\sum_{k=1}^{l}c_kd_k=\sum_{q=1}^{p}c'_qd'_q$, gives

$x+a+\sum_{i=1}^{m}a_ib_i=b+\sum_{i=1}^{m}a_ib_i$. Thus $x+\sum_{j=1}^{n}a'_jb'_j=b+\sum_{i=1}^{m}a_ib_i$ and consequently,

$$x+\sum_{j=1}^{n}a'_jb'_j+\sum_{k=1}^{l}c_kd_k$$

$$=b+\sum_{k=1}^{l}c_kd_k+\sum_{i=1}^{m}a_ib_i$$

$$=\sum_{q=1}^{p}c'_qd'_q+\sum_{i=1}^{m}a_ib_i$$

therefore

$$x+\sum_{j=1}^{n}a'_jb'_j+\sum_{k=1}^{l}c_kd_k=\sum_{q=1}^{p}c'_qd'_q+\sum_{i=1}^{m}a_ib_i.$$

Now we have

$(\mu_A\odot_k\mu_B)(a)\wedge(\mu_A\odot_k\mu_B)(b)\wedge\frac{1-k}{2}=$

$$\left[\bigvee_{a+\sum_{i=1}^{m}a_ib_i=\sum_{j=1}^{n}a'_jb'_j}\left\{\begin{array}{c}(\wedge_{i=1}^{m}\mu_A(a_i))\wedge\\(\wedge_{i=1}^{m}\mu_B(b_i))\wedge\\\left(\wedge_{j=1}^{n}\mu_A(a'_j)\right)\wedge\\\left(\wedge_{j=1}^{n}\mu_B(b'_j)\right)\end{array}\right\}\wedge\frac{1-k}{2}\right]\wedge\left[\bigvee_{b+\sum_{k=1}^{l}c_kd_k=\sum_{q=1}^{p}c'_ld'_l}\left\{\begin{array}{c}(\wedge_{k=1}^{l}\mu_A(c_k))\wedge\\(\wedge_{k=1}^{l}\mu_B(d_k))\wedge\\\left(\wedge_{q=1}^{p}\mu_A(c'_l)\right)\wedge\\\left(\wedge_{q=1}^{p}\mu_B(d'_l)\right)\end{array}\right\}\wedge\frac{1-k}{2}\right]\wedge\frac{1-k}{2}$$

$$=\left[\bigvee_{\substack{a+\sum_{i=1}^{m}a_ib_i=\sum_{j=1}^{n}a'_jb'_j\\b+\sum_{k=1}^{l}c_kd_k=\sum_{q=1}^{p}c'_ld'_l}}\left[\begin{array}{c}(\wedge_{i=1}^{m}\mu_A(a_i))\wedge\\(\wedge_{i=1}^{m}\mu_B(b_i))\wedge\\\left(\wedge_{j=1}^{n}\mu_A(a'_j)\right)\wedge\\\left(\wedge_{j=1}^{n}\mu_B(b'_j)\right)\wedge\\(\wedge_{k=1}^{l}\mu_A(c_k))\wedge\\(\wedge_{k=1}^{l}\mu_B(d_k))\wedge\\\left(\wedge_{q=1}^{p}\mu_A(c'_l)\right)\wedge\\\left(\wedge_{q=1}^{p}\mu_B(d'_l)\right)\end{array}\right]\wedge\frac{1-k}{2}\right]\wedge\frac{1-k}{2}$$

$$\leq\bigvee_{x+\sum_{s=1}^{u}g_sh_s=\sum_{t=1}^{v}g'_th'_t}\cdot\left\{\begin{array}{c}(\wedge_{s=1}^{u}\mu_A(g_s))\wedge\\(\wedge_{s=1}^{u}\mu_B(h_s))\wedge\\(\wedge_{t=1}^{v}\mu_A(g'_t))\wedge\\(\wedge_{t=1}^{v}\mu_B(h'_t))\end{array}\right\}\wedge\frac{1-k}{2}$$

$=(\mu_A\odot_k\mu_B)(x)$

And

$(\lambda_A\odot_k\lambda_B)(a)\vee(\lambda_A\odot_k\lambda_B)(b)\vee\frac{1-k}{2}=$

$$\left[\bigwedge_{a+\sum_{i=1}^{m}a_ib_i=\sum_{j=1}^{n}a'_jb'_j}\left\{\begin{array}{c}(\vee_{i=1}^{m}\lambda_A(a_i))\vee\\(\vee_{i=1}^{m}\lambda_B(b_i))\vee\\\left(\vee_{j=1}^{n}\lambda_A(a'_j)\right)\vee\\\left(\vee_{j=1}^{n}\lambda_B(b'_j)\right)\end{array}\right\}\vee\frac{1-k}{2}\right]\vee\left[\bigwedge_{b+\sum_{k=1}^{l}c_kd_k=\sum_{q=1}^{p}c'_qd'_q}\left\{\begin{array}{c}(\vee_{k=1}^{l}\lambda_A(c_k))\vee\\(\vee_{k=1}^{l}\lambda_B(d_k))\vee\\\left(\vee_{q=1}^{p}\lambda_A(c'_q)\right)\vee\\\left(\vee_{q=1}^{p}\lambda_B(d'_q)\right)\end{array}\right\}\vee\frac{1-k}{2}\right]\vee\frac{1-k}{2}$$

$$=\left[\bigwedge_{\substack{a+\sum_{i=1}^{m}a_ib_i=\sum_{j=1}^{n}a'_jb'_j\\b+\sum_{k=1}^{l}c_kd_k=\sum_{q=1}^{p}c'_qd'_q}}\left[\begin{array}{c}(\vee_{i=1}^{m}\lambda_A(a_i))\vee\\(\vee_{i=1}^{m}\lambda_B(b_i))\vee\\\left(\vee_{j=1}^{n}\lambda_A(a'_j)\right)\vee\\\left(\vee_{j=1}^{n}\lambda_B(b'_j)\right)\vee\\(\vee_{k=1}^{l}\lambda_A(c_k))\vee\\(\vee_{k=1}^{l}\lambda_B(d_k))\vee\\\left(\vee_{q=1}^{p}\lambda_A(c'_q)\right)\vee\\\left(\vee_{q=1}^{p}\lambda_B(d'_q)\right)\end{array}\right]\vee\frac{1-k}{2}\right]\vee\frac{1-k}{2}$$

$$\geq\bigwedge_{x+\sum_{s=1}^{u}g_sh_s=\sum_{t=1}^{v}g'_th'_t}\left\{\begin{array}{c}(\vee_{s=1}^{u}\lambda_A(g_s))\vee\\(\vee_{s=1}^{u}\lambda_B(h_s))\vee\\(\vee_{t=1}^{v}\lambda_A(g'_t))\vee\\(\vee_{t=1}^{v}\lambda_B(h'_t))\end{array}\right\}\vee\frac{1-k}{2}$$

$=(\lambda_A\odot_k\lambda_B)(x)$.

Hence $A\odot_k B$ is an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy k-ideal of R.

By simple calculations we can prove that $A\odot_k B\subseteq A\cap_k B$.

3.8 Corollary

If A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right k-ideal of R and B is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left k-ideal of R, then $A \odot_k B \subseteq A \cap_k B$.

3.9 Definition

The k-sum $A +_k B$ of an intuitionistic fuzzy subsets A and B is defined by

$A +_k B = \langle \mu_A +_k \mu_B,\ \lambda_A +_k \lambda_B \rangle$, where

$$(\mu_A +_k \mu_B)(x) = \bigvee_{x+(a_1+b_1)=(a_2+b_2)} \begin{bmatrix} \mu_A(a_1) \wedge \mu_A(a_2) \wedge \\ \mu_B(b_1) \wedge \mu_B(b_2) \end{bmatrix} \wedge \frac{1-k}{2}$$

and

$$(\lambda_A +_k \lambda_B)(x) = \bigwedge_{x+(a_1+b_1)=(a_2+b_2)} \begin{bmatrix} \lambda_A(a_1) \vee \lambda_A(a_2) \vee \\ \lambda_B(b_1) \vee \lambda_B(b_2) \end{bmatrix} \vee \frac{1-k}{2}$$

where $x, a_1, b_1, a_2, b_2 \in R$.

3.10 Theorem

The k-sum of $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideals of R is also an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideal of R.

Proof. Let A and B be $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideals of R. Then for $x, y \in R$, we have

$$(\mu_A +_k \mu_B)(x) \wedge (\mu_A +_k \mu_B)(y) \wedge \tfrac{1-k}{2}$$

$$= \begin{bmatrix} \bigvee_{x+(a_1+b_1)=(a_2+b_2)} \\ \begin{bmatrix} \mu_A(a_1) \wedge \mu_A(a_2) \wedge \\ \mu_B(b_1) \wedge \mu_B(b_2) \end{bmatrix} \\ \wedge \frac{1-k}{2} \end{bmatrix} \wedge \begin{bmatrix} \bigvee_{y+(a'_1+b'_1)=(a'_2+b'_2)} \\ \begin{bmatrix} \mu_A(a'_1) \wedge \mu_A(a'_2) \wedge \\ \mu_B(b'_1) \wedge \mu_B(b'_2) \end{bmatrix} \\ \wedge \frac{1-k}{2} \end{bmatrix} \wedge \frac{1-k}{2}$$

$$= \begin{bmatrix} \bigvee_{\substack{x+(a_1+b_1)=(a_2+b_2) \\ y+(a'_1+b'_1)=(a'_2+b'_2)}} \\ \begin{bmatrix} \mu_A(a_1) \wedge \mu_A(a_2) \wedge \\ \mu_B(b_1) \wedge \mu_B(b_2) \wedge \\ \mu_A(a'_1) \wedge \mu_A(a'_2) \wedge \\ \mu_B(b'_1) \wedge \mu_B(b'_2) \end{bmatrix} \\ \wedge \frac{1-k}{2} \end{bmatrix} \wedge \frac{1-k}{2}$$

$$\leq \bigvee_{\substack{x+(a_1+b_1)=(a_2+b_2) \\ y+(a'_1+b'_1)=(a'_2+b'_2)}} \begin{bmatrix} \mu_A(a_1+a'_1) \wedge \\ \mu_B(b_1+b'_1) \wedge \\ \mu_A(a_2+a'_2) \wedge \\ \mu_B(b_2+b'_2) \end{bmatrix} \wedge \frac{1-k}{2}$$

$$\leq \bigvee_{(x+y)+(c_1+d_1)=(c_2+d_2)} \begin{bmatrix} \mu_A(c_1) \wedge \mu_B(d_1) \\ \mu_A(c_2) \wedge \mu_B(d_2) \end{bmatrix} \wedge \frac{1-k}{2}$$

$$= (\mu_A +_k \mu_B)(x+y).$$

And

$$(\lambda_A +_k \lambda_B)(x) \vee (\lambda_A +_k \lambda_B)(y) \vee \tfrac{1-k}{2}$$

$$= \begin{bmatrix} \bigwedge_{x+(a_1+b_1)=(a_2+b_2)} \\ \begin{bmatrix} \lambda_A(a_1) \vee \lambda_A(a_2) \\ \vee \lambda_B(b_1) \vee \lambda_B(b_2) \end{bmatrix} \\ \vee \frac{1-k}{2} \end{bmatrix} \vee \begin{bmatrix} \bigwedge_{y+(a'_1+b'_1)=(a'_2+b'_2)} \\ \begin{bmatrix} \lambda_A(a'_1) \vee \lambda_A(a'_2) \vee \\ \lambda_B(b'_1) \vee \lambda_B(b'_2) \end{bmatrix} \\ \vee \frac{1-k}{2} \end{bmatrix} \vee \frac{1-k}{2}$$

$$= \begin{bmatrix} \bigwedge_{\substack{x+(a_1+b_1)=(a_2+b_2) \\ y+(a'_1+b'_1)=(a'_2+b'_2)}} \\ \begin{bmatrix} \lambda_A(a_1) \vee \lambda_A(a_2) \vee \\ \lambda_B(b_1) \vee \lambda_B(b_2) \\ \lambda_A(a'_1) \vee \lambda_A(a'_2) \vee \\ \lambda_B(b'_1) \vee \lambda_B(b'_2) \end{bmatrix} \\ \vee \frac{1-k}{2} \end{bmatrix} \vee \frac{1-k}{2}$$

$$\geq \bigwedge_{\substack{x+(a_1+b_1)=(a_2+b_2) \\ y+(a'_1+b'_1)=(a'_2+b'_2)}} \begin{bmatrix} \lambda_A(a_1+a'_1) \vee \lambda_B(b_1+b'_1) \\ \lambda_A(a_2+a'_2) \vee \lambda_B(b_2+b'_2) \end{bmatrix} \vee \frac{1-k}{2}$$

$$\geq \bigwedge_{(x+y)+(c_1+d_1)=(c_2+d_2)} \begin{bmatrix} \lambda_A(c_1) \vee \lambda_B(d_1) \\ \lambda_A(c_2) \vee \lambda_B(d_2) \end{bmatrix} \vee \frac{1-k}{2}$$

$$= (\lambda_A +_k \lambda_B)(x+y).$$

Similarly we can prove the conditions $(2b)$ and $(2'b)$.

Let $x+a=b$. Now to prove the condition $(3b)$, let $a+(a_1+b_1)=(a_2+b_2)$ and $b+(c_1+d_1)=(c_2+d_2)$. Then

$$x+a+(c_1+d_1)=(c_2+d_2)$$

where

$$x+a+(c_1+d_1)+(a_1+b_1)=(a_1+b_1)+(c_2+d_2)$$

and

$$x+(a_2+b_2)+(c_1+d_1)=(a_1+b_1)+(c_2+d_2).$$

Thus

$$x+(a_2+c_1)+(b_2+d_1)=(a_1+c_2)+(b_1+d_2).$$

Therefore $(\mu_A+_k\mu_B)(a)\wedge(\mu_A+_k\mu_B)(b)\wedge\frac{1-k}{2}$

$$=\left[\begin{array}{c}\bigvee_{a+(a_1+b_1)=(a_2+b_2)}\\ \left[\begin{array}{c}\mu_A(a_1)\wedge\mu_A(a_2)\wedge\\ \mu_B(b_1)\wedge\mu_B(b_2)\end{array}\right]\\ \wedge\frac{1-k}{2}\end{array}\right]\wedge\left[\begin{array}{c}\bigvee_{b+(c_1+d_1)=(c_2+d_2)}\\ \left[\begin{array}{c}\mu_A(c_1)\wedge\mu_A(c_2)\wedge\\ \mu_B(d_1)\wedge\mu_B(d_2)\end{array}\right]\\ \wedge\frac{1-k}{2}\end{array}\right]\wedge\frac{1-k}{2}$$

$$=\left[\begin{array}{c}\bigvee_{\substack{a+(a_1+b_1)=(a_2+b_2)\\ b+(c_1+d_1)=(c_2+d_2)}}\\ \left[\begin{array}{c}\mu_A(a_1)\wedge\mu_A(a_2)\wedge\\ \mu_B(b_1)\wedge\mu_B(b_2)\wedge\\ \mu_A(c_1)\wedge\mu_A(c_2)\wedge\\ \mu_B(d_1)\wedge\mu_B(d_2)\end{array}\right]\wedge\frac{1-k}{2}\end{array}\right]\wedge\frac{1-k}{2}$$

$$\leq\bigvee_{\substack{a+(a_1+b_1)=(a_2+b_2)\\ b+(c_1+d_1)=(c_2+d_2)}}\left[\begin{array}{c}\mu_A(a_1+c_2)\wedge\mu_A(a_2+c_1)\wedge\\ \mu_B(b_1+d_2)\wedge\mu_B(b_2+d_1)\end{array}\right]\wedge\frac{1-k}{2}$$

$$\leq\bigvee_{x+(a'+b')=(c'+d')}\left[\begin{array}{c}\mu_A(a')\wedge\mu_A(c')\wedge\\ \mu_B(b')\wedge\mu_B(d')\end{array}\right]\wedge\frac{1-k}{2}$$

$$=(\mu_A+_k\mu_B)(x).$$

And

$$(\lambda_A+_k\lambda_B)(a)\vee(\lambda_A+_k\lambda_B)(b)\vee\frac{1-k}{2}$$

$$=\left[\begin{array}{c}\bigwedge_{a+(a_1+b_1)=(a_2+b_2)}\\ \left[\begin{array}{c}\lambda_A(a_1)\vee\lambda_A(a_2)\vee\\ \lambda_B(b_1)\vee\lambda_B(b_2)\end{array}\right]\\ \vee\frac{1-k}{2}\end{array}\right]\vee\left[\begin{array}{c}\bigwedge_{b+(c_1+d_1)=(c_2+d_2)}\\ \left[\begin{array}{c}\lambda_A(c_1)\vee\lambda_A(c_2)\vee\\ \lambda_B(d_1)\vee\lambda_B(d_2)\end{array}\right]\\ \vee\frac{1-k}{2}\end{array}\right]\vee\frac{1-k}{2}$$

$$=\left[\begin{array}{c}\bigwedge_{\substack{a+(a_1+b_1)=(a_2+b_2)\\ b+(c_1+d_1)=(c_2+d_2)}}\\ \left[\begin{array}{c}\lambda_A(a_1)\vee\lambda_A(a_2)\vee\\ \lambda_B(b_1)\vee\lambda_B(b_2)\vee\\ \lambda_A(c_1)\vee\lambda_A(c_2)\vee\\ \lambda_B(d_1)\vee\lambda_B(d_2)\end{array}\right]\\ \vee\frac{1-k}{2}\end{array}\right]\vee\frac{1-k}{2}$$

$$\geq\bigwedge_{\substack{a+(a_1+b_1)=(a_2+b_2)\\ b+(c_1+d_1)=(c_2+d_2)}}\left[\begin{array}{c}\lambda_A(a_2+c_1)\vee\lambda_A(a_1+c_2)\vee\\ \lambda_B(b_2+d_1)\vee\lambda_B(b_1+d_2)\end{array}\right]\vee\frac{1-k}{2}$$

$$\geq\bigwedge_{x+(a'+b')=(c'+d')}\left[\begin{array}{c}\lambda_A(a')\vee\lambda_A(c')\vee\\ \lambda_B(b')\vee\lambda_B(d')\end{array}\right]\vee\frac{1-k}{2}$$

$$=(\lambda_A+_k\lambda_B)(x).$$

Hence $A+_kB$ is an $(\in,\in\vee q_k)^*$-intuitionistic fuzzy k-ideal of R.

3.11 Theorem

An intuitionistic fuzzy subset $A=(\mu_A,\lambda_A)$ of a hemiring R satisfies condition $(1b)$ and $(3b)$ if and only if it satisfies the condition

(4) $A+_kA\subseteq A_k$

*Proof.*Suppose $A=(\mu_A,\lambda_A)$ satisfies conditions $(1b)$ and $(3b)$ and $x\in R$. Then

$(\mu_A+_k\mu_A)(x)=$

$$\left(\begin{array}{c}\bigvee_{x+(a_1+b_1)=(a_2+b_2)}\\ \left[\begin{array}{c}\mu_A(a_1)\wedge\mu_A(a_2)\wedge\\ \mu_A(b_1)\wedge\mu_A(b_2)\end{array}\right]\end{array}\right)\wedge\frac{1-k}{2}$$

$$= \left(\begin{array}{c} \vee_{x+(a_1+b_1)=(a_2+b_2)} \\ \left[\begin{array}{c} \mu_A(a_1) \wedge \mu_A(b_1) \wedge \frac{1-k}{2} \\ \wedge \mu_A(a_2) \wedge \mu_A(b_2) \wedge \frac{1-k}{2} \end{array} \right] \end{array} \right) \wedge \frac{1-k}{2}$$

$$\leq \left(\begin{array}{c} \vee_{x+(a_1+b_1)=(a_2+b_2)} \\ [\mu_A(a_1+b_1) \wedge \mu_A(a_2+b_2)] \end{array} \right) \wedge \frac{1-k}{2}$$

by condition $(1b)$

$$\leq \mu_A(x) \wedge \frac{1-k}{2}$$

by condition $(3b)$ and
$(\lambda_A +_k \lambda_A)(x) =$

$$\left(\begin{array}{c} \wedge_{x+(a_1+b_1)=(a_2+b_2)} \\ \left[\begin{array}{c} \lambda_A(a_1) \vee \lambda_A(a_2) \vee \\ \lambda_A(b_1) \vee \lambda_A(b_2) \end{array} \right] \end{array} \right) \vee \frac{1-k}{2}$$

$$= \left(\begin{array}{c} \wedge_{x+(a_1+b_1)=(a_2+b_2)} \\ \left[\begin{array}{c} \lambda_A(a_1) \vee \lambda_A(b_1) \vee \frac{1-k}{2} \\ \vee \lambda_A(a_2) \vee \lambda_A(b_2) \vee \frac{1-k}{2} \end{array} \right] \end{array} \right) \vee \frac{1-k}{2}$$

$$= \left(\begin{array}{c} \wedge_{x+(a_1+b_1)=(a_2+b_2)} \\ \left[\begin{array}{c} \lambda_A(a_1) \vee \lambda_A(b_1) \vee \frac{1-k}{2} \\ \vee \lambda_A(a_2) \vee \lambda_A(b_2) \vee \frac{1-k}{2} \end{array} \right] \end{array} \right) \vee \frac{1-k}{2}$$

$$\geq \left(\begin{array}{c} \wedge_{x+(a_1+b_1)=(a_2+b_2)} \\ [\lambda_A(a_1+b_1) \vee \lambda_A(a_2+b_2)] \end{array} \right) \vee \frac{1-k}{2}$$

by condition $(1b)$

$$\geq \lambda_A(x) \vee \frac{1-k}{2}$$

by condition $(3b)$.
Thus condition (4) is holds.
Conversely assume that $A +_k A \subseteq A_k$. Then for each $x \in R$ we have

$$\mu_A(0) \geq \mu_A(0) \wedge \frac{1-k}{2} \geq (\mu_A +_k \mu_A)(0)$$

$$= \left[\begin{array}{c} \vee_{0+(a_1+b_1)=(a_2+b_2)} \\ \left[\begin{array}{c} \mu_A(a_1) \wedge \mu_A(a_2) \wedge \\ \mu_A(b_1) \wedge \mu_A(b_2) \end{array} \right] \end{array} \right] \wedge \frac{1-k}{2}$$

$$\geq \mu_A(x) \wedge \frac{1-k}{2}$$

because $0+x+x = x+x$
Thus $\mu_A(0) \geq \mu_A(x) \wedge \frac{1-k}{2}$ and

$$\lambda_A(0) \leq \lambda_A(0) \vee \frac{1-k}{2} \leq (\lambda_A +_k \lambda_A)(0)$$

$$= \left[\begin{array}{c} \wedge_{0+(a_1+b_1)=(a_2+b_2)} \\ \left[\begin{array}{c} \lambda_A(a_1) \vee \lambda_A(a_2) \vee \\ \lambda_A(b_1) \vee \lambda_A(b_2) \end{array} \right] \end{array} \right] \vee \frac{1-k}{2}$$

$$\leq \lambda_A(x) \vee \frac{1-k}{2}$$

because $0+x+x = x+x$
Thus $\lambda_A(0) \leq \lambda_A(x) \vee \frac{1-k}{2}$.
Let $x, y \in R$. Then

$$\mu_A(x+y) \geq \mu_A(x+y) \wedge \frac{1-k}{2} \geq (\mu_A +_k \mu_A)(x+y)$$

$$= \left[\begin{array}{c} \vee_{(x+y)+(a_1+b_1)=(a_2+b_2)} \\ \left[\begin{array}{c} \mu_A(a_1) \wedge \mu_A(a_2) \wedge \\ \mu_A(b_1) \wedge \mu_A(b_2) \end{array} \right] \end{array} \right] \wedge \frac{1-k}{2}$$

$$\geq \left\{ \begin{array}{c} \mu_A(0) \wedge \mu_A(x) \wedge \\ \mu_A(0) \wedge \mu_A(y) \end{array} \right\} \wedge \frac{1-k}{2}$$

because $(x+y)+(0+0) = (x+y)$.

$$= \left\{ \mu_A(x) \wedge \mu_A(y) \wedge \frac{1-k}{2} \right\}$$

because $\mu_A(0) \geq \mu_A(x) \wedge \frac{1-k}{2}$ and

$$\lambda_A(x+y) \leq \lambda_A(x+y) \vee \frac{1-k}{2} \leq (\lambda_A +_k \lambda_A)(x+y)$$

$$= \left[\begin{array}{c} \wedge_{(x+y)+(a_1+b_1)=(a_2+b_2)} \\ \left[\begin{array}{c} \lambda_A(a_1) \vee \lambda_A(a_2) \vee \\ \lambda_A(b_1) \vee \lambda_A(b_2) \end{array} \right] \end{array} \right] \vee \frac{1-k}{2}$$

$$\leq \left\{ \begin{array}{c} \lambda_A(0) \vee \lambda_A(x) \vee \\ \lambda_A(0) \vee \lambda_A(y) \end{array} \right\} \vee \frac{1-k}{2}$$

$$= \left\{ \lambda_A(x) \vee \lambda_A(y) \vee \frac{1-k}{2} \right\}$$

because $\lambda_A(0) \le \lambda_A(x) \vee \frac{1-k}{2}$.
Thus A satisfies condition $(1b)$.
Let $a, b, x \in R$ be such that
$x + (a+0) = (b+0)$. Then

$$\mu_A(x) \ge \mu_A(x) \wedge \frac{1-k}{2} \ge (\mu_A +_k \mu_A)(x)$$

$$= \left[\bigvee_{x+(a_1+b_1)=(a_2+b_2)} \left[\begin{array}{c} \mu_A(a_1) \wedge \mu_A(a_2) \wedge \\ \mu_A(b_1) \wedge \mu_A(b_2) \end{array} \right] \right] \wedge \frac{1-k}{2}$$

$$\ge \left\{ \begin{array}{c} \mu_A(0) \wedge \mu_A(a) \wedge \\ \mu_A(0) \wedge \mu_A(b) \end{array} \right\} \wedge \frac{1-k}{2}$$

$$= \left\{ \mu_A(a) \wedge \mu_A(b) \wedge \frac{1-k}{2} \right\}$$

because $\mu_A(0) \ge \mu_A(x) \wedge \frac{1-k}{2}$.
And

$$\lambda_A(x) \le \lambda_A(x) \vee \frac{1-k}{2} \le (\lambda_A +_k \lambda_A)(x)$$

$$= \left[\bigwedge_{x+(a_1+b_1)=(a_2+b_2)} \left[\begin{array}{c} \lambda_A(a_1) \vee \lambda_A(a_2) \vee \\ \lambda_A(b_1) \vee \lambda_A(b_2) \end{array} \right] \right] \vee \frac{1-k}{2}$$

$$\le \left\{ \begin{array}{c} \lambda_A(0) \vee \lambda_A(a) \vee \\ \lambda_A(0) \vee \lambda_A(b) \end{array} \right\} \vee \frac{1-k}{2}$$

$$= \left\{ \lambda_A(a) \vee \lambda_A(b) \vee \frac{1-k}{2} \right\}$$

because $\lambda_A(0) \le \lambda_A(x) \vee \frac{1-k}{2}$.
This shows that A satisfies condition $(3b)$.

3.12 Theorem

An intuitionistic fuzzy subset $A = (\mu_A, \lambda_A)$ of a hemiring R is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left(right) k-ideal of R if and only if
(1) $A +_k A \subseteq A_k$
(2) $\overline{1} \odot_k A \subseteq A_k \left(A \odot_k \overline{1} \subseteq A_k \right)$.

*Proof.*Let $A = (\mu_A, \lambda_A)$ be an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left k-ideal of R. Then by Theorem 3.11 A satisfied (1). Let $x \in R$. If $(\chi_R \odot_k \mu_A)(x) = 0$ and $(\chi_R^c \odot_k \lambda_A)(x) = 1$, then
$(\chi_R \odot_k \mu_A)(x) \le (\mu_A)(x) \wedge \frac{1-k}{2}$, and
$(\chi_R^c \odot_k \lambda_A)(x) \ge (\lambda_A)(x) \vee \frac{1-k}{2}$, otherwise there exist elements $a_i.b_i.a'_j, b'_j \in R$ such that
$x + \sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j$. Then we have
$(\chi_R \odot_k \mu_A)(x) =$

$$\bigvee_{x+\sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j} \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \chi_R(a_i)\right) \wedge \\ \left(\wedge_{i=1}^m \mu_A(b_i)\right) \wedge \\ \left(\wedge_{j=1}^n \chi_R(a'_j)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(b'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

$$= \left(\bigvee_{x+\sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j} \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \mu_A(b_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(b'_j)\right) \end{array} \right\} \right) \wedge \frac{1-k}{2}$$

$$= \left(\bigvee_{x+\sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j} \left\{ \begin{array}{c} \left(\begin{array}{c} \wedge_{i=1}^m \mu_A(b_i) \\ \wedge \frac{1-k}{2} \end{array} \right) \wedge \\ \left(\begin{array}{c} \wedge_{j=1}^n \mu_A(b'_j) \\ \wedge \frac{1-k}{2} \end{array} \right) \end{array} \right\} \right) \wedge \frac{1-k}{2}$$

$$\le \left(\bigvee_{x+\sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j} \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \mu_A(a_i b_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(a'_j b'_j)\right) \end{array} \right\} \right) \wedge \frac{1-k}{2}$$

$$= \left(\bigvee_{x+\sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j} \left\{ \begin{array}{c} \left(\begin{array}{c} \wedge_{i=1}^m \mu_A(a_i b_i) \\ \wedge \frac{1-k}{2} \end{array} \right) \wedge \\ \left(\begin{array}{c} \wedge_{j=1}^n \mu_A(a'_j b'_j) \\ \wedge \frac{1-k}{2} \end{array} \right) \end{array} \right\} \right) \wedge \frac{1-k}{2}$$

$$\le \left(\bigvee_{x+\sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j} \left\{ \begin{array}{c} \mu_A(\sum_{i=1}^m a_i b_i) \wedge \\ \mu_A(\sum_{j=1}^n a'_j b'_j) \end{array} \right\} \right) \wedge \frac{1-k}{2}$$

$$\le \mu_A(x) \wedge \frac{1-k}{2}.$$

and $(\chi_R^c \odot_k \lambda_A)(x) =$

$$\wedge_{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j} \left\{ \begin{array}{c} \left(\vee_{i=1}^m \chi_R^c(a_i)\right) \vee \\ \left(\vee_{i=1}^m \lambda_A(b_i)\right) \vee \\ \left(\vee_{j=1}^n \chi_R^c(a'_j)\right) \vee \\ \left(\vee_{j=1}^n \lambda_A(b'_j)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$= \wedge_{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j} \left\{ \begin{array}{c} \left(\vee_{i=1}^m \lambda_A(b_i)\right) \vee \\ \left(\vee_{j=1}^n \lambda_A(b'_j)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$= \wedge_{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j} \left\{ \begin{array}{c} \left(\vee_{i=1}^m \lambda_A(b_i) \vee \frac{1-k}{2}\right) \vee \\ \left(\vee_{j=1}^n \lambda_A(b'_j) \vee \frac{1-k}{2}\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$\geq \wedge_{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j} \left\{ \begin{array}{c} \left(\vee_{i=1}^m \lambda_A(a_ib_i)\right) \vee \\ \left(\vee_{j=1}^n \lambda_A(a'_jb'_j)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$= \wedge_{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j} \left\{ \begin{array}{c} \left(\begin{array}{c} \vee_{i=1}^m \lambda_A(a_ib_i) \\ \vee \frac{1-k}{2} \end{array}\right) \vee \\ \left(\begin{array}{c} \vee_{j=1}^n \lambda_A(a'_jb'_j) \\ \vee \frac{1-k}{2} \end{array}\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$\geq \wedge_{x+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j} \left\{ \begin{array}{c} \lambda_A\left(\sum_{i=1}^m a_ib_i\right) \vee \\ \lambda_A\left(\sum_{j=1}^n a'_jb'_j\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$\geq \lambda_A(x) \vee \frac{1-k}{2}.$$

This implies that $\overline{1} \odot_k A \subseteq A_k$.

Conversely assume that given conditions (1) and (2) hold. Then by Theorem 3.11 A satisfies $(1b)$ and $(3b)$. Now we show that A satisfies the condition $(2b)$. Let $x, y \in R$. Then we have

$$\mu_A(xy) \geq \mu_A(xy) \wedge \frac{1-k}{2}$$
$$\geq (\chi_R \odot_k \mu_A)(xy)$$

$$= \left(\begin{array}{c} \vee_{xy+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j} \\ \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \chi_R(a_i)\right) \wedge \\ \left(\wedge_{i=1}^m \mu_A(b_i)\right) \wedge \\ \left(\wedge_{j=1}^n \chi_R(a'_j)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(b'_j)\right) \end{array} \right\} \end{array} \right) \wedge \frac{1-k}{2}$$

$$= \left(\begin{array}{c} \vee_{xy+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j} \\ \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \mu_A(b_i)\right) \\ \wedge \left(\wedge_{j=1}^n \mu_A(b'_j)\right) \end{array} \right\} \end{array} \right) \wedge \frac{1-k}{2}$$

$\geq \mu_A(y) \wedge \frac{1-k}{2}$ because $xy + 0y = xy$ and

$$\lambda_A(xy) \leq \lambda_A(xy) \vee \frac{1-k}{2}$$
$$\leq (\chi_R^c \odot_k \lambda_A)(xy)$$

$$= \left(\begin{array}{c} \wedge_{xy+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j} \\ \left\{ \begin{array}{c} \left(\vee_{i=1}^m \chi_R^c(a_i)\right) \vee \\ \left(\vee_{i=1}^m \lambda_A(b_i)\right) \vee \\ \left(\vee_{j=1}^n \chi_R^c(a'_j)\right) \vee \\ \left(\vee_{j=1}^n \lambda_A(b'_j)\right) \end{array} \right\} \end{array} \right) \vee \frac{1-k}{2}$$

$$= \left(\begin{array}{c} \wedge_{xy+\sum_{i=1}^m a_ib_i=\sum_{j=1}^n a'_jb'_j} \\ \left\{ \begin{array}{c} \left(\vee_{i=1}^m \lambda_A(b_i)\right) \vee \\ \left(\vee_{j=1}^n \lambda_A(b'_j)\right) \end{array} \right\} \end{array} \right) \vee \frac{1-k}{2}$$

$\leq \lambda_A(y) \vee \frac{1-k}{2}$ because $xy + 0y = xy$.

This shows that A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left k-ideal of R. Similarly we can prove the case of an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right k-ideal of R.

3.13 Theorem

A non empty subset A of R is a k-ideal of R if and only if its intuitionistic characteristic function C_A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideal of R.

*Proof.*Proof is straightforward.

4 k-regular hemiring

4.1 Definition [1,14]

A hemiring R is said to be k-regular if for each $a \in R$, there exist $x, y \in R$ such that $a + axa = aya$.

4.2 Theorem [14]

A hemiring R is k-regular if and only if for any right k-ideal A and any left k-ideal B, we have $\overline{AB} = A \cap B$.

4.3 Theorem

A hemiring R is k-regular if and only if for any $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right k-ideal A and any $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left k-ideal B of R, we have $A \odot_k B = A \cap_k B$.

*Proof.*Let R be a k-regular hemiring and A, B be $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right k-ideal and $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left k-ideal of R, respectively. Then by corollary 3.8 $A \odot_k B \subseteq A \cap_k B$. To show the reverse inclusion, let $x \in R$. Since R is k-regular, so there exist $a, a' \in R$ such that $x + xax = xa'x$. Then we have

$$(\mu_A \odot_k \mu_B)(x) = \bigvee_{x+\sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j} \left\{ \begin{array}{c} \left(\wedge_{i=1}^m \mu_A(a_i)\right) \wedge \\ \left(\wedge_{i=1}^m \mu_B(b_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(a'_j)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_B(b'_j)\right) \end{array} \right\} \wedge \frac{1-k}{2}$$

$$\geq \min\{\mu_A(xa),\ \mu_A(xa'),\ \mu_B(x)\} \wedge \tfrac{1-k}{2}$$
$$\geq \min\left\{\mu_A(x)\ , \mu_B(x),\ \tfrac{1-k}{2}\right\} \wedge \tfrac{1-k}{2}$$
$$= (\mu_A \wedge_k \mu_B)(x) \text{ and}$$

$$(\lambda_A \odot_k \lambda_B)(x) = \bigwedge_{x+\sum_{i=1}^m a_i b_i = \sum_{j=1}^n a'_j b'_j} \left\{ \begin{array}{c} \left(\vee_{i=1}^m \lambda_A(a_i)\right) \vee \\ \left(\vee_{i=1}^m \lambda_B(b_i)\right) \vee \\ \left(\vee_{j=1}^n \lambda_A(a'_j)\right) \vee \\ \left(\vee_{j=1}^n \lambda_B(b'_j)\right) \end{array} \right\} \vee \frac{1-k}{2}$$

$$\leq \max\{\lambda_A(xa),\ \lambda_A(xa'),\ \lambda_B(x)\} \vee \tfrac{1-k}{2}$$
$$\leq \max\left\{\lambda_A(x),\ \lambda_B(x),\ \tfrac{1-k}{2}\right\}$$
$$= (\lambda_A \vee_k \lambda_B)(x).$$

This implies that $A \odot_k B \supseteq A \cap_k B$. Therefore $A \odot_k B = A \cap_k B$.

Conversely, let A and B be any right k-ideal and left k-ideal of R, respectively. Then the intuitionistic characteristic functions C_A and C_B are $(\in, \in \vee q_k)^*$-intuitionistic fuzzy right k-ideal and $(\in, \in \vee q_k)^*$-intuitionistic fuzzy left k-ideal of R, respectively. Now, by assumption $(C_{\overline{AB}})_k = (C_A \odot C_B)_k = C_A \odot_k C_B = C_A \cap_k C_B = (C_A \cap C_B)_k = (C_{A \cap B})_k$. This follows that $\overline{AB} = A \cap B$. Hence by Theorem 4.2, R is k-regular hemiring.

From Theorem 4.2, it follows that in a k-regular hemiring every k-ideal A is k-idempotent, that is $\overline{AA} = A$. On the other hand, in such hemirings we have $A \odot_k A = A_k$ for all $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideals.

4.4 Theorem

Let R be a hemiring with identity 1. Then the following assertions are equivalent:

(1) Each k-ideal of R is idempotent.

(2) $A \odot_k A = A_k$ for every $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideal of R.

(3) $A \odot_k B = A \cap_k B$ for all $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideals $A = (\mu_A, \lambda_A)$ and $B = (\mu_B, \lambda_B)$ of R.

Proof.(1) $\Longrightarrow$ (2) Let $x \in R$. The smallest k-ideal of R containing x has the form $\overline{RxR}$. By hypothesis, we have

$$\overline{RxR} = \overline{(\overline{RxR})(\overline{RxR})} = \overline{RxRRxR}.$$

Thus $x \in \overline{RxRRxR}$, this implies

$$x + \sum_{i=1}^m r_i x s_i u_i x t_i = \sum_{j=1}^n r'_j x s'_j u'_j x t'_j$$

for some $r_i, s_i, u_i, t_i, r'_j, s'_j, u'_j, t'_j \in R$.

As $\mu_A(x) \wedge \frac{1-k}{2}$

$$= \mu_A(x) \wedge \mu_A(x) \wedge \frac{1-k}{2}$$
$$= \left(\mu_A(x) \wedge \frac{1-k}{2}\right) \wedge \left(\mu_A(x) \wedge \frac{1-k}{2}\right) \wedge \frac{1-k}{2}$$
$$\leq \mu_A(r_i x s_i) \wedge \mu_A(u_i x t_i) \wedge \frac{1-k}{2}$$

$(1 \leq i \leq m)$, so

$\mu_A(x) \wedge \frac{1-k}{2} \leq \left(\wedge_{i=1}^m \mu_A(r_i x s_i)\right) \wedge \left(\wedge_{i=1}^m \mu_A(u_i x t_i)\right) \wedge \frac{1-k}{2}$

Similarly $\mu_A(x) \wedge \frac{1-k}{2} \leq \left(\wedge_{j=1}^n \mu_A(r'_j x s'_j)\right) \wedge \left(\wedge_{j=1}^n \mu_A(u'_j x t'_j)\right) \wedge \frac{1-k}{2}$

Therefore $\mu_A(x) \wedge \frac{1-k}{2} \leq$

$$\left[\begin{array}{c} \left(\wedge_{i=1}^m \mu_A(r_i x s_i)\right) \wedge \\ \left(\wedge_{i=1}^m \mu_A(u_i x t_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(r'_j x s'_j)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(u'_j x t'_j)\right) \end{array} \right] \wedge \frac{1-k}{2}$$

$$\leq \bigvee_{x+\sum_{i=1}^m r_i x s_i u_i x t_i = \sum_{j=1}^n r'_j x s'_j u'_j x t'_j} \left[\begin{array}{c} \left(\wedge_{i=1}^m \mu_A(r_i x s_i)\right) \wedge \\ \left(\wedge_{i=1}^m \mu_A(u_i x t_i)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(r'_j x s'_j)\right) \wedge \\ \left(\wedge_{j=1}^n \mu_A(u'_j x t'_j)\right) \end{array} \right] \wedge \frac{1-k}{2}$$

$= (\mu_A \odot_k \mu_A)(x)$.

And $\lambda_A(x) \vee \frac{1-k}{2} = \lambda_A(x) \vee \lambda_A(x) \vee \frac{1-k}{2}$
$= \left(\lambda_A(x) \vee \frac{1-k}{2}\right) \vee \left(\lambda_A(x) \vee \frac{1-k}{2}\right) \vee \frac{1-k}{2}$
$\geq \lambda_A(r_i x s_i) \vee \lambda_A(u_i x t_i) \vee \frac{1-k}{2}$ $(1 \leq i \leq m)$, so
$\lambda_A(x) \vee \frac{1-k}{2} \geq \left(\vee_{i=1}^m \lambda_A(r_i x s_i)\right) \vee \left(\vee_{i=1}^m \lambda_A(u_i x t_i)\right) \vee \frac{1-k}{2}$
Similarly $\lambda_A(x) \vee \frac{1-k}{2} \geq$
$\left(\vee_{j=1}^n \lambda_A(r'_j x s'_j)\right) \vee \left(\vee_{j=1}^n \lambda_A(u'_j x t'_j)\right) \vee \frac{1-k}{2}$
Therefore $\lambda_A(x) \vee \frac{1-k}{2} \geq$

$$\begin{bmatrix} \left(\vee_{i=1}^m \lambda_A(r_i x s_i)\right) \vee \\ \left(\vee_{i=1}^m \lambda_A(u_i x t_i)\right) \vee \\ \left(\vee_{j=1}^n \lambda_A(r'_j x s'_j)\right) \vee \\ \left(\vee_{j=1}^n \lambda_A(u'_j x t'_j)\right) \end{bmatrix} \vee \frac{1-k}{2}$$

$$\geq \wedge_{x+\sum_{i=1}^m r_i x s_i u_i x t_i = \sum_{j=1}^n r'_j x s'_j u'_j x t'_j} \begin{bmatrix} \left(\vee_{i=1}^m \lambda_A(r_i x s_i)\right) \vee \\ \left(\vee_{i=1}^m \lambda_A(u_i x t_i)\right) \vee \\ \left(\vee_{j=1}^n \lambda_A(r'_j x s'_j)\right) \vee \\ \left(\vee_{j=1}^n \lambda_A(u'_j x t'_j)\right) \end{bmatrix} \vee \frac{1-k}{2}$$

$= (\lambda_A \odot_k \lambda_A)(x)$.

Hence $A_k \subseteq A \odot_k A$ and by Theorem 3.7, $A \odot_k A \subseteq A_k$. Thus $A \odot_k A = A_k$.

(2) $\Longrightarrow$ (1) Let A be a k-ideal of R. Then the intuitionistic characteristic function C_A of A is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideal of R. Hence by hypothesis

$$(C_A)_k = C_A \odot_k C_A = (C_A \odot C_A)_k = (C_{\overline{AA}})_k.$$

This follows that $A = \overline{AA}$.

(2) $\Longrightarrow$ (3) Let A and B be $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideals of R. We know that

$$(A \cap_k B) \odot_k (A \cap_k B) \subseteq (A_k) \odot_k (B_k) = (A \odot_k B).$$

Since $(A \cap_k B)$ is an $(\in, \in \vee q_k)^*$-intuitionistic fuzzy k-ideal of R, so by hypothesis $(A \cap_k B)$ is idempotent. Thus $(A \cap_k B) = (A \cap_k B) \odot_k (A \cap_k B) \subseteq (A \odot_k B)$. By Theorem 3.7, $A \odot_k B \subseteq A \cap_k B$. Hence $A \odot_k B = A \cap_k B$.

(3) $\Longrightarrow$ (2) Obvious.

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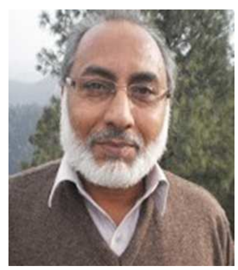

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