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Extended Eigenvalues of Direct Sum of Operators

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Abstract: In this paper a connection between extended eigenvalues of direct sum of operators in the direct sum of Hilbert spaces and their coordinate operators has been investigated. Moreover, the structure of the set of extended eigenvalues of normal compact operators has been researched.

Keywords: Extended eigenvalue, direct sum of operators, compact operator.

1 Introduction

It is known that a complex number λ is called an extended eigenvalue of the linear bounded operator A in the Hilbert space H if there exists nonzero operator $T \in L(H)$ such that $TA = \lambda AT$, where $L(H)$ is a space of linear bounded operators. A set of extended eigenvalues of the linear bounded operators is denoted by $\Sigma(\cdot)$ [1].

Note that the structure of the set of extended eigenvalues in complex plane for operators in $L(H)$ have many different forms (for example, see [1-7]). One of the fundamental problems in this theory is to describe a structure of the set of extended eigenvalues in complex plane for linear bounded operators in Hilbert spaces.

2 Extended eigenvalues of direct sum of operators

In this section the structure of the set of eigenvalues of direct sum of operators will be investigated. Firstly, we prove the following result.

Theorem 2.1. Let H_n be a Hilbert space, $A_n : H_n \longrightarrow H_n$, $A_n \in L(H_n)$ for any $n \geq 1$, $H = \bigoplus_{n=1}^{\infty} H_n$, $A = \bigoplus_{n=1}^{\infty} A_n \in L(H)$. In this case for the set of extended eigenvalues it is true that

$$\bigcup_{n=1}^{\infty} \Sigma(A_n) \subset \Sigma(A).$$

Proof. Firstly we assume that $\lambda \in \bigcup_{n=1}^{\infty} \Sigma(A_n)$. In this case there exists at least one natural number n_λ and nonzero operator $T_{n_\lambda} \in L(H_{n_\lambda})$ such that

$$T_{n_\lambda} A_{n_\lambda} = \lambda A_{n_\lambda} T_{n_\lambda}.$$

If we choose an operator $T_\lambda : H \longrightarrow H$ in form

$$T_\lambda = \{0, ..., 0, T_{n_\lambda}, 0, ...\},$$

where operator T_{n_λ} is placed in n_λ-th index, then operator $T_\lambda \in L(H), T_\lambda \neq 0$ and $T_\lambda A = \lambda A T_\lambda$. Hence, $\lambda \in \Sigma(A)$, i.e.

$$\bigcup_{n=1}^{\infty} \Sigma(A_n) \subset \Sigma(A)$$

Theorem 2.2. Let H_n be a Hilbert space, $A_n : H_n \longrightarrow H_n$, $A_n \in L(H_n)$ for any $n \geq 1$, $H = \bigoplus_{n=1}^{\infty} H_n$, $A = \bigoplus_{n=1}^{\infty} A_n \in L(H), \lambda \in \Sigma(A)$ such that $TA = \lambda AT$ and $TH_{n_\lambda} \subset H_{n_\lambda}$ for some $n_\lambda \in \mathbb{N}$. Then $\lambda \in \bigcup_{n=1}^{\infty} \Sigma(A_n)$, i.e. $\{\lambda \in \Sigma(A) : TA = \lambda AT$ and there exists $n_\lambda \in \mathbb{N}$ such

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that $TH_{n_\lambda} \subset H_{n_\lambda}\} \subset \bigcup_{n=1}^{\infty} \Sigma(A_n)$

Proof. Let us $\lambda \in \Sigma(A)$. Then there exists nonzero operator $T \in L(H)$ such that $TA = \lambda AT$. In this case from relation $TA = \lambda AT$ it is obtained that

$$TA_n = \lambda AT_n, n \geq 1,$$

where T_n is a restriction of the operator T to the space $H_n, n \geq 1$. Since $TH_{n_\lambda} \subset H_{n_\lambda}$ for some $n_\lambda \in \mathbb{N}$ and $T_{n_\lambda} \neq 0$, then it is established that

$$T_{n_\lambda} A_{n_\lambda} = \lambda A_{n_\lambda} T_{n_\lambda},$$

i.e. $\lambda \in \Sigma(A_{n_\lambda})$. Consequently

$$\lambda \in \bigcup_{n=1}^{\infty} \Sigma(A_n).$$

From these theorems we have the following corollary.

Corollary 2.3. Let H_n be a Hilbert space, $A_n : H_n \longrightarrow H_n$, $A_n \in L(H_n)$ for any $n \geq 1$, $H = \bigoplus_{n=1}^{\infty} H_n$, $A = \bigoplus_{n=1}^{\infty} A_n \in L(H)$. If for each $\lambda \in \Sigma(A)$ there exists $T_\lambda \in L(H)$ such that $T_\lambda A = \lambda AT_\lambda$ and there exists $n_\lambda \in \mathbb{N}$ such that $\{0\} \neq T_\lambda H_{n_\lambda} \subset H_{n_\lambda}$, then

$$\Sigma(A) = \bigcup_{n=1}^{\infty} \Sigma(A_n).$$

Theorem 2.4. Let H_n be a Hilbert space, $A_n : H_n \longrightarrow H_n$, $A_n \in L(H_n)$ for any $n \geq 1$, $H = \bigoplus_{n=1}^{\infty} H_n$, $A = \bigoplus_{n=1}^{\infty} A_n \in L(H)$. If $A_m = 0$ for some $m \in \mathbb{N}$, then $\Sigma(A) = \mathbb{C}$.

Proof. Indeed, if we choose the operator $T = \bigoplus_{n=1}^{\infty} T_n : H \longrightarrow H$ in following form

$$T_m \neq 0 \quad \text{and} \quad T_n = 0, n \neq m, n \geq 1, T_n \in L(H_n),$$

then it is clear that an equation $TA = \lambda AT$ is hold for any $\lambda \in \mathbb{C}$ and $T \in L(H)$.

Theorem 2.5. Let H_n be a Hilbert space, $A_n : H_n \longrightarrow H_n$, $A_n \in L(H_n)$ for any $n \geq 1$, $H = \bigoplus_{n=1}^{\infty} H_n$, $A = \bigoplus_{n=1}^{\infty} A_n \in L(H)$. If for some $m \in \mathbb{N}$ an operator A_m is a nilpotent in H_m, then $\Sigma(A) = \mathbb{C}$.

Proof. In this case the operator T can be choosen

$$T = 0 \oplus \cdots \oplus 0 \oplus A_m^{k-1} \oplus 0 \oplus \cdots,$$

where $k \in \mathbb{N}$ is a nilpotency index of the operator A_m.

It is known that if A is a normal compact operator in a Hilbert space H, then it can be written as a direct sum of operators, i.e. $A = \bigoplus_{n=0}^{\infty} \mu_n E_n$, where $E_n : H_n \longrightarrow H_n$ is a identity operator, $H_n = H_{\mu_n}(A)$ is a subspace of eigenelements corresponding to eigenvalue μ_n of the operator A for any $n \geq 0, \mu_0 = 0$. Moreover $H = \bigoplus_{n=0}^{\infty} H_n$, where $H_0 = \ker A$.

Theorem 2.6. If A is a normal compact operator in a Hilbert space H, $\ker A \neq 0, \sigma_p(A) = \{\mu_n : n \geq 0\}$ is point spectrum of A, $H_n = H_{\mu_n}(A), n \geq 0$ is a subspace of eigenelements corresponding to eigenvalue μ_n of the operator A, then $\Sigma(A) = \mathbb{C}$.

Proof. Since $A_0 = 0$ in the direct sum $A = \bigoplus_{n=0}^{\infty} \mu_n E_n$ in the Hilbert space $H = \bigoplus_{n=0}^{\infty} H_n$, then from the Theorem 2.4 it is obtained that $\Sigma(A) = \mathbb{C}$.

The following examples are some applications of the last theorem.

Example 1. Let us

$$A : l_2(\mathbb{N}) \longrightarrow l_2(\mathbb{N}), A(x_n) = \{0, w_2x_2, w_3x_3, \ldots\}, (x_n) \in l_2(\mathbb{N}),$$

where (w_n) is a sequence of complex numbers such that $\lim_{n \longrightarrow \infty} w_n = 0$.

It is clear that A is normal compact operator. Then from above theorem we have $\Sigma(A) = \mathbb{C}$.

Example 2. Let H be any Hilbert space, M be any linear manifold in H, $M \neq H, \dim M < \infty$ and $P : H \longrightarrow M$ is an orthogonal projection compact operator in H.

In this case from Theorem 2.6 it is clear that $\Sigma(P) = \mathbb{C}$.

Example 3. In Banach space B there exists a compact operator A such that $\Sigma(A) = \mathbb{C}$.

Indeed, if we assume that $B = l_p(\mathbb{N}), 1 \leq p < \infty$ and for any $x = (x_n) \in l_p(\mathbb{N})$

$$A(x_n) = \{w_1x_1, 0, w_3x_3, 0, w_5x_5, \ldots, 0, w_{2n-1}x_{2n-1}, 0, \ldots\},$$

where (w_n) is a sequence of complex numbers such that $\lim_{n \longrightarrow \infty} w_n = 0$, then A is compact operator in $l_p(\mathbb{N})$. Moreover an operator defined in form

$T : l_p(\mathbb{N}) \longrightarrow l_p(\mathbb{N}), 1 \leq p < \infty,$

$T(x_n) = \{0, x_2, 0, x_4, 0, \dots, 0, x_{2n}, 0, \dots\}, x = (x_n) \in l_p(\mathbb{N})$, it is hold that $TA = AT = 0$. Consequently, $\Sigma(A) = \mathbb{C}$.

Theorem 2.7. Let A be a normal compact operator in a Hilbert space H, $\sigma_c(A) = \{0\}, \sigma_p(A) = \{\mu_n : n \geq 1\}$ is point spectrum of $A = \bigoplus_{n=1}^{\infty} \mu_n E_n$, $H_n = H_{\mu_n}(A), n \geq 1$ is a subspace of eigenelements corresponding to eigenvalue μ_n of the operator A, $H = \bigoplus_{n=1}^{\infty} H_n$, $\lambda \in \Sigma(A), TA = \lambda AT$ and T_n is a resrtriction of the operator T to the space $H_n, n \geq 1$. In this case if $T_n H_n \cap H_m \neq \{0\}$ and $T_n H_n \cap H_k \neq \{0\}$ for some $m, k \in \mathbb{N}$, then $m = k$ and $\lambda = \frac{\mu_n}{\mu_m}$.

Proof. In this case there exists an element $x_n^m \in H_n \setminus \{0\}$ such that

$$T x_n^m \in H_m \setminus \{0\} \quad \text{and} \quad T_n A_n x_n^m = \lambda A_m T_n x_n^m.$$

Therefore

$$(\mu_n - \lambda \mu_m) T_n x_n^m = 0.$$

Consequently

$$\lambda = \frac{\mu_n}{\mu_m}.$$

In a similar way it can be obtained that $\lambda = \frac{\mu_n}{\mu_k}$. From this we have $\mu_m = \mu_k$, then $m = k$.

Theorem 2.8. For a normal compact operator A with $\ker A = \{0\}$ it is true that

$$\Sigma(A) \subset \bigcup_{n,m=1}^{\infty} \left\{ \frac{\mu_n}{\mu_m} \right\},$$

where $\mu_n, n \geq 1$ is a nonzero eigenvalue of A.

Proof. Let $\lambda \in \Sigma(A)$. Then there exists nonzero operator $T \in L(H)$ such that $TA = \lambda AT$. Since $T \neq 0$, then there exists $n_\star \in \mathbb{N}$ such that $T_{n_\star} \neq 0$, $T_{n_\star}$ is a restriction of the operator T to the space $H_{n_\star}, n_\star \geq 1$. From last relation there exists $x_{n_\star} \in H_{n_\star} \setminus \{0\}$ such that $T_{n_\star} x_{n_\star} \neq 0$. Additionally, there exists $m_\star \in \mathbb{N}$ such that

$$y_{m_\star} = T_{n_\star} x_{n_\star} \in H_{m_\star} \setminus \{0\}.$$

In this case from relation $TA = \lambda AT$ it is obtained that

$$TA_{n_\star} = \lambda AT_{n_\star}, n_\star \geq 1.$$

Then for $x_{n_\star} \in H_{n_\star} \setminus \{0\}, n_\star \geq 1$

$$TA_{n_\star} x_{n_\star} = \lambda AT_{n_\star} x_{n_\star}.$$

From this

$$\mu_{n_\star} (T_{n_\star} x_{n_\star}) = \lambda \mu_{m_\star} (T_{n_\star} x_{n_\star}).$$

Then

$$(\mu_{n_\star} - \lambda \mu_{m_\star}) T_{n_\star} x_{n_\star} = 0.$$

Since $T_{n_\star}, n_\star \geq 1$, then $\lambda = \dfrac{\mu_{n_\star}}{\mu_{m_\star}}$.

Actually, the last results are true for the large class of linear bounded operators too.

Theorem 2.9. If $A \in L(H), \ker A = \{0\}, \sigma(A) = \sigma_p(A) = \{\mu_n : n \geq 0\}, H_n = H_{\mu_n}(A)$ is a subspace of eigenelements corresponding to eigenvalue μ_n of the operator A for any $n \geq 1$ and $H = \bigoplus_{n=1}^{\infty} H_n$. In this case it is true that

$$\Sigma(A) = \bigcup_{m,n=1}^{\infty} \left\{ \frac{\mu_m}{\mu_n} \right\}.$$

Proof. For the simplicity of explanations let us $Ax_n = \mu_n x_n, \dim H_n = 1$. In work [5] it has been proved that

$$\Sigma(A) \subset \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \varnothing\}$$

From this and the structure of the spectrum A it is implied that

$$\Sigma(A) \subset \bigcup_{n,m=1}^{\infty} \left\{ \frac{\mu_n}{\mu_m} \right\}.$$

Now we assume that $T \in L(H)$, T_n is a restriction of the operator T to the subspace H_n for $n \geq 1$. If we choose operator $T = \bigoplus_{n=1}^{\infty} T_n : H \longrightarrow H$ in form

$$T_n x_n = x_m, n \geq 1 \quad \text{and} \quad T_k = 0, k \neq n,$$

then for $x = (x_n) \in H$ it is clear that

$$\begin{aligned} TAx &= TA(x_n) = T(A_n x_n) = T(\{A_1 x_1, A_2 x_2, \dots, A_n x_n, \dots\}) \\ &= T(\{\mu_1 x_1, \mu_2 x_2, \dots, \mu_n x_n, \dots\}) \\ &= \{0, 0, \dots, 0, \mu_n x_m, 0, \dots\} \end{aligned}$$

In a similar way it can be shown that

$$\begin{aligned} ATx &= AT(\{x_1, x_2, \dots, x_n, \dots\}) = A(\{0, 0, \dots, 0, x_m, 0, \dots\}) \\ &= \{0, 0, \dots, 0, \mu_m x_m, 0, \dots\} \end{aligned}$$

Then for the $\lambda = \frac{\mu_m}{\mu_n} \in \mathbb{C}$ and $T \in L(H), T \neq 0$ we have

$$TAx = \left(\frac{\mu_m}{\mu_n} \right) ATx.$$

This means that

$$\frac{\mu_m}{\mu_n} \in \Sigma(A), m \geq 1, n \geq 1.$$

Consequently

$$\bigcup_{m,n=1}^{\infty} \left\{ \frac{\mu_m}{\mu_n} \right\} \subset \Sigma(A).$$

Hence

$$\Sigma(A) = \bigcup_{m,n=1}^{\infty} \left\{ \frac{\mu_m}{\mu_n} \right\}.$$

Corollary 2.10. If $A \in C_\infty(H), \ker A = \{0\}, \sigma_p(A) = \{\mu_n : n \geq 1\}$, then

$$\Sigma(A) = \bigcup_{m,n=1}^{\infty} \left\{ \frac{\mu_m}{\mu_n} \right\}.$$

Theorem 2.9 and Corollary 2.10 give some ideas on the form of the set of extended eigenvalues in complex plane (see [7]).

Example 4. Unfortunately these results are not true for nonnormal compact operators. For example, the Volterra operator

$$V : L_2(0,1) \longrightarrow L_2(0,1), Vf(x) = \int_0^x f(t)dt, f \in L_2(0,1)$$

is a nonnormal compact operator in $L_2(0,1)$ and $\Sigma(V) = (0,+\infty)$ (see [1]).

On the other hand in Hilbert space H there exists nonnormal compact operator A such that $\Sigma(A) = \{1\}$ [7].

Theorem 2.11. Let H_n be a Hilbert space, $A_n : H_n \longrightarrow H_n, \quad A_n \in L(H_n)$ for any $n \geq 1$, $H = \bigoplus_{n=1}^{\infty} H_n$, $A = \bigoplus_{n=1}^{\infty} A_n \in L(H)$. If $\dim H < \infty$, then

$$\Sigma(A) = \bigcup_{n,m=1}^{\infty} \{\lambda \in \mathbb{C} : \sigma(A_n) \cap \sigma(\lambda A_m) \neq \varnothing\}.$$

In general case

$$\Sigma(A) \subset \bigcup_{n,m=1}^{\infty} \{\lambda \in \mathbb{C} : \sigma(A_n) \cap \sigma(\lambda A_m) \neq \varnothing\}.$$

Proof. Under these conditions it is easy to prove that

$$\sigma(A) = \sigma_p(A) = \bigcup_{n=1}^{\infty} \sigma_p(A_n).$$

On the other hand for the case $\dim H < \infty$ it is known that

$$\Sigma(A) = \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \varnothing\}.$$

But for any Hilbert space

$$\Sigma(A) \subset \{\lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \varnothing\} \quad [5].$$

Hence the above relations are true.

Remark 2.12. Let H_n be a Hilbert space, $A_n : H_n \longrightarrow H_n, \quad A_n \in L(H_n)$ for any $n \geq 1$, $H = \bigoplus_{n=1}^{\infty} H_n$, $A = \bigoplus_{n=1}^{\infty} A_n \in L(H)$. In general

$$\Sigma(A) \neq \bigcup_{n=1}^{\infty} \Sigma(A_n).$$

Proof. It is sufficient that to give an example for the validity of this claim. Assume that H_1 and H_2 are any Hilbert spaces,

$A_1 \in L(H_1), A_2 \in L(H_2), H = H_1 \oplus H_2, A = A_1 \oplus A_2,$

$\sigma(A_1) = \{1,3\}, \sigma(A_2) = \{2,4\}$

In this case

$\{\lambda \in \mathbb{C} : \sigma(A_1) \cap \sigma(\lambda A_1) \neq \varnothing\} = \{1,3,1/3\},$
$\{\lambda \in \mathbb{C} : \sigma(A_2) \cap \sigma(\lambda A_2) \neq \varnothing\} = \{1,2,1/2\},$
$\{\lambda \in \mathbb{C} : \sigma(A_1) \cap \sigma(\lambda A_2) \neq \varnothing\} = \{1/2,1/4,3/4,3/2\},$
$\{\lambda \in \mathbb{C} : \sigma(A_2) \cap \sigma(\lambda A_1) \neq \varnothing\} = \{2/3,4/3,2,4\}.$

Then from Theorem 2.11 it is clear that

$$\Sigma(A) = \{1,1/4,1/3,1/2,2/3,3/4,4/3,3/2,2,3,4\} \neq \Sigma(A_1) \cup \Sigma(A_2) = \{1,1/3,1/2,2,3\}$$

3 Conclusions

In this paper the inner structure of the set of extended eigenvalues of direct sum of operators defined in the direct sum of Hilbert spaces has been researched. In particular this problem for normal compact operators has been investigated more deeply. In corresponding places the obtained results have been supplemented by examples.

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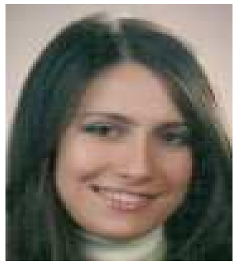

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