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The Concept of weak (ψ,α,β) for Some Well Known Fixed Point Theorems

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Abstract: Eslamian and Abkar [8] introduced fixed point of the weak-(ψ,α,β) contractive mappings in complete metric space (also see [9]). In this paper we established a generalized concept of the weak-(ψ,α,β) contractive condition depended on another mapping. Thus, we obtained a new generalization of Banach fixed point theorem, Kannan fixed point theorem and Chatterjea fixed point theorem.

Keywords: fixed point, Kannan fixed point theorem,Chatterjea fixed point theorem, contraction mappings

1 Introduction and Preliminaries

Banach contraction principle is one of the most important result in mathematics. Due to the importance, generalizations of Banach fixed point theorem have been investigated broadly by many mathematicians.

A mapping $T:X\to X$ where (X,d) is a metric space, is said to be a contraction if there exists $k\in[0,1)$ such that for all $x,y\in X$,

$$d(Tx,Ty)\le kd(x,y). \tag{1}$$

In [1], Banach proved that a contraction mapping has a unique fixed point in complete metric space.

Kannan [2], established the following result.

Theorem 1. *[2] If a mapping $T:X\to X$ where (X,d) is a complete metric space, satisfies the inequality*

$$d(Tx,Ty)\le a[d(x,Tx)+d(y,Ty)] \tag{2}$$

where $a\in\left[0,\frac{1}{2}\right)$ and $x,y\in X$, then T has a unique fixed point. The mappings satisfying (2) are called Kannan type mappings.

A similar contractive condition has been introduced by Chatterjea [3] as following:

Theorem 2. *[3] If a mapping $T:X\to X$ where (X,d) is a complete metric space, satisfies the inequality*

$$d(Tx,Ty)\le b[d(x,Ty)+d(y,Tx)] \tag{3}$$

such that $b\in\left[0,\frac{1}{2}\right)$ and $x,y\in X$, then T has a unique fixed point. The mappings satisfying (3) are called Chatterjea type mapping.

In 2011, Moradi and Davood [4] introduced a new extension of Kannan type contractive mapping depended on another function T which is continuous, one to one and subsequentially convergent.

Definition 1. *[4] Let (X,d) be a metric space.*

SSC) A mapping $T:X\to X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ also is convergence.

SC) A mapping $T:X\to X$ is said to be subsequentially convergent if we have, for every sequence $\{y_n\}$, if $\{Ty_n\}$ is convergence then $\{y_n\}$ has a convergent subsequence.

Theorem 3. *[4] Let (X,d) be a complete metric space and $T,S:X\to X$ be mappings such that T is continuous, one to one and subsequentially convergent. If $\lambda\in\left[0,\frac{1}{2}\right)$ and $x,y\in X$, S satisfying*

$$d(TSx,TSy)\le\lambda[d(Tx,TSx)+d(Ty,TSy)] \tag{4}$$

then, S has a unique fixed point. Also if T is sequentially convergent then for every $x_0\in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

In 2013, Razani and Parvaneh, generating result of Moradi and Davood, gave fixed point theorems for weakly T-Chatterjea and weakly T-Kannan-contractive mappings in complete metric spaces [13]. Also, Eslamian

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and Abkar introduced the fixed point theorem of the $(\psi,\alpha,\beta)-$weak contractive mappings in complete metric space [8],[9].

In this paper, a generalized weak-(ψ,α,β) contractive condition is considered for Banach fixed point theorem, Kannan fixed point theorem and Chatterjea fixed point theorem. Therefore we obtain some results which are more general than the result of Razani and Parvaneh [13], Eslamian and Abkar [8].

2 Main Results

We denote by Ψ the set of functions $\psi:[0,\infty)\to[0,\infty)$ satisfying the properties: $a)$ ψ is continuous and monotone nondecreasing, $b)$ $\psi(t)=0$ if and only if $t=0$.

We denote by Φ the set of functions $\alpha:[0,\infty)\to[0,\infty)$ satisfying the properties: $a)$ α is continuous, $b)$ $\alpha(t)=0$ if and only if $t=0$.

We donete by Γ_1 set of the function $\beta:[0,\infty)\to[0,\infty)$ satisfying the properties: $a)$ β is lower semi-continuous, $b)$ $\beta(t)=0$ if and only if $t=0$.

We denote by Γ_2 the set of functions $\beta:[0,\infty)^2\to[0,\infty)$ satisfying the properties : $a)$ β is continuous, $b)$ $\beta(a,b)=0$ if and only if $a=b=0$.

Also, we denote by $SSC(X)$ the set of all mappings $T:X\to X$ such that T is one to one , continuous and subsequentially convergent, by $SC(X)$ the set of all mappings $T:X\to X$ such that T is one to one , continuous and sequentially convergent.

Theorem 4. *Let (X,d) be a complete metric space and $f:X\to X$ be a mapping. Let $T\in SSC(X)$ and f satisfying the inequality*

$$\psi(d(Tfx,Tfy))\leq\alpha(d(Tx,Ty))-\beta(d(Tx,Ty)) \tag{5}$$

such that for $\psi\in\Psi,\alpha\in\Phi,\beta\in\Gamma_1$ we have

$$\psi(t_1)\leq\alpha(t_2),\text{ implies } t_1\leq t_2 \tag{6}$$

and for all

$$t>0,\ \psi(t)-\alpha(t)+\beta(t)>0. \tag{7}$$

Then, f has a unique fixed point.

Proof. *Let x_0 be an arbitrary point in X. We define the iterative sequence $\{x_n\}$ by $x_{n+1}=fx_n$ (equivelently, $x_n=f^nx_0$), $n=1,2,...$ From (5), we have*

$$\begin{aligned}\psi(d(Tx_n,Tx_{n+1})) &= \alpha(d(Tfx_{n-1},Tfx_n))\\ &\leq \alpha(d(Tx_{n-1},Tx_n))-\beta(d(Tx_{n-1},Tx_n)) \quad (8)\\ &\leq \alpha(d(Tx_{n-1},Tx_n)). \quad (9)\end{aligned}$$

The inequality (9) implies that $\{d(Tx_n,Tx_{n+1})\}$ is a monotone decreasing sequence and consequently, there exists $r\geq0$ such that

$$d(Tx_n,Tx_{n+1})\to r\ \text{ as } n\to\infty.$$

Letting $n\to\infty$ in (8), we obtain that $\psi(r)\leq\alpha(r)-\beta(r)$. From (7), we have $r=0$. Now, we prove that $\{Tx_n\}$ is a Cauchy sequence. If possible, let $\{Tx_n\}$ be not a Cahucy sequence. Then, there exists $\varepsilon>0$ for which we can find subsequences $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ with $n(k)>m(k)>k$ such that

$$d\left(Tx_{m(k)},Tx_{n(k)}\right)\geq\varepsilon. \tag{10}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k)>m(k)$ and

$$d\left(Tx_{m(k)},Tx_{n(k)-1}\right)<\varepsilon. \tag{11}$$

Also, using (10) we have

$$\begin{aligned}\varepsilon &\leq d\left(Tx_{m(k)},Tx_{n(k)}\right)\\ &\leq d\left(Tx_{m(k)},Tx_{n(k)-1}\right)+d\left(Tx_{n(k)},Tx_{n(k)-1}\right)\end{aligned} \tag{12}$$

letting $k\to\infty$ in (12),

$$\lim_{k\to\infty}d\left(Tx_{m(k)},Tx_{n(k)}\right)=\varepsilon. \tag{13}$$

From triangle inequality,

$$d\left(Tx_{m(k)},Tx_{n(k)-1}\right)\leq d\left(Tx_{m(k)},Tx_{n(k)}\right)+d\left(Tx_{n(k)},Tx_{n(k)-1}\right) \tag{14}$$

letting $k\to\infty$ in (14)

$$\lim_{k\to\infty}d\left(Tx_{m(k)},Tx_{n(k)-1}\right)=\varepsilon. \tag{15}$$

Again,

$$d\left(Tx_{m(k)-1},Tx_{n(k)}\right)\leq d\left(Tx_{m(k)-1},Tx_{m(k)}\right)+d\left(Tx_{m(k)},Tx_{n(k)}\right) \tag{16}$$

letting $k\to\infty$ in (16)

$$\lim_{k\to\infty}d\left(Tx_{m(k)-1},Tx_{n(k)}\right)=\varepsilon. \tag{17}$$

For the last,

$$d\left(Tx_{m(k)-1},Tx_{n(k)-1}\right)\leq d\left(Tx_{m(k)-1},Tx_{m(k)}\right)+d\left(Tx_{m(k)},Tx_{n(k)-1}\right) \tag{18}$$

letting $k\to\infty$ in (18)

$$\lim_{k\to\infty}d\left(Tx_{m(k)-1},Tx_{n(k)-1}\right)=\varepsilon.$$

Now, consider the (10) with (5)

$$\begin{aligned}\psi(\varepsilon) &\leq \psi\left(d\left(Tx_{m(k)},Tx_{n(k)}\right)\right)\\ &\leq \alpha\left(d\left(Tx_{m(k)-1},Tx_{n(k)-1}\right)\right)-\beta\left(d\left(Tx_{m(k)-1},Tx_{n(k)-1}\right)\right)\end{aligned} \tag{19}$$

and letting $k\to\infty$ in (19), we have $\psi(\varepsilon)\leq\alpha(\varepsilon)-\beta(\varepsilon)$. From (7), we get $\varepsilon=0$. But this case is in conflict with $\varepsilon>0$. Thus, $\{Tx_n\}$ is a Cauchy sequence in complete metric space X. Hence, there is $v\in X$ such that

$$\lim_{n\to\infty}Tx_n=v. \tag{20}$$

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Note that T is a subsequentially convergent, $\{x_n\}$ has a convergent subsequence. Thus, there is $u \in X$ and a subsequence $\{x_{n(k)}\}$ such that

$$\lim_{k\to\infty} x_{n(k)} = u. \tag{21}$$

Also, T is continous and $x_{n(k)} \to u$, therefore

$$\lim_{n\to\infty} Tx_{n(k)} = Tu. \tag{22}$$

Note that $\{Tx_{n(k)}\}$ is a subsequence of $\{Tx_n\}$, so $Tu = v$. Now, we show that $u \in X$ is fixed of f.

$$\begin{aligned}\psi\left(d\left(Tx_{n(k)+1}, Tfu\right)\right) &= \psi\left(d\left(Tfx_{n(k)}, Tfu\right)\right)\\ &\le \alpha\left(d\left(Tx_{n(k)}, Tu\right)\right) - \beta\left(d\left(Tx_{n(k)}, Tu\right)\right)\end{aligned} \tag{23}$$

letting $k \to \infty$ in (23)

$$\psi\left(d\left(Tu, Tfu\right)\right) \le 0. \tag{24}$$

This implies that $Tu = Tfu$. Since T is one -to-one so $fu = u$. This shows $u \in X$ is a fixed of f.

To prove the uniqueness of the fixed point, if possible, let u and u' be two fixed points of f. We have $fu' = u'$ and

$$\begin{aligned}\psi\left(d\left(Tu, Tu'\right)\right) &= \psi\left(d\left(Tfu, Tfu'\right)\right)\\ &\le \alpha\left(d\left(Tu, Tu'\right)\right) - \beta\left(d\left(Tu, Tu'\right)\right).\end{aligned} \tag{25}$$

Inequality (25) is in conflict with the (7) unless $d(Tu, Tu') = 0$. This implies that $Tu = Tu'$. Since T is one -to-one, we get $u = u'$. Thus, the fixed point is unique.

Remark. In Theorem 4, if T is sequentially convergent ($T \in SC(X)$), by replacing $\{n\}$ with $\{n(k)\}$ we obtain that

$$\lim_{n\to\infty} x_n = u.$$

This implies that $\{x_n\}$ converges to the fixed point of f.

In Theorem 4, if we take $\beta(t) = 0$ and $\alpha(t) = kF(t) = k\psi(t)$ then we obtain fixed point of T_F- type contractive mappings given by Moradi and Beiranvand in [4].

Corollary 1. *Let (X, d) be a complete metric space and $f : X \to X$ be a mapping. If for $k \in [0, 1)$ and for all $x, y \in X$,*

$$F\left(d\left(Tfx, Tfy\right)\right) \le kF\left(d\left(Tx, Ty\right)\right)$$

where

1) $F : [0,\infty) \to [0,\infty)$, F is nondecreasing continuous from the right and $F^{-1}(0) = \{0\}$.

2) T is one to one and graph closed (or subsequentially convergent and continuous).

Then, f has a unique fixed point. Also, if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to the fixed point.

If we take $T = x$ we obtain the following result given by Eslamian and Abkar in [8],[9].

Corollary 2. *Let (X, d) be a complete metric space and $f : X \to X$ be mapping.*

$$\psi\left(d\left(fx, fy\right)\right) \le \alpha\left(d\left(x, y\right)\right) - \beta\left(d\left(x, y\right)\right) \tag{26}$$

such that .

$$\psi(t_1) \le \alpha(t_2), \text{ implies } t_1 \le t_2 \tag{27}$$

and for all $t > 0$

$$\psi(t) - \alpha(t) + \beta(t) > 0.$$

Then, f has a unique fixed point.

If we take $\alpha(t) = \psi(t)$ and $Tx = x$, then we obtain the following result given by Dutta and Choudhury in [6].

Corollary 3. *Let (X, d) be a complete metric space and let $f : X \to X$ be a self-mapping satisfying the inequality*

$$\psi\left(d\left(fx, fy\right)\right) \le \psi\left(d\left(x, y\right)\right) - \phi\left(d\left(x, y\right)\right) \tag{28}$$

where $\psi, \phi : [0,\infty) \to [0,\infty)$ are both continuous and monotone nondecreasing functions with $\psi(t) = 0 = \phi(t)$ if and only if $t = 0$.

Then f has a unique fixed point.

If we take $\psi(t) = t$, $\beta(t) = 0$, $\alpha(t) = \varphi(t)t$ and $Tx = x$, then we obtain the following result given by Geraghty in [10].

Corollary 4. *Let (X, d) be a complete metric space and let $f : X \to X$ be a self-mapping satisfying the inequality*

$$d\left(fx, fy\right) \le \varphi\left(d\left(x, y\right)\right) d\left(x, y\right)$$

such that $\varphi : [0,\infty) \to [0, 1)$ and

$$\varphi(t_n) \to 1 \text{ implies } t_n \to 0.$$

Then f has a unique fixed point

If we take $\psi(s) = \int_0^s \varphi(t)\,dt$,$\alpha(t) = k\psi(t)$, $k \in (0, 1)$ and $\beta(t) = 0$,then, we obtain the following result that more general than the result of Branciari [11].

Corollary 5. *Let (X, d) be a complete metric spaces. Let $f : X \to X$ be a mapping and $T \in SSC$ such that for each $x, y \in X, k \in (0, 1)$*

$$\int_0^{d(Tfx, Tfy)} \varphi(t)\,dt \le k \int_0^{d(Tx, Ty)} \varphi(t)\,dt$$

where $\varphi : [0,\infty) \to [0,\infty)$ is a Lebesgue-integrable mapping which summeble (i.e., with finite integral) on each compact subset of $[0,\infty)$, nonnegative, and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)\,dt > 0$; then f has a unique fixed point $a \in X$ such that for each $x \in X$, $\lim_{n\to\infty} f^n x = a$.

The following example is neither satisfying Banach contraction principle nor Corollary 1 but satisfies *Theorem 4.*

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Example 1. Let $X=[0,1]$ endowed with Euclidean metric. Also, let $S:X\to X$ be given as $Sx=x-x^2$. Then S is a not contraction mapping. If we take $Tx=\frac{x}{2}$ *such that T is continuous, one to one subsequentially convergent then we obtain that*

$$\begin{aligned} d(TSx,TSy) &= \left|\frac{x-x^2}{2}-\frac{y-y^2}{2}\right| \\ &\le \left|\frac{x}{2}-\frac{y}{2}\right|+\left|\frac{x^2}{2}-\frac{y^2}{2}\right| \\ &\le d(Tx,Ty). \end{aligned} \tag{29}$$

Thus, (29) shows that S doesn't satisfies neither Banach Contraction Principle nor Corollary 1. Thus we can not guarantee the existence of the fixed point of S. Now, we take $\psi(t)=t=\alpha(t),\ \beta(t)=2t^2$. Assume that $x>y$, we have

$$\begin{aligned} d(TSx,TSy) &= \left|\frac{x-x^2}{2}-\frac{y-y^2}{2}\right| \\ &= \frac{1}{2}\left[(x-y)-\left(x^2-y^2\right)\right] \\ &\le \frac{1}{2}\left[(x-y)-(x-y)^2\right] \\ &= d(Tx,Ty)-2d(Tx,Ty)^2 \\ &= d(Tx,Ty)-\beta(d(Tx,Ty)). \end{aligned} \tag{30}$$

Thus, S satisfies the inequality *(5) with T and β. Hence, (according to the Theorem 4)* S has a unique fixed point in X. In fact, $p=0\in X$ is unique fixed point of S.

Now, we introduce the concept of weak-(ψ,α,β) contractive condition for Kannan fixed point theorem.

Theorem 5. *Let (X,d) be a complete metric space and $f:X\to X$ be a mapping. Let $T\in SSC(X)$ and f satisfying the inequality*

$$\psi(d(Tfx,Tfy))\le\alpha\left(\tfrac{1}{2}\left[d(Tx,Tfx)+d(Ty,Tfy)\right]\right)-\beta(d(Tx,Tfx),d(Ty,Tfy)) \tag{31}$$

such that for $\psi\in\Psi,\alpha\in\Phi,\beta\in\Gamma_2$, we have

$$\psi(t_1)\le\alpha(t_2),\ \textit{implies}\ t_1\le t_2 \tag{32}$$

and for all $t>0$

$$\psi(t)-\alpha(t)+\beta(t,t)>0. \tag{33}$$

Then, f has a unique fixed point.

Proof. *Let x_0 be an arbitrary point in X. We define the iterative sequence $\{x_n\}$ by $x_{n+1}=fx_n$ (equivelently, $x_n=f^nx_0$), $n=1,2,...$From (31), we have*

$$\begin{aligned} \psi d(Tx_n,Tx_{n+1}) &= \psi(d(Tfx_{n-1},Tfx_n)) \\ &\le \alpha\left(\frac{1}{2}\left[d(Tx_{n-1},Tfx_{n-1})+d(Tx_n,Tfx_n)\right]\right) \\ &\quad -\beta(d(Tx_{n-1},Tfx_{n-1}),d(Tx_n,Tfx_n)) \\ &\le \alpha\left(\frac{1}{2}\left[d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})\right]\right) \end{aligned} \tag{34}$$

From (32), we see that $\{d(Tx_n,Tx_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence, there is $r\in\mathbb{R}$ such that

$$\lim_{n\to\infty} d(Tx_n,Tx_{n+1})=r. \tag{35}$$

Letting $n\to\infty$ in (34), then we have $\psi(r)\le\alpha(r)-\beta(r,r)$. From (33), we have $r=0$.

Next, we prove that $\{Tx_n\}$ is a Cauchy sequence. If possible, let $\{Tx_n\}$ be not a Cahucy sequence. Then there exists $\varepsilon>0$ for which we can find subsequences $\{Tx_{m(k)}\}$ and $\{Tx_{n(k)}\}$ of $\{Tx_n\}$ with $n(k)>m(k)>k$ such that

$$d\left(Tx_{m(k)},Tx_{n(k)}\right)\ge\varepsilon. \tag{36}$$

Take advantage of (31), we have $\psi(\varepsilon)\le\alpha(0)-\beta(0,0)$. But this case is in conflict with $\varepsilon>0$. Thus, $\{Tx_n\}$ is a Cauchy sequence in complete metric space X. Hence, there is $u\in X$ such that $x_{n(k)}\to u$, as $k\to\infty$ and $\lim_{n\to\infty}Tx_{n(k)}=Tu$.

Also, we have

$$\begin{aligned} \psi\left(d\left(Tfu,Tfx_{n(k)-1}\right)\right) &\le \alpha\left(\frac{1}{2}\left[d(Tu,Tfu)+d\left(Tx_{n(k)-1},Tx_{n(k)}\right)\right]\right) \\ &\quad -\beta\left(d(Tfu,Tu),d\left(Tx_{n(k)-1},Tx_{n(k)}\right)\right). \end{aligned}$$

Letting $k\to\infty$ in the last inequality, we have $\psi(d(Tfu,Tu))\le\le\alpha\left(\frac{1}{2}d(Tu,Tfu)\right)$. From (32), we obtain $d(Tu,Tfu)=0$ and as T is one to we get $u=fu$. It is easy to see the uniqueness of the fixed point.

Example 2. Let $X=[0,1]$ endowed with $d(x,y)=|x-y|$. Let $fx=\frac{x}{3}$ and $Tx=x^2$. If we consider $x=0$ and $y=1$, then f doesn't satisfies the condition of Kannan fixed point theorem. On the other hand if we take $\psi(t)=t=\alpha(t)$ and $\beta(t,t)=0$,then for all $x,y\in X$, we have

$$d(Tfx,Tfy)\le\frac{1}{8}\left[d(Tx,Tfx))+d(Ty,Tfy)\right]$$

Thus, *(according to the Theorem 5)* f has a unique fixed point in X. Really, $p=0\in X$ is unique fixed point of f.

In Theorem 5, if we consider $\psi(t)=\alpha(t)$, then we obtain the following result given by Razani and Parvaneh [13].

Corollary 6. *(Weak $T-K$ Contraction Mapping Theorem) Let (X,d) be a complete metric space and $T,f:X\to X$ be mappings such that T is one to one and graph closed (or subsequentially convergent and continuous). Let f satisfying the inequality*

$$\psi(d(Tfx,Tfy))\le\psi\left(\tfrac{1}{2}\left[d(Tx,Tfx))+d(Ty,Tfy)\right]\right)-\beta(d(Tx,Tfx),d(Ty,Tfy))$$

where;

1) $\beta:[0,\infty)\times[0,\infty)\to[0,\infty)$ is a continuous function such that $\beta(x,y)=0$ if and only if $x=y=0$

2) $\psi:[0,\infty)\to[0,\infty)$ is continuous and strictly increasing and $\psi(0)=0$.

Then, f has a unique fixed point. Also, if T is sequentially convergent then for $x_0\in X$ the sequence of iterates $\{f^nx_0\}$ converges to this fixed point.

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Now, we introduce the concept of weak-(ψ,α,β) contractive condition for Chatterjea fixed point theorem.

Theorem 6. *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. Let $T \in SSC(X)$ and f satisfying the inequality*

$$\psi(d(Tfx,Tfy)) \le \alpha\left(\tfrac{1}{2}[d(Tx,Tfx)+d(Ty,Tfy)]\right) - \beta(d(Tx,Tfx),d(Ty,Tfy))$$

such that for $\psi \in \Psi, \alpha \in \Phi, \beta \in \Gamma_2$, we have

$$\psi(t_1) \le \alpha(t_2), \text{ implies } t_1 \le t_2 \tag{37}$$

and for all $t > 0$

$$\psi(t) - \alpha(t) + \beta(t,t) > 0. \tag{38}$$

Then, f has a unique fixed point.

Proof. *Let x_0 be an arbitrary point in X. We define the iterative sequence $\{x_n\}$ by $x_{n+1} = fx_n$ (equivelently, $x_n = f^n x_0$), $n = 1,2,\ldots$.*

$$\psi(d(Tx_n,Tx_{n+1})) = \psi(d(Tfx_{n-1},Tfx_n)) \le \alpha\left(\tfrac{1}{2}[d(Tx_{n-1},Tx_n)+d(Tx_n,Tx_{n+1})]\right). \tag{39}$$

From (37), we obtain that $\{d(Tx_n,Tx_{n+1})\}$ is a monotone decreasing sequence of non-negative real numbers. Hence there is $r \in \mathbb{R}$ such that as $n \to \infty$ $d(Tx_n,Tx_{n+1}) \to r$ and $\psi(r) \le \alpha(r) - \beta(r,0)$. *From (38), we obtain $r = 0$. As in the proof of the Theorem 4, we get $\{Tx_n\}$ is a Cauchy sequence. To complete the proof, the similar process in theorem 4* and *5 would be used.*

In Theorem 6, if we consider $\psi(t) = \alpha(t)$, then we obtain the following result given by Razani and Parvaneh [13].

Corollary 7. *(Weak $T-C$ Contraction Mapping Theorem) Let (X,d) be a complete metric space and $T, f : X \to X$ be mappings such that T is one to one and graph closed (or subsequentially convergent and continuous). Let f satisfying the inequality*

$$\psi(d(Tfx,Tfy)) \le \psi\left(\tfrac{1}{2}[d(Tx,Tfy)+d(Ty,Tfx)]\right) - \beta(d(Tx,Tfy),d(Ty,Tfx))$$

where;

1) $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ *is a continuous function such that $\beta(x,y) = 0$ if and only if $x = y = 0$*

2) $\psi : [0,\infty) \to [0,\infty)$ *is continuous and strictly increasing and $\psi(0) = 0$.*

Then, f has a unique fixed point. Also, if T is sequentially convergent then for $x_0 \in X$ the sequence of iterates $\{f^n x_0\}$ converges to this fixed point.

Remark. Also, the Theorem 6 is more general than the theorems given in [3] and [14].

3 Conclusion

We established some fixed point theorems which are more general than the results of Razani and Parvaneh [13], Eslamian and Abkar [8].

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