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Fixed Point Theorem of Gregus Type in d-Metric Space

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Abstract: In this article, a common fixed point theorem for a pair of self-mapping is establish in d-metric space using (CLR) property. Our establish theorem extend, generalize and improve similar type of results of the literature in the setting of d-metric space.

Keywords: d-metric space, (CLR) property, weakly compatible mappings, property (E.A).

1 Introduction

In 1922, Banach established a fixed point result in complete metric space for a contraction mapping, which is one of the most important result of functional analysis. As a part of study of denotational semitics and data flow network Mathews [1] generalized Banach contraction principle in partial metric space (pms).

Hitzler [2], initiated the idea of dislocated metric (d-metric) space and established fixed point theorem of Banach type in such a space. Results on fixed point for compatible and weakly compatible mappings introduced by Jungck in [3,4] are established in [5,6]. In [7], Aamri and EI-Moutawakil initiated the idea of property (E.A), while Sintunavarat and Kuman in [8] introduced the concept of (CLR) property. In the above mentioned concepts the later one is superior then the previous one.

Gregus [9] established a result on fixed point in Banach space. Several authors generalized such a theorem in different spaces (see [10,11,12]). Using the idea of weakly compatible, (CLR) property and property (E.A) there we have proved fixed point theorem of Gregous type in d-metric space. For the support of our constructed results an example is provided.

2 Preliminaries

Definition. [2]. Consider $d_1 : X_0 \times X_0 \to \mathbb{R}^+ \cup \{0\}$ be a function on a non-empty set X_0 satisfying

1) $d_1(x_1,y_1) = d_1(y_1,x_1) = 0$ implies $x_1 = y_1$;
2) $d_1(x_1,y_1) = d_1(y_1,x_1)$;
3) $d_1(x_1,y_1) \leq d_1(x_1,z_1) + d_1(z_1,y_1)$ for all $x_1,y_1,z_1 \in X_0$.

Then d_1 is a d-metric on X_0 and (X_0,d_1) is a d-metric space.

Example. Suppose $X_0 = \mathbb{R}^+$. A function $d_1 : X_0 \times X_0 \to \mathbb{R}^+ \cup \{0\}$ defined by

$$d_1(x_0,y_0) = x_0 + y_0 \quad \text{for all } x_0,y_0 \in X_0.$$

Definition. Suppose S_0 and T_0 be self-mappings on X_0 which is non-empty then

1. A point $x_0 \in X_0$ is called fixed point of T_0 if $T_0x_0 = x_0$.
2. A point $x_0 \in X_0$ is known as coincidence point of S_0 and T_0 if $S_0x_0 = T_0x_0$ and we said $u_0 = S_0x_0 = T_0x_0$ is a point of coincidence.
3. A point $x_0 \in X_0$ is known as fixed point of both S_0 and T_0 if $S_0x_0 = T_0x_0 = x_0$.

Definition. Mappings S_0 and T_0 of a d-metric space (X_0,d_0) are known to be compatible if

$$\lim_{n\to\infty} d_0(S_0T_0x_n, T_0S_0x_n) = 0$$

when there exists a sequence $\{x_n\}$ in X_0 such that

$$\lim_{n\to\infty} S_0x_n = \lim_{n\to\infty} T_0x_n = t_0$$

for some t_0 in X_0.

Definition. Mappings S_0 and T_0 on a d-metric space (X_0,d_0) are said to be weakly compatible if they commute at all of their coincidence points i.e if $S_0u_0 = T_0u_0$ for some $u_0 \in X_0$ then $S_0T_0u_0 = T_0S_0u_0$.

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Definition. [7]. Mappings S_0 and T_0 on a d-metric space (X_0,d_0) are said to satisfy property (E.A) if there exists a sequence $\{t_n\}$ in X_0 such that

$$\lim_{n\to\infty} S_0t_n = \lim_{n\to\infty} T_0t_n = t_0$$

for some t_0 in X_0.
Example. Consider $X_0 = [0,1]$ with d-metric given by

$$d_0(x_0,y_0) = x_0 + y_0 \quad \text{for all } x_0,y_0 \in X_0.$$

The self-mappings S_0 and T_0 on X_0 are defined by

$$S_0x_0 = \{\begin{matrix} 1-x_0 & \text{if } x_0 \in [0,\frac{1}{2}] \\ 0 & \text{if } x_0 \in (\frac{1}{2},1] \end{matrix}$$

and

$$T_0x_0 = \begin{cases} \frac{1}{2} & \text{if } x_0 \in [0,\frac{1}{2}] \\ \frac{3}{4} & \text{if } x_0 \in (\frac{1}{2},1] \end{cases}$$

for a sequence $t_n = \frac{1}{2} - \frac{1}{n}$ in X_0 with $n \geq 2$ hold property (E.A) as

$$\lim_{n\to\infty} S_0t_n = \lim_{n\to\infty} T_0t_n = \frac{1}{2} \in X_0.$$

Also S_0 and T_0 are weakly compatible as they commute at $\frac{1}{2}$ which is the only coincidence point of S_0 and T_0 but not compatible because

$$\lim_{n\to\infty} d_0(S_0T_0t_n, T_0S_0t_n) \neq 0.$$

Definition. [8]. Mappings S_0 and T_0 on a d-metric space (X_0,d_0) are said to satisfy (CLR) property if there exists a sequence $\{t_n\}$ in X_0 such that

$$\lim_{n\to\infty} S_0t_n = \lim_{n\to\infty} T_0t_n = T_0u_0$$

for some u_0 in X_0.
Example. Suppose $X_0 = \mathbb{R}^+ \cup 0$, with d-metric space on X_0 is given by

$$d_0(x_0,y_0) = x_0 + y_0 \quad \text{for all } \ x_0,y_0 \in X_0.$$

Mappings S_0 and T_0 are given by

$$S_0x_0 = \frac{x_0}{2} \quad \text{and} \quad T_0x_0 = 2x_0 \ \ \forall \ \ x_0 \in X_0.$$

Suppose a sequence $t_n = \frac{1}{n}$

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t_0.$$

Thus S_0 and T_0 hold (CLR) property.
Remark. It is clear from Jungck [3] definition that two self-mappings are said to be non-compatible if there exist at least one sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t \ \text{ for some } t \in X.$$

but $\lim_{n\to\infty} d(STx_n, TSx_n)$ either not equal to zero or does not exist. Therefore, two non-compatible mappings satisfy property (E.A).

3 Main Results

Theorem. Suppose S_0 and T_0 be weakly compatible mappings on d-metric space (X_0,d_0) satisfying

1.S_0 and T_0 satisfy (CLR) property;
2.$d_0^p(S_0x_0,S_0y_0) \leq \alpha d_0^p(T_0x_0,T_0y_0) +$
$\beta \max\left\{ d_0^p(S_0x_0,T_0x_0), d_0^p(S_0y_0,T_0y_0) \right\} +$
$\gamma \max\left\{ d_0^p(T_0x_0,T_0y_0), d_0^p(S_0x_0,T_0x_0), d_0^p(S_0y_0,T_0y_0) \right\};$

for all $x_0,y_0 \in X_0$, $\alpha,\beta,\gamma \geq 0$ for $2(\alpha+\beta+\gamma) < 1$ and $p \geq 1$. Then S_0 and T_0 have a fixed point which is unique and common to both of the mappings.
Proof. Since S_0 and T_0 hold (CLR) property therefore there exists a sequence $\{l_n\}$ in X_0 such that

$$\lim_{n\to\infty} S_0l_n = \lim_{n\to\infty} T_0l_n = T_0u_0 \tag{1}$$

for any u_0 in X_0.
To show that $S_0u_0 = T_0u_0$ for this suppose

$$d_0^p(S_0l_n,S_0u_0) \leq \alpha d_0^p(T_0l_n,T_0u_0) +$$
$$\beta \max\left\{ d_0^p(S_0l_n,l_n), d_0^p(S_0u_0,T_0u_0) \right\} +$$
$$\gamma \max\left\{ d_0^p(T_0l_n,T_0u_0), d_0^p(S_0l_n,T_0l_n), d_0^p(S_0u_0,T_0u_0) \right\}.$$

Taking limit $n \to \infty$ and using (1) we have

$$\begin{aligned} d_0^p(T_0u_0,S_0u_0) &\leq \alpha d_0^p(T_0u_0,T_0u_0) + \\ &\beta \max\left\{ d_0^p(T_0u_0,T_0u_0), d_0^p(S_0u_0,T_0u_0) \right\} + \\ \gamma \max&\left\{ d_0^p(T_0u_0,T_0u_0), d_0^p(T_0u_0,T_0u_0), d_0^p(S_0u_0,T_0u_0) \right\}. \end{aligned} \tag{2}$$

$$d_0^p(T_0u_0,T_0u_0) \leq d_0^p(T_0u_0,S_0u_0) + d_0^p(S_0u_0,T_0u_0).$$

Using symmetric property we have

$$d_0^p(T_0u_0,T_0u_0) \leq 2d_0^p(T_0u_0,S_0u_0). \tag{3}$$

Using (3) in (2) we have

$$d_0^p(T_0u_0,S_0u_0) \leq 2(\alpha+\beta+\gamma) d_0^p(T_0u_0,S_0u_0)$$

which create a contradiction because $2(\alpha+\beta+\gamma) < 1$. Therefore, the above inequality is hold only if $d_0^p(T_0u_0,S_0u_0) = 0$ using symmetric property $d_0^p(S_0u_0,T_0u_0) = 0$ we get $S_0u_0 = T_0u_0$.
Because S_0 and T_0 are mappings which are weakly compatible thus

$$S_0u_0 = T_0u_0 \ \Rightarrow \ S_0T_0u_0 = T_0S_0u_0.$$

Therefore

$$S_0S_0u_0 = T_0S_0u_0 = S_0T_0u_0. \tag{4}$$

Now to prove that S_0u_0 is the fixed point of S_0 and T_0 common to both of them. Assume

$$d_0^p(S_0u_0, S_0S_0u_0) \leq \alpha d_0^p(T_0u_0, T_0S_0u_0) + \beta \max\left\{d_0^p(S_0u_0, T_0u_0), d_0^p(S_0S_0u_0, T_0S_0u_0)\right\} + \gamma \max\left\{d_0^p(T_0u_0, T_0S_0u_0), d_0^p(S_0u_0, T_0u_0), d_0^p(S_0S_0u_0, T_0S_0u_0)\right\}.$$

Using (4) and the fact that $S_0u_0 = T_0u_0$ we have

$$d_0^p(S_0u_0, S_0S_0u_0) \leq \alpha d_0^p(S_0u_0, S_0S_0u_0) + \beta \max\left\{d_0^p(S_0u_0, S_0u_0), d_0^p(S_0S_0u_0, S_0S_0u_0)\right\} + \gamma \max\left\{d_0^p(S_0u_0, S_0S_0u_0), d_0^p(S_0u_0, S_0u_0), d_0^p(S_0S_0u_0, S_0S_0u_0)\right\}. \tag{5}$$

Since

$$d_0^p(S_0u_0, S_0u_0) \leq d_0^p(S_0u_0, S_0S_0u_0) + d_0^p(S_0S_0u_0, S_0u_0).$$

By symmetric property we have

$$d_0^p(S_0u_0, S_0u_0) \leq 2d_0^p(S_0u_0, S_0S_0u_0). \tag{6}$$

Similarly we can show that

$$d_0^p(S_0S_0u_0, S_0S_0u_0) \leq 2d_0^p(S_0u_0, S_0S_0u_0). \tag{7}$$

Using (6) and (7) in (5) we have

$$d_0^p(S_0u_0, S_0S_0u_0) \leq (\alpha + 2(\beta + \gamma))d_0^p(S_0u_0, S_0S_0u_0)$$

which is a contradiction therefore $d_0^p(S_0u_0, S_0S_0u_0) = 0$ also by symmetric property $d_0^p(S_0S_0u_0, S_0u_0) = 0$ implies $S_0S_0u_0 = S_0u_0$. Also by (4) $T_0S_0u_0 = S_0u_0$. Thus S_0u_0 is the fixed point of S_0 and T_0 which is common to both of them.

Uniqueness. Suppose $u_0 \neq v_0$ be differen fixed points of S_0 and T_0 and common to both of theses mappings. Using (2) we get

$$d_0^p(u_0, v_0) = d_0^p(S_0u_0, S_0v_0) \leq \alpha d_0^p(T_0u_0, T_0v_0) + \beta \max\left\{d_0^p(S_0u_0, T_0u_0), d_0^p(S_0v_0, T_0v_0)\right\} + \gamma \max\left\{d_0^p(T_0u_0, T_0v_0), d_0^p(S_0u_0, T_0u_0), d_0^p(S_0v_0, T_0v_0)\right\}$$

$$\leq \alpha d_0^p(u_0, v_0) + \beta \max\left\{d_0^p(u_0, u_0), d_0^p(v_0, v_0)\right\} + \gamma \max\left\{d_0^p(u_0, v_0), d_0^p(u_0, u_0), d_0^p(v_0, v_0)\right\}.$$

Again since

$$d_0^p(u_0, u_0) \leq 2d_0^p(u_0, v_0) \text{ and } d_0^p(v_0, v_0) \leq 2d_0^p(u_0, v_0).$$

Hence the above inequality takes the form

$$d_0^p(u_0, v_0) \leq (\alpha + 2(\beta + \gamma))d_0^p(u_0, v_0)$$

which create again a contradiction which implies $d_0^p(u_0, v_0) = 0$ and using symmetric property we get $d_0^p(v_0, u_0) = 0$ implies $u_0 = v_0$. Thus fixed point of S_0 and T_0 is unique.

The following corollaries are deduced from the above theorem.

Corollary. Suppose S_0 and T_0 be mappings which are weakly compatible on d-metric space (X_0, d_0) satisfying

1. S_0 and T_0 hold (CLR) property;
2. $d_0(S_0x_0, S_0y_0) \leq \alpha d_0(T_0x_0, T_0y_0) + \beta \max\left\{d_0(S_0x_0, T_0x_0), d_0(S_0y_0, T_0y_0)\right\} + \gamma \max\left\{d_0(T_0x_0, T_0y_0), d_0(S_0x_0, T_0x_0), d_0(S_0y_0, T_0y_0)\right\}$;

for all $x_0, y_0 \in X_0$, $\alpha, \beta, \gamma \geq 0$ for $2(\alpha + \beta + \gamma) < 1$. Then S_0 and T_0 have fixed point which is unique and common to both of the mappings.

Corollary. Consider S_0 and T_0 be mappings which are weakly compatible on d-metric space (X_0, d_0) satisfying

1. S_0 and T_0 hold (CLR) property;
2. $d_0^p(S_0x_0, S_0y_0) \leq \alpha d_0^p(T_0x_0, T_0y_0) + \beta \max\left\{d_0^p(S_0x_0, T_0x_0), d_0^p(S_0y_0, T_0y_0)\right\}$;

for all $x_0, y_0 \in X_0$, $\alpha, \beta \geq 0$ for $2(\alpha + \beta) < 1$ and $p \geq 1$. Then S_0 and T_0 have fixed point which is unique and common to both of the mappings.

Corollary. Suppose S_0 and T_0 be mappings on d-metric space (X_0, d_0) which are weakly compatible satisfying

1. S_0 and T_0 hold (CLR) property;
2. $d_0^p(S_0x_0, S_0y_0) \leq \alpha d_0^p(T_0x_0, T_0y_0)$;

for all $x_0, y_0 \in X_0$, $\alpha \geq 0$ for $2\alpha < 1$ and $p \geq 1$. Then S_0 and T_0 have fixed point which is unique and common to both of the mappings.

Example. Consider $X_0 = [0, 1]$ with d-metric on X_0 is given by

$$d_0(x_0, y_0) = x_0 + y_0 \text{ for all } x_0, y_0 \in X_0.$$

The self-mappings S_0 and T_0 are defined by

$$S_0x_0 = \frac{x_0}{4} \text{ and } T_0x_0 = x_0 \text{ for all } x_0 \in X_0.$$

Clearly S_0 and T_0 satisfy (CLR) property by selecting $\{l_n\} = \frac{1}{n}$.

$$\lim_{n\to\infty} S_0 l_n = \lim_{n\to\infty} T_0 l_n = t_0.$$

Also S_0 and T_0 are weakly compatible because

$$S_0 0 = T_0 0 \ \Rightarrow \ S_0 T_0 0 = T_0 S_0 0.$$

$$d_0^p(S_0x_0, S_0y_0) = \frac{x_0}{4} + \frac{y_0}{4} \le \frac{1}{6}(x_0 + y_0) = \alpha d_0^p(x_0, y_0).$$

Thus all the conditions of last corollary are satisfied for $\frac{1}{6} \le \alpha < 1$ having 0 is the fixed point of S_0 and T_0 which is unique and common to both the mappings.

Now we prove a common fixed point theorem for a pair of weakly compatible mappings using property (E.A) with additional condition of closeness of the subspace.

Theorem. Consider S_0 and T_0 be mappings on d-metric space (X_0, d_0) which are weakly compatible satisfying

1.S_0 and T_0 hold property (E.A);
2.$d_0^p(S_0x_0, S_0y_0) \le \alpha d_0^p(T_0x_0, T_0y_0) + \beta \max\left\{ d_0^p(S_0x_0, T_0x_0), d_0^p(S_0y_0, T_0y_0) \right\} + \gamma \max\left\{ d_0^p(T_0x_0, T_0y_0), d_0^p(S_0x_0, T_0x_0), d_0^p(S_0y_0, T_0y_0) \right\}$;

for all $x_0, y_0 \in X_0$, $\alpha, \beta, \gamma \ge 0$ for $2(\alpha + \beta + \gamma) < 1$ and $p \ge 1$. If $T_0(X_0)$ is a closed subspace of X_0. Then S_0 and T_0 have fixed point which is unique and common to both the mappings.

Proof. Because S_0 and T_0 hold property (E.A), so there must exists a sequence $\{l_n\}$ in X_0 such that

$$\lim_{n\to\infty} S_0 l_n = \lim_{n\to\infty} T_0 l_n = t_0 \ \text{ for some } \ t_0 \in X_0.$$

$T_0(X_0)$ is a closed subspace of X_0, thus there must exists $u_0 \in X_0$ such that $T_0u_0 = t_0$. Thus S_0 and T_0 hold (CLR) property and so by previous S_0 and T_0 have fixed point which is unique and common to both the mappings.

The following corollaries are deduced from the above theorem.

Corollary. Suppose S_0 and T_0 be mappings on d-metric space (X_0, d_0) which are weakly compatible satisfying

1.S_0 and T_0 hold property (E.A);
2.$d_0(S_0x_0, Sy) \le \alpha d_0(T_0x_0, T_0y_0) + \beta \max\left\{ d_0(S_0x_0, T_0x_0), d_0(S_0y_0, T_0y_0) \right\} + \gamma \max\left\{ d_0(T_0x_0, T_0y_0), d_0(S_0x_0, T_0x_0), d_0(S_0y_0, T_0y_0) \right\}$;
3.$S_0(X_0) \subset T_0(X_0)$;

for all $x_0, y_0 \in X_0$, $\alpha, \beta, \gamma \ge 0$ for $2(\alpha + \beta + \gamma) < 1$. If $T_0(X_0)$ is a closed subspace of X_0. Then S_0 and T_0 have a fixed point which is common to both mappings and unique.

Corollary. Suppose S_0 and T_0 be mappings on d-metric space (X_0, d_0) which are weakly compatible satisfying

1.S_0 and T_0 hold property (E.A);
2.$d_0(S_0x_0, S_0y_0) \le \alpha d_0(T_0x_0, T_0y_0) + \beta \max\left\{ d_0(S_0x_0, T_0x_0), d_0(S_0y_0, T_0y_0) \right\}$;
3.$S_0(X_0) \subset T_0(X_0)$;

for all $x_0, y_0 \in X_0$, $\alpha, \beta \ge 0$ for $2(\alpha + \beta) < 1$. If $T_0(X_0)$ is a closed subspace of X_0. Then S_0 and T_0 have fixed point which is unique and common to both the mappings.

Corollary. Consider S_0 and T_0 be mappings on d-metric space (X_0, d_0) which are weakly compatible satisfying

1.S_0 and T_0 hold property (E.A);
2.$d_0(S_0x_0, S_0y_0) \le \alpha d_0(T_0x_0, T_0y_0)$;
3.$S_0(X_0) \subset T_0(X_0)$;

for all $x_0, y_0 \in X_0$, $\alpha \ge 0$ for $2\alpha < 1$. If $T_0(X_0)$ is a closed subspace of X_0. Then S_0 and T_0 have fixed point which is unique and common to both the mappings.

4 Conclusion

Our constructed theorems extend, generalize and improve the results established by Gregus [9], Fisher and Sessa [10], Jungck [11] and Diwan and Gupta [12] in the frame work of d-metric space. In case of (CLR) property completeness (closeness) of the space or subspace is not necessary. Moreover, in case of using (CLR) property containment of ranges of the involved mappings is not necessarily required.

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