Journal of Ecology of Health & Environment
An International Journal

http://dx.doi.org/10.18576/jehe/040203

Analytical Solution of One Dimension Time Dependent Advection- Diffusion Equation

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Received: 5 May 2015, Revised: 15 Feb. 2016, Accepted: 18 Feb. 2016.
Published online: 1 May 2016.

Abstract: An analytical model for the crosswind integrated concentrations released from a source in an inversion layer is formulated by considering the wind speed as a linear profile of vertical height and eddy diffusivity as a power law profile of vertical height, the separation of variables technique is used to solve the advection–diffusion equation. The analytical model is compared with data collected from nine experiments conducted at Inshas, Cairo (Egypt). The model shows a good agreement between observed and calculated concentration.

Keywords: Advection–Diffusion Equation, Bessel Function, Separation of Variables Technique, Eigen Value Problem.

1 Introduction

As a result of the huge development and the big progress in industry there are more requests on the energy sources that reduces air pollution, Furthermore Air pollutants released from various sources affect directly or indirectly man and his environment. In nature, transport of pollutants occurs in through the combination of advection and diffusion. The concentration of a contaminant released into the air may therefore be described by the advection diffusion equation (ADE) which is a second order differential equation of parabolic type [1].

Essa et al. solved the ADE in two dimensional spaces (x-z) depending on time using Laplace transform to find cross wind integrated normalized concentration [1]. An analytical solution of two dimensional ADE for a semi-infinite medium (half plane) with one-dimensional flow using a double integral expression is studied by [2].

In this study we derived the solution of the ADE in one dimension which depends on time using the separation of variables technique to evaluate the cross wind integrated of pollutants per emission rate, taking the eddy diffusivity kz is expressed as a function of power law in the vertical height "z". Comparison between estimating model and the observed from nine experiments at Inshas site, Cairo-Egypt [3].

2 Mathematical Description

The dispersion of pollutants in the atmosphere is governed by the basic atmospheric diffusion equation. Under the assumption of incompressible flow, atmospheric diffusion equation based on the Gradient transport theory can be written in the rectangular coordinate system as:

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}+v\frac{\partial C}{\partial y}+w\frac{\partial C}{\partial z}=\frac{\partial}{\partial x}\left(K_x\frac{\partial C}{\partial x}\right)+\frac{\partial}{\partial y}\left(K_y\frac{\partial C}{\partial y}\right)+\frac{\partial}{\partial z}\left(K_z\frac{\partial C}{\partial z}\right)+S+R \qquad (1)$$

where C(x, y, z) is the mean concentration of a pollutant (Bq/m^3), ($\mu g/m^3$) and (ppm); in which t is the time, S and R are the source and removal terms, respectively; (u, v, w) and (k_x, k_y, k_z) are the components of wind and diffusivity vectors in x , y and w directions , respectively , in an Eulerian frame of reference.

The following assumptions are made in order to simplify equation (1):

1) We are going to study Eq. (1) in case, when the components of wind (u, v, w) tends to zero
2) Source and removal (physical / chemical) pollutants are ignored so that S=0 and R=0.
3) Under the moderate to strong winds, the transport due advection dominates over that due to longitudinal diffusion: $\left|u\frac{\partial C}{\partial x}\right| >> \left|\frac{\partial}{\partial x}\left(k_x\frac{\partial C}{\partial x}\right)\right|$.
4) Taking the cross wind integration of Eq. (1) in which $\tilde{C}$ is the cross wind concentration.

With the above assumptions, equation (1) reduces to:

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$$\frac{\partial \tilde{C}(t,z,h)}{\partial t} = \frac{\partial}{\partial z}\left(K_z \frac{\partial \tilde{C}(t,z,h)}{\partial z} \right) \tag{2}$$

Under the following boundary conditions:

$$\tilde{C}(t,z,h) = \delta(z - z_s) \quad at \quad t = 0 \tag{3}$$

Where δ (…) is Dirac's delta function, and h is the mixing height.

- The eddy diffusivity k_z is expressed as a functions of power law of z as:

$$k_z = \alpha z^m ; \qquad m < 1; \qquad \alpha \neq 0 \tag{4}$$

Where α, m are turbulence parameters and depend on atmospheric stability.

- $\tilde{C}(t,z,h)\, is\ finite \qquad \forall \qquad z \to \infty$

Eq. (2) can solve analytically by putting the solution $\tilde{C}$ in the form:

$$\tilde{C}(t,z,h) = S(t).R(z,h) \tag{5}$$

We are going to prove that the above solution exists in view of the condition that, there exist orthogonal eigen-functions which can be made a bases.

Now let introducing Eq. (5) in Eq. (2) to lead:

$$R(z,h)\frac{\partial S(t)}{\partial t} - \alpha S(t)\frac{\partial}{\partial z}\left(z^m \frac{\partial R(z,h)}{\partial z} \right) = 0 \tag{6}$$

the above equation is simplified to give:

$$\frac{1}{S(t)}\frac{\partial S(t)}{\partial t} = \frac{\alpha}{R(z,h)}\frac{\partial}{\partial z}\left(z^m \frac{\partial R(z,h)}{\partial z} \right) \tag{7}$$

Each side of Eq. (7) is thus a constant which assumed to be λ, hence:

$$\frac{1}{S(t)}\frac{\partial S(t)}{\partial t} = \lambda \tag{8}$$

And.

$$\frac{\alpha}{R(z,h)}\frac{\partial}{\partial z}\left(z^m \frac{\partial R(z,h)}{\partial z} \right) = \lambda \tag{9}$$

Eq. (8) is first order differential equation in t, and has a general solution, namely:

$$S(t) = ce^{\lambda t} \tag{10}$$

In which c is constant.

In order to solve the second order differential equation Eq. (9), one can study the different cases of λ, which has three cases:

Case I: $\lambda > 0$

Then we can write $\lambda = p^2$ for some positive real *p*. Then Eq. (9) can be written as:

$$z^2 \frac{d^2 R(z,h)}{dz^2} + mz\frac{dR(z,h)}{dz} - \frac{p^2}{\alpha} z^{2-m} R(z,h) = 0 \qquad : \qquad \alpha \neq 0 \tag{11}$$

Which reduces to modified Bessel's equation; namely [4]:

$$z_*^2 \frac{d^2 R_*}{dz_*^2} + z_* \frac{dR_*}{dz_*} - \left[\left(\frac{2pz_*}{(2-m)\sqrt{\alpha}} \right)^2 + n^2 \right] R_* = 0 \tag{12}$$

On changing the independent and dependent variables by means of the substitutions:

$$\left.\begin{aligned} z_* &= z^{\frac{2-m}{2}} \\ and & \\ R(z,h) &= z_*^n R_*(z,h) \end{aligned}\right\} \tag{13}$$

Eq. (12) has a solution which is a linear combination of modified Bessel's functions of first kind $I_n\left(\frac{2pz_*}{(2-m)\sqrt{\alpha}}\right)$ and second kind $K_n\left(\frac{2pz_*}{(2-m)\sqrt{\alpha}}\right)$ of order n where [14]:

$$n = \frac{1-m}{2-m} \qquad ; \qquad n > 0\ or\ m < 1 \frac{n!}{r!(n-r)!} \tag{15}$$

Then the general solution $R(z,h)$ of Eq. (11) for $\lambda > 0$ can be written as:

$$R(z,h) = z^{\frac{1-m}{2}}\left[A(h) I_n\left(\frac{2pz_*}{(2-m)\sqrt{\alpha}}\right) + B(h) K_n\left(\frac{2pz_*}{(2-m)\sqrt{\alpha}}\right) \right] \tag{16}$$

Consequently, and by virtue of Eq. (10) the solution of Eq. (5), can be expressed as:

$$\tilde{C}(t,z,h) = e^{p^2 t} z^{\frac{1-m}{2}}\left[A(h) I_n\left(\frac{2pz_*}{(2-m)\sqrt{\alpha}}\right) + B(h) K_n\left(\frac{2pz_*}{(2-m)\sqrt{\alpha}}\right) \right] \tag{17}$$

At this point, we are going to find the constants $A(h)$ and $B(h)$. By virtue of properties of modified Bessel's functions $I_n(z)$ and $K_n(z)$, in which I_n tends to ∞ as $z \to \infty$, and K_n tends to 0 as $z \to \infty$ and hence $\tilde{C}$ is finite at $z \to \infty$, then $A(h) = 0$, and Eq. (17) becomes:

$$\tilde{C}(t,z,h) = e^{p^2 t} z^{\frac{1-m}{2}} B(h) K_n\left(\frac{2pz_*}{(2-m)\sqrt{\alpha}}\right) \tag{18}$$

From properties of $K_n(p_j z)$, namely:

$$\int_0^\infty zK_n(p_j z)K_n(p_i z)dz = \frac{\pi(p_j p_i)^{-n}[p_j^{2n} - p_i^{2n}]}{2\sin n\pi[p_j^2 - p_i^2]} \neq 0 \qquad i \neq j \tag{19}$$

Which make the functions $K_n(p_j z)$ not orthogonal, and since the functions $e^{p_j^2 t}$ not orthogonal, then the solution $\tilde{C}$ cannot expand as a linear combination of orthogonal eigen functions when $\lambda > 0$, which imply that λ cannot greater than zero.

Case II: $\lambda = 0$

In the case of $\lambda = 0$ and by virtue of Eq. (10), *S(t)* is constant. Consequently from Eq. (5) $\tilde{C}$ is independent on time t, and therefore it cannot satisfies the boundary conditions as given by Eq. (3).

On the other hand from Eq. (9), at the case of $\lambda = 0$ make the solution is zero at $z = 0$, therefore $\lambda = 0$ and $z \neq 0$ we have no solution and $\lambda = 0$, $z = 0$ the solution is zero, so λ cannot be zero.

Case III: $\lambda < 0$

At this case λ written as:

$$\lambda = -p^2 \quad ; \quad p_j > 0 \tag{20}$$

The solution $S(t)$ becomes:

$$S(t) = ce^{-p^2 t} \tag{21}$$

and Eq. (9) can be expressed as:

$$z^2 \frac{d^2R(z,h)}{dz^2} + mz\frac{dR(z,h)}{dz} + \frac{p^2}{\alpha} z^{2-m} R(z,h) = 0 \quad ; \quad \alpha \neq 0 \tag{22}$$

Which after employing the transformation as given by Eq. (13), and Eq. (14), becomes:

$$z_*^2 \frac{d^2R_*(z,h)}{dz_*^2} + z_* \frac{dR_*(z,h)}{dz} + \left[\left(\frac{2pz_*}{(2-m)\sqrt{\alpha}}\right)^2 - n^2\right] R_*(z,h) = 0 \tag{23}$$

Eq. (23) is Bessel's equation, and its general solution is a linear combination of independent Bessel's functions of first kind J_n and J_{-n}, of order *n*, where *n* is given by Eq. (15).

There from the Solution $R(z,h)$ of Eq. (22) for $\lambda < 0$ can be written as:

$$R(z,h) = z^{\frac{1-m}{2}} \left[A(h)J_n\left(\frac{2pz_*}{(2-m)\sqrt{\alpha}}\right) + B(h)J_{-n}\left(\frac{2pz_*}{(2-m)\sqrt{\alpha}}\right)\right] \tag{24}$$

Consequently, and by virtue of Eq. (21) the solution of Eq. (5) for $\lambda < 0$, can be expressed as:

$$\tilde{C}(t,z,h) = e^{-p^2 t} z^{\frac{1-m}{2}} \left[A(h)J_n\left(\frac{2pz^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}}\right) + B(h)J_{-n}\left(\frac{2pz^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}}\right)\right] \tag{25}$$

Now, we are going to study the Newman boundary condition, namely:

$$K_z \frac{\partial \tilde{C}(t,z,h)}{\partial z} = 0 \qquad at \qquad z = 0, z = h \tag{26}$$

Upon introducing the above condition at $z = 0$, in Eq. (25) and by virtue of the two relations [4]

$$\left[\frac{\partial}{\partial x}\left(x^n J_n(x)\right) = x^n J_n(x) \quad and \quad \frac{\partial}{\partial x}\left(x^n J_{-n}(x)\right) = -x^n J_{-n+1}(x)\right]$$

One gets:

$$0 = K_z \frac{\partial \tilde{C}(t,z,h)}{\partial z} = e^{-p^2 t} \left[\begin{array}{l} A(h)\left(\dfrac{(2-m)^{1-n} p^n \alpha^{1-\frac{n}{2}}}{\Gamma(n)}\right) \\ +B(h)\left(\dfrac{p}{\sqrt{\alpha}}.\alpha.\displaystyle\sum_{r=0}^{\infty} \dfrac{(-1)^r \left(\dfrac{p}{(2-m)\sqrt{\alpha}}\right)^{-n+2r} z^{1+(\frac{2-m}{2})r}}{r!\Gamma(1-n+r+1)}\right) \end{array} \right] \tag{27}$$

The first bracket in the right hand side of the above equation not equal zero at $z = 0$ although the second term equal zero which implies that $A(h) = 0$, then Eq. (25) becomes:

$$\tilde{C}(t,z,h) = B(h)e^{-p^2 t} z^{\frac{1-m}{2}} J_{-n}\left(\frac{2pz^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}}\right) \tag{28}$$

Again upon introducing the boundary condition as given by Eq. (26) in the above equation at $z = h$, one get:

$$0 = K_z \left.\frac{\partial \tilde{C}(t,z,h)}{\partial z}\right|_{z=h} = K_z \frac{\partial}{\partial z}\left[B(h)e^{-p^2 t} z^{\frac{1-m}{2}} J_{-n}\left(\frac{2pz^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}}\right)\right]\Bigg|_{z=h}$$

$$= -p\sqrt{\alpha}B(h)e^{-p^2 t} z^{\frac{1}{2}} J_{1-n}\left(\frac{2pz^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}}\right)\Bigg|_{z=h}$$

Upon introducing Eq. (20) in which $p > 0$, and since $\alpha \neq 0$ from Eq. (4), one gets:

$$J_{-n+1}\left(\frac{2ph^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}}\right)=0 \qquad (29)$$

On putting in the above equation $p=p_j$ we have the Eigenvalue equation, namely:

$$J_{-n+1}(\Gamma_j)=0 \quad ; \qquad \Gamma_j=\frac{2p_jh^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}} \qquad ; \qquad j\neq 0,\ p_0=0 \qquad (30)$$

By using the relation, namely: {if σ is the rout of $J_\upsilon(\sigma l)=0$ then [5]:

$$\int_0^h xJ_\upsilon(\sigma_j l)J_\upsilon(\sigma_i l)dx=\begin{cases}0 & i\neq j\\ \frac{l^2}{2}J^2_{\upsilon+1}(\sigma_i l)=\frac{l^2}{2}J'^2_\upsilon(\sigma_i l)=\frac{l^2}{2}J^2_{\upsilon-1}(\sigma_i l)\end{cases} \qquad (31)$$

Then from Eq. (30) we have the Eigen function $J_{-n}\left(\frac{2p_iz^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}}\right)$ which satisfying orthogonally relation namely:

$$\int_0^h \frac{2z^{1-m}}{(2-m)\sqrt{\alpha}}J_{-n}\left(\frac{2p_iz^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}}\right)J_{-n}\left(\frac{2p_jz^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}}\right)dz=\begin{cases}0 & i\neq j\\ \frac{2h^{2-m}}{(2-m)^2\alpha}J^2_{-n+1}\left(\frac{2p_ih^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}}\right) \quad ; & i=j\end{cases} \qquad (32)$$

Therefore, we can expand $\tilde{C}(t,z,h)$ as a linear combination with expansion coefficient λ_i as:

$$\tilde{C}(t,z,h)=\lambda_0+\sum_{i=1}^{\infty}\lambda_i z^{\frac{1-m}{2}}J_{-n}\left(\frac{2p_iz^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}}\right) \qquad (33)$$

in which λ_0 corresponding to $p_0=0$ which is the trivial solution of Eq. (30) and can be determined by multiplying the above equation with $\frac{2-m}{2}\left(\frac{1}{\sqrt{\alpha}}\right)^{-n+2}$ and integrating from *0* to h after invoking the boundary condition at *t=0* as given by Eq. (3) together with Eq. (30), we get:

$$\lambda_0=\frac{1}{h} \qquad (34)$$

Which the summation in Eq. (33) vanish by virtue of the integration namely [5]:

$$\int_0^c x^{n+1}J_n(\alpha x)dx=\frac{1}{\alpha c}J_{n+1}(\alpha x) \qquad n>-1 \qquad (35)$$

Which is equivalent to :

$$\int_0^h \frac{2-m}{2}\left(\frac{1}{\sqrt{\alpha}}\right)^{-n+2}z^{\left(\frac{2-m}{2}\right)^n}J_{-n}\left(\frac{p_iz^{\frac{2-m}{2}}}{\sqrt{\alpha}}\right)dz=\frac{\sqrt{\alpha}}{p_ih^{\frac{2-m}{2}}}J_{-n+1}\left(\frac{p_ih^{\frac{2-m}{2}}}{\sqrt{\alpha}}\right) \quad ; \qquad -n>-1 \qquad (36)$$

To find λ_i $i\neq 0$, multiplying Eq. (33) by $\frac{2-m}{2\alpha}z^{\frac{1-m}{2}}J_{-n}\left(\frac{p_iz^{\frac{2-m}{2}}}{\sqrt{\alpha}}\right)$ and integrating from *0* to *h*, we get:

$$\lambda_i=\frac{(2-m)h^{\frac{m-3}{2}}}{J_{-n}\left(\frac{p_ih^{\frac{2-m}{2}}}{\sqrt{\alpha}}\right)} \qquad (37)$$

Introducing Eq. (34) and Eq. (37) in Eq. (33) to get:

$$\tilde{C}(t,z,h)=\frac{1}{h}+z^{\frac{1-m}{2}}\sum_{i=1}^{\infty}(2-m)h^{\frac{m-3}{2}}e^{-p^2t}\frac{J_{-n}\left(\frac{p_iz^{\frac{2-m}{2}}}{\sqrt{\alpha}}\right)}{J_{-n}\left(\frac{p_ih^{\frac{2-m}{2}}}{\sqrt{\alpha}}\right)} \qquad (38)$$

Let us now evaluate $\tilde{C}(t,z,h)$ at surface of ground.

$$\tilde{C}(t,z,h)\Big|_{z=0}=\frac{1}{h}+(2-m)h^{\frac{m-3}{2}}\sum_{i=1}^{\infty}\frac{e^{-p^2t}}{J_{-n}\left(\frac{p_ih^{\frac{2-m}{2}}}{\sqrt{\alpha}}\right)} \qquad (39)$$

At $m=1 \quad \Rightarrow \quad n=0$:

$$\tilde{C}(t,0,h)\Big|_{z=0} = \frac{1}{h}\left(1+\sum_{i=1}^{\infty}\frac{e^{-p^2 t}}{J_0(p_i\sqrt{\frac{h}{\alpha}})}\right) \quad (40)$$

In which $\frac{2p_j h^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}}$ is given as

$$J_{-n+1}\left(\frac{2ph^{\frac{2-m}{2}}}{(2-m)\sqrt{\alpha}}\right)=0$$

3 Source Data

The diffusion data for the estimating were gathered during ^{135}**I** isotope tracer nine experiments in moderate wind with unstable conditions at Inshas, Cairo. During each run, the tracer was released from source has height 43m for twenty four hours working, where the air samples were collected during half hour at a height 0.7m. We collected air samples from 92m to 184m around the source in AEA, Egypt. The study area is flat, dominated by sandy soil with poor vegetation cover. The air samples collected were analyzed in Radiation Protection Department, NRC, AEA, Cairo, Egypt using a high volume air sampler with 220V /50Hz bias [7]. Meteorological data have been provided by the measurements done at 10 and 60 m. Table 1 gives the data information about the diffusion tests and the wind vectors. In addition, it contains values of vertical velocity scale (w*) and mixing height (z_i). The data from these nine unstable test runs have been utilized for the following analysis.

Table 1. Meteorological data of the nine convective test runs at Inshas site in March and May 2006.

Run no.	Working hours	Release rate (Bq)	Wind speed (m s^{-1})	Wind direction (deg)	W* (ms^{-1})	Z_i (m)	P-G stability class
1	48	1028571	4	301.1	2.27	600.85	A
2	49	1050000	4	278.7	3.05	801.13	A
3	1.5	42857.14	6	190.2	1.61	973	B
4	22	471428.6	4	197.9	1.23	888	C
5	23	492857.1	4	181.5	0.958	921	A
6	24	514285.7	4	347.3	1.3	443	D
7	28	1007143	4	330.8	1.51	1271	C
8	48.7	1043571	4	187.6	1.64	1842	C
9	48.25	1033929	4	141.7	2.1	1642	A

4 Model parameters

For the concentration computations, we require the knowledge of wind speed, wind direction, source strength, the dispersion parameters, mixing height and the vertical scale velocity. Wind speeds are greater than 3m/s most of the time even at 10m level. Further the variation wind direction with time is also visible. Thus in the present study, we have adopted dispersion parameters for urban terrain which are based on power law functions. The analytical expressions depend upon downwind distance, vertical distance and atmospheric stability. The atmospheric stability has been calculated from Monin-Obukhov length scale (1/L) [6] based on friction velocity, temperature, and surface heat flux.

5 Results and Discussion

The concentration is computed using data collected at vertical distance of a 30m multi-level micrometeorological tower. In all a test runs were conducted for the purpose of computation. The concentration at a receptor can be computed in the following way:

Applying formula (38) which contains eddy diffusivities as function with power law at y = 0.0 for half hourly averaging.

Table 2. Observed and predicted concentrations for Run 9 experiments

Test	Downwind distance (m)	Vertical distance (m)	Observed conc. (Bq/m^3)	Predicted conc. (Bq/m^3)
1	100	5	0.025	0.051
2	98	10	0.037	0.031
3	115	5	0.091	0.070
4	135	5	0.197	0.160
5	99	2	0.272	0.234
6	184	11	0.188	0.138
7	165	12	0.447	0.339
8	134	7.5	0.123	0.107
9	96	5.0	0.032	0.034

As an illustration, results computed from these approaches are shown in Table 2, for nine typical tests conducted at Inshas site, Cairo-Egypt [6]. This Table shows that the observed and predicted concentrations for ^{135}**I** using Eq.

(38) with power law of eddy diffusivities and the wind speed in linear form of "z" are very near to each other of **^{135}I**.

Fig. 1 shows the variation of predicted and observed concentration of **^{135}I** with the downwind distance. One gets good agreement between observed and predicted concentration.

Fig. 2 shows that the predicted concentrations which are estimated from Eq. (38) are a factor of two with the observed concentration.

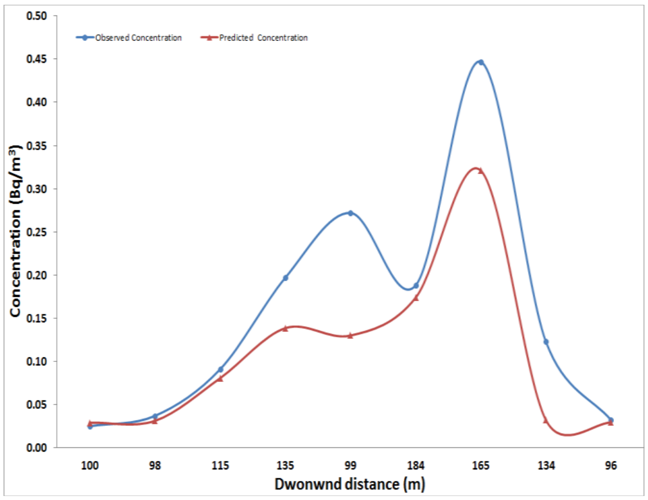

Fig. 1. Maximum computed concentrations compared with observed maximum value for each test run.

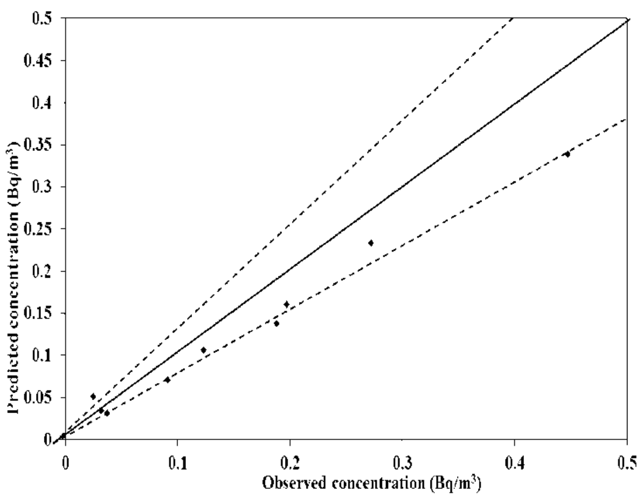

Fig. 2. Diagram of Predicted model for Eq. (38) with corresponding observation. Solid lines indicate one to one and dashed lines a factor of two.

6 Statistical methods

Now, the statistical method is presented and comparison among analytical, statically and observed results will be offered [15]. The following standard statistical performance measures that characterize the agreement between prediction (C_p=C_{pred}) and observations (C_o=C_{obs}):

1- Normalized mean square error (NMSE): It is an estimator of the overall deviations between predicted and observed concentrations. Smaller values of NMSE indicate a better model performance. It is defined as:

$$NMSE = \frac{\overline{(C_o - C_p)^2}}{\overline{C}_o \overline{C}_p}$$

2- Fractional bias (FB): It provides information on the tendency of the model to overestimate or underestimate the observed concentrations. The values of FB lie between -2 and +2 and it has a value of zero for an ideal model. It is expressed as:

$$\mathrm{FB} = \frac{(\overline{\mathrm{C}}_o - \overline{\mathrm{C}}_p)}{0.5(\overline{\mathrm{C}}_o + \overline{\mathrm{C}}_p)}$$

3- Correlation coefficient (R): It describes the degree of association between predicted and observed concentrations and is given by:

$$R = \frac{\overline{(C_o - \overline{C}_o)(C_p - \overline{C}_p)}}{\sigma_o \sigma_p}$$

4- Fraction within a factor of two (FAC2) is defined as:

FAC2= fraction of the data for which

$0.5 \leq (C_p/C_o) \leq 2$

Where σ_p and σ_o are the standard deviations of C_p and C_o respectively. Here the over bars indicate the average over all measurements (Nm). A perfect model would have the following idealized performance: NMSE = FB = 0 and COR = FAC2 = 1.0

Table (3): Comparison between averages predicted isotopes for ^{135}I and observed concentrations.

Statistical functions	^{135}I			
	NMSE	FB	COR	FAC2
Predicated Concentrations model	0.32	0.38	0.94	0.76

From the statistical method of Table (3), we find that the predicted concentrations for **^{135}I** lie inside factor of 2 with observed data. Regarding to NMSE, FB and COR the predicted concentrations for **^{135}I** are better with observed data.

7 Conclusions

In this paper, we have formulated a mathematical model for dispersion of air pollutants in moderated winds. The diffusion in vertical height direction and advection along the mean wind are taking into account. The eddy diffusivity is assumed to be power law in the vertical height "z".

The analytical model is compared with data collected from nine experiments conducted at Inshas, Cairo (Egypt). One gets the predicted concentrations are in a best agreement with the corresponding observation.

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