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Some Properties of Homology Groups of Khalimsky Spaces

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Abstract: In this paper we introduce the digital singular homology groups of the digital spaces topologized by the Khalimsky topology by constructing the digital standard n-simplexes. Then we'll compute the digital singular homology groups of some basic digital spaces up to the dimension 2 and investigate that the digital singular homology theory for the digital spaces is a functor from the category KDTC of KD-topological category to the category Ab of abelian groups.

Keywords: Khalimsky topology, digital topology, singular homology

1 Introduction

The theory of homology was given by Poincaré [16] and many different homology theories (e.g. simplicial homology, singular homology, Čech homology, etc.) were developped by many mathematicians (e.g. Alexander, Čech, Eilenberg, Vietoris, etc.). Eilenberg and Steenrod [8] formally defined the features of homology theory by giving a set of certain axioms which a homology theory should satisfy. Simplicial homology was defined for the simplicial complexes and the homology groups depend only on the geometric realization of the simplicial complex.

Arslan et al. [2] introduce the digital simplicial homology groups of n-dimensional digital images. In the work of [6] the concept of the simplicial homology groups of digital images and the earlier definiton of Euler characteristics of digital images have been expanded and some certain minimal simple closed surfaces have been studied to compute their simplicial homology groups. In addition to those works, Karaca and Ege [7] investigate the Eilenberg-Steenrod axioms for the simplicial homology groups of digital images. They state the universal coefficient theorem for digital images and conclude that the Künneth formula for the simplicial homology doesn't hold and the Hurewicz theorem need not be hold in digital images.

For each dimension n, the singular homology counts the n-dimensional holes of a space. The resulting homology groups are the same for all homotopically equivalent spaces. The construction of the singular homology can be applied to all topological spaces and is preserved by the continuous functions. Thus, according to the category theory, homology group becomes a functor from the category of topological spaces to the category of graded abelian groups.

In this paper we define the digital standard n-simplexes and introduce the digital singular homology groups in digital spaces topologized by the Khalimsky topology. Then we'll compute the digital singular homology groups of some basic digital spaces up to the dimension 2 and investigate that the digital singular homology theory in digital spaces is a functor from the category KDTC of KD-topological category to the category Ab of abelian groups.

2 Preliminaries

Let $\mathbb{Z}$ be the set of integers and $X \subset \mathbb{Z}^m$ for some positive integer m. Let κ indicates some adjacency relation for the members of X. The generalization of the adjacency is as follows [5]. Let l, m be positive integers, $1 \leq l \leq m$ and two distinct points $p = (p_1, \ldots, p_m)$ and $q = (q_1, \ldots, q_m)$ in $\mathbb{Z}^m$, p and q are k_l-*adjacent* if there are at most l

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distinct coordinates j for which $|p_j - q_j| = 1$, and for all other coordinates j, $p_j = q_j$. That is, two points p and q in $\mathbb{Z}$ are 2-adjacent if $q = p \pm 1$. Two points p and q in $\mathbb{Z}^2$ are 8-adjacent if they are distinct and differ by at most 1 in each coordinate; they are 4-adjacent if they are 8-adjacent and differ in exactly one coordinate. Two points p and q in $\mathbb{Z}^3$ are 26-adjacent if they are distinct and differ by at most 1 in each coordinate; they are 18-adjacent if they are 26-adjacent and differ in at most two coordinates; they are 6-adjacent if they are 18-adjacent and differ in exactly one coordinate. We call the pair (X, κ) as a *digital image*.

Let κ be an adjacency relation defined on $\mathbb{Z}^m$ and p be a point in $\mathbb{Z}^m$. Then the point in $\mathbb{Z}^m$ which is κ-adjacent to p is called a κ-*neighbor* of p [11]. Let (X, κ) be a digital image in $\mathbb{Z}^m$. Then X is κ-*connected* [14] if and only if for every pair of different points p and q in X, there is a sequence $\{p_0, p_1, \ldots, p_s\}$ of points of X such that $p = p_0$, $q = p_s$ and p_i and p_{i+1} are κ-adjacent where $i \in \{0, 1, \ldots, s-1\}$ [11]. In this case, we call the sequence as κ-*path* between the points p and q in X. Let $\ell_\kappa(p,q)$ denote the length of a shortest κ-path between p and q. If there is no κ-path between the points p and q, take $\ell_\kappa(p,q) = \infty$. Then, let

$$N_\kappa(p,\varepsilon) := \{q \in X \ : \ \ell_\kappa(p,q) \leq \varepsilon\} \cup \{p\}$$

where $\varepsilon \in \mathbb{N}$ [9].

For each $m \in \mathbb{Z}$, define the sets

$$B(m) = \begin{cases} \{m\}, & \text{if } m \text{ is odd} \\ \{m-1, m, m+1\}, & \text{if } m \text{ is even.} \end{cases}$$

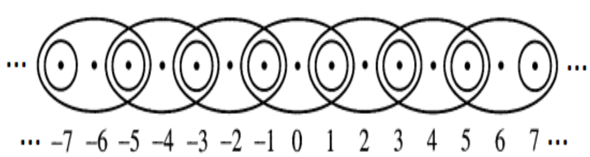


Fig. 1: The illustration of $B(m)$

Then the collection

$$\mathscr{B} = \{B(n) \ : \ n \in \mathbb{Z}\}$$

is a basis for a topology on $\mathbb{Z}$ and the topology generated by this basis is called Khalimsky digital line topology [12]. Note that the product topology on $\mathbb{Z}^m$ for $m > 1$ is the topology generated by the basis

$$\mathscr{B} = \{\prod_{i=1}^{m} B(n_i) \ : \text{each } B(n_i) \text{ is a basis in } \mathbb{Z}\}.$$

Let (X, κ) be a digital image in $\mathbb{Z}^m$. Then X has the subspace topology inherited from $\mathbb{Z}^m$ where the basis of the subspace topology is

$$\mathscr{B} = \{X \cap \prod_{i=1}^{m} B(n_i) \ : \ \text{each } B(n_i) \text{ is a basis in } \mathbb{Z}\}.$$

We will denote such spaces by $(X_{m,\kappa}, \tau_X)$.

If $(X_{m_1,\kappa_1}, \tau_X)$ is a space and x is a point in $(X_{m,\kappa}, \tau_X)$, then a neighbourhood of x is a subset O_x of X that includes an open set U containing x. Let $(X_{m_1,\kappa_1}, \tau_X)$ and $(Y_{m_2,\kappa_2}, \tau_Y)$ be spaces and let

$$f : (X_{m_1,\kappa_1}, \tau_X) \to (Y_{m_2,\kappa_2}, \tau_Y)$$

be a function. Then we say that f is *continuous at* x [10], if for all open subsets $O_{f(x)}$ of Y containing $f(x)$, the preimage of the open set $O_{f(x)}$ is an open subset of X containing x.

Definition 2.1. [10] Let $f : (X_{m_1,\kappa_1}, \tau_X) \to (Y_{m_2,\kappa_2}, \tau_Y)$ be a function. If

1. f is continuous at x and
2. for any $N_{\kappa_2}(f(x), \varepsilon) \subset Y$, there is $N_{\kappa_1}(x, \delta) \subset X$ such that $f(N_{\kappa_1}(x,\delta)) \subset N_{\kappa_2}(f(x), \varepsilon)$, where $\varepsilon, \delta \in \mathbb{N}$,

then we say that f is *KD-*(κ_1, κ_2) *continuous function at* $x \in X$. Moreover if f is KD-(κ_1, κ_2) continuous at any point in X, the we call f as a *KD-*(κ_1, κ_2) *continuous*.

A KD-(κ_1, κ_2)-continuous bijective function is *KD-*(κ_1, κ_2)*-isomorphism* [10], if the inverse of f is KD-(κ_2, κ_1)-continuous.

Let S be a set of nonempty subsets of a digital image (X, κ). We call the members s of S as the *simplices* of (X, κ) [18] if the following two statements hold:

1. if p and q are two distinct points of S, then they are κ-adjacent,
2. if $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$.

If the number of elements of S is $n+1$, then S is called an *n-simplex*.

Let (K, κ) be a finite collection of digital n-simplices ranging over $0 \leq n \leq d$ for some integer d. Then (K, κ) is called *a finite digital simplicial complex* [2] if

i) S belongs to K, then every face of S also belongs to K,
ii) S and P in K, then $S \cap P$ is either empty or a common face of S and P.

The dimension of K is the biggest integer n such that K has an n-simplex.

3 Digital Singular Homology Groups

For $n \geq 0$ let $e_0 = (0,...,0)$ and for $1 < i \leq n$, let

$$e_i = (i_1, i_2, ..., i_n)$$

be the point in $\mathbb{Z}^n$ where components of e_i are defined by

$$i_m = \begin{cases} 1, & \text{if } m \leq i \\ 0, & \text{if } m > i. \end{cases}$$

For example in $\mathbb{Z}^2$,

$$e_0 = (0,0), \quad e_1 = (1,0), \quad e_2 = (1,1)$$

and in $\mathbb{Z}^3$,

$$e_0 = (0,0,0), \quad e_1 = (1,0,0), \quad e_2 = (1,1,0), \quad e_3 = (1,1,1).$$

We denote the digital standard n-simplex by

$$\Delta^n = [e_0, e_1, ..., e_n].$$

Example 3.1.

- For $n = 0$, the Khalimsky topology on $\Delta^0 = [e_0]$ is
$$\tau_{\Delta^0} = \{\emptyset, \Delta^0\}.$$
- For $n = 1$, the Khalimsky topology on $\Delta^1 = [e_0, e_1]$ is
$$\tau_{\Delta^1} = \{\emptyset, \Delta^1, \{e_1\}\}.$$
- For $n = 2$, the Khalimksy topology on $\Delta_2 = [e_0, e_1, e_2]$ is
$$\tau_{\Delta^2} = \{\emptyset, \Delta^2, \{e_2\}, \{e_1, e_2\}\}.$$
- For $n = 3$, the Khalimsky topology on $\Delta^3 = [e_0, e_1, e_2, e_3]$ is
$$\tau_{\Delta^3} = \{\emptyset, \Delta^3, \{e_3\}, \{e_2, e_3\}, \{e_1, e_2, e_3\}\}.$$

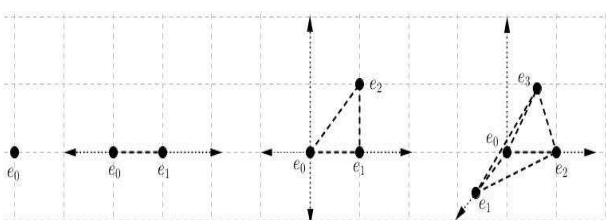


Fig. 2: Δ^0, Δ^1, Δ^2 and Δ^3

We give a linear ordering of its vertices, called *orientation*. In that case, let

$$e_0 < e_1 < ... < e_n$$

be the orientation of $\Delta^n = [e_0, e_1, ..., e_n]$. Under this orientation, the induced orientation of its faces defined by orienting the ith face in the sense

$$(-1)^i [e_0, ..., \widehat{e_i}, ..., e_n]$$

where $\widehat{e_i}$ means that it is deleted and $-[e_0, ..., \widehat{e_i}, ..., e_n]$ is the ith face with orientation opposite to the one with the vertices ordered as $e_0 < e_1 < ... < e_n$. Then the boundary of Δ^n is

$$\cup_{i=1}^{n} [e_0, ..., \widehat{e_i}, ..., e_n]$$

and the oriented boundary of Δ^n is

$$\cup_{i=1}^{n} (-1)^i [e_0, ..., \widehat{e_i}, ..., e_n].$$

Definition 3.2. Let $(X_{m,\kappa}, \tau_X)$ be a digital space. A *digital singular n-simplex* in X is a KD-($3^n - 1, \kappa$)-continuous map

$$\sigma^n : \Delta^n \to X.$$

For $n \geq 0$, let $S_n(X)$ denote the free abelian group with basis of all digital singular n-simplexes in a digital space X and define

$$S_{-1}(X) = 0.$$

The elements of $S_n(X)$ are called *the digital singular n-chains* in X.

Let $\varepsilon_i := \varepsilon_i^n : \Delta^{n-1} \to \Delta^n$ to be a map taking the vertices $\{e_0, e_1, ..., e_{n-1}\}$ to the vertices $\{e_0, ..., \widehat{e_i}, ..., e_n\}$ and preserving the orderings.

Note that the superscript n indicates that the target of ε_i^n is Δ^n. We call ε_i as *ith face map*.

For instance, there are 4 face maps

$$\varepsilon_i^3 : \Delta^2 \to \Delta^3,$$

such that

- $\varepsilon_0 : [e_0, e_1, e_2] \mapsto [e_1, e_2, e_3]$
- $\varepsilon_1 : [e_0, e_1, e_2] \mapsto [e_0, e_2, e_3]$
- $\varepsilon_2 : [e_0, e_1, e_2] \mapsto [e_0, e_1, e_3]$
- $\varepsilon_3 : [e_0, e_1, e_2] \mapsto [e_0, e_1, e_2]$.

Definition 3.3. Let $(X_{m,\kappa}, \tau_X)$ be a space. If $\sigma^n : \Delta^n \to X$ is a singular digital n-simplex, then its boundary is

$$\partial_n(\sigma^n) = \sum_{j=0}^{n} (-1)^j \sigma^n \varepsilon_j^n \in S_{n-1}(X)$$

and if $n = 0$, define

$$\partial_0(\sigma^n) = 0.$$

By the linearity of ∂_n, note that for each $n \geq 0$, there is a unique homomorphism

$$\partial_n : S_n \to S_{n-1}(X)$$

with

$$\partial_n(\sigma^n) = \sum_{j=0}^{n} (-1)^j \sigma^n \varepsilon_j^n$$

for every singular digital n-simplex σ^n in X.

Definition 3.4. The homomorphisms

$$\partial_n : S_n(X) \to S_{n-1}(X)$$

are called *boundary operators*. For each digital space $(X_{m,\kappa}, \tau_X)$, a sequence of free abelian groups and homomorphisms

$$\dots \xrightarrow{\partial_{n+1}} S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_2} S_1(X)$$

$$\xrightarrow{\partial_1} S_0(X) \xrightarrow{\partial_0} 0$$

called *the digital singular complex* of the digital space $(X_{m,\kappa}, \tau_X)$ and it is denoted by $S_*(X)$.

Lemma 3.5. If $k < j$, then face maps satisfy

$$\varepsilon_j^{n+1}\varepsilon_k^n = \varepsilon_k^{n+1}\varepsilon_{j-1}^n.$$

Theorem 3.6. For all $n \geq 0$, we have $\partial_n\partial_{n+1} = 0$.

Proof. It suffices to show that $\partial_n\partial_{n+1}(\sigma) = 0$ for the generators $\sigma \in S_{n+1}(X)$.

$$\begin{aligned}
\partial_n\partial_{n+1}(\sigma) &= \partial_n(\sum_{j=0}^{n+1}(-1)^j\sigma\varepsilon_j^{n+1}) \\
&= \sum_{k=0}^{n}(-1)^k\sum_{j=0}^{n+1}(-1)^j\sigma\varepsilon_j^{n+1}\varepsilon_k^n \\
&= \sum_{j,k}(-1)^{j+k}\sigma\varepsilon_j^{n+1}\varepsilon_k^n \\
&= \sum_{j\leq k}(-1)^{j+k}\sigma\varepsilon_j^{n+1}\varepsilon_k^n + \sum_{k<j}(-1)^{j+k}\sigma\varepsilon_j^{n+1}\varepsilon_k^n \\
&= \sum_{j\leq k}(-1)^{j+k}\sigma\varepsilon_j^{n+1}\varepsilon_k^n + \sum_{k<j}(-1)^{j+k}\sigma\varepsilon_k^{n+1}\varepsilon_{j-1}^n
\end{aligned}$$

In the second sum of the right-hand side of the equation, take $p = k$ and $q = j-1$. Then

$$\begin{aligned}
\partial_n\partial_{n+1}(\sigma) = &\sum_{j\leq k}(-1)^{j+k}\sigma\varepsilon_j^{n+1}\varepsilon_k^n \\
&+ \sum_{p\leq q}(-1)^{p+q+1}\sigma\varepsilon_p^{n+1}\varepsilon_q^n.
\end{aligned}$$

We see that each term $\sigma\varepsilon_j^{n+1}\varepsilon_k^n$ occurs twice. From the opposite signs of these sums terms cancel in pairs. □

Definition 3.7. In a digital space $(X_{m,\kappa}, \tau_X)$, the group of the digital singular *n-cyles* is the kernel of the boundary operator ∂_n

$$Z_n(X) := \text{Kernel } \partial_n$$

and the group of the digital singular *n-boundaries* is the image of the boundary operator ∂_{n+1} in X is

$$B_n(X) := \text{Image } \partial_{n+1}.$$

Note that as $\partial_n\partial_{n+1} = 0$, for every digital space $(X_{m,\kappa}, \tau_X)$ and $n \geq 0$, we have

$$B_n(X) \subset Z_n(X) \subset S_n(X).$$

Definition 3.8. For each $n \geq 0$, *the nth digital singular homology group* of a digital space $(X_{m,\kappa}, \tau_X)$ is

$$H_n(X) := \frac{Z_n(X)}{B_n(X)} = \frac{\text{Kernel } \partial_n}{\text{Image } \partial_{n+1}}.$$

The coset $z_n + B_n(X)$ where $z_n \in Z_n(X)$ is called *the homology class* of z_n and it is denoted by $\overline{z_n}$.

Theorem 3.9. Let $X = \{x\}$ be a one point space in $\mathbb{Z}^m$. Then for all $n > 0$,

$$H_n(X) = 0.$$

Proof. Since X is a one point space, there will be only one digital singular n-simplex

$$\sigma^n : \Delta^n \to X$$

which is the constant map for all $n \geq 0$. Therefore

$$S_n(X) \cong \mathbb{Z}$$

Computing the boundary operations

$$\partial_n(\sigma^n) = \sum_{i=0}^{n}(-1)^i\sigma^n\varepsilon_i = \sum_{i=0}^{n}(-1)^i\sigma^{n-1}$$

yields that

$$\partial_n(\sigma^n) = \begin{cases} 0, & n \text{ is odd} \\ \sigma^{n-1}, & n \text{ is even and positive.} \end{cases}$$

Therefore if n is odd, then

$$S_n(X) = \text{Ker } \partial_n = Z_n(X)$$

and since $n+1$ is even, ∂_{n+1} will be an isomorphism, so that

$$S_n(X) = \text{Image } \partial_{n+1} = B_n(X).$$

Thus $H_n(X) = 0$.

If n is even then ∂_n will be an isomorphism so that

$$S_n(X) = \text{Kernel } \partial_n = 0.$$

Thus $H_n(X) = 0$ as well. □

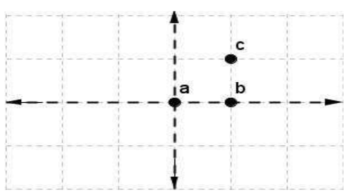
Theorem 3.10. Let $X = \{a = (0,0), b = (1,0), c = (1,1)\}$ be a subset of $\mathbb{Z}^2$ as shown in the Figure 3. Then the digital singular homology groups of X up to the dimension 2 are as follows:

$$H_0(X) \cong \mathbb{Z}, \quad H_1(X) = 0, \quad H_2(X) = 0.$$

Proof. We have already shown that the Khalimsky topology on X is

$$\tau_X = \{\emptyset, X, \{c\}, \{b,c\}\}.$$

Now we will compute the singular digital homology groups of X. The digital singular chain maps are as follows:



$S_0(X)$ has for a basis

$\sigma_1^0 : e_0 \mapsto a$ $\quad \sigma_2^0 : e_0 \mapsto b$ $\quad \sigma_3^0 : e_0 \mapsto c.$

$S_1(X)$ has for a basis

$\sigma_1^1 : e_0 \mapsto a,\ e_1 \mapsto a$

$\sigma_2^1 : e_0 \mapsto b,\ e_1 \mapsto b$

$\sigma_3^1 : e_0 \mapsto c,\ e_1 \mapsto c$

$\sigma_4^1 : e_0 \mapsto a,\ e_1 \mapsto b$

$\sigma_5^1 : e_0 \mapsto a,\ e_1 \mapsto b$

$\sigma_6^1 : e_0 \mapsto b,\ e_1 \mapsto c.$

$S_2(X)$ has for a basis

$\sigma_1^2 : e_0 \mapsto a,\ e_1 \mapsto a,\ e_2 \mapsto a$

$\sigma_2^2 : e_0 \mapsto b,\ e_1 \mapsto b,\ e_2 \mapsto b$

$\sigma_3^2 : e_0 \mapsto c,\ e_1 \mapsto c,\ e_2 \mapsto c$

$\sigma_4^2 : e_0 \mapsto a,\ e_1 \mapsto b,\ e_2 \mapsto c$

$\sigma_5^2 : e_0 \mapsto a,\ e_2 \mapsto a,\ e_1 \mapsto b$

$\sigma_6^2 : e_0 \mapsto a,\ e_1 \mapsto b,\ e_2 \mapsto b$

$\sigma_7^2 : e_0 \mapsto a,\ e_1 \mapsto a,\ e_2 \mapsto c$

$\sigma_8^2 : e_0 \mapsto a,\ e_1 \mapsto c,\ e_2 \mapsto c$

$\sigma_9^2 : e_0 \mapsto b,\ e_1 \mapsto b,\ e_2 \mapsto c$

$\sigma_{10}^2 : e_0 \mapsto b,\ e_1 \mapsto c,\ e_2 \mapsto c.$

$S_3(X)$ has for a basis

$\sigma_1^3 : e_0 \mapsto a,\ e_1 \mapsto a,\ e_2 \mapsto a,\ e_3 \mapsto a$

$\sigma_2^3 : e_0 \mapsto b,\ e_1 \mapsto b,\ e_2 \mapsto b,\ e_3 \mapsto b$

$\sigma_3^3 : e_0 \mapsto c,\ e_1 \mapsto c,\ e_2 \mapsto c,\ e_3 \mapsto c$

$\sigma_4^3 : e_0 \mapsto a,\ e_1 \mapsto a,\ e_2 \mapsto a,\ e_3 \mapsto b$

$\sigma_5^3 : e_0 \mapsto a,\ e_2 \mapsto a,\ e_1 \mapsto a,\ e_3 \mapsto c$

$\sigma_6^3 : e_0 \mapsto a,\ e_1 \mapsto b,\ e_2 \mapsto b,\ e_3 \mapsto b$

$\sigma_7^3 : e_0 \mapsto b,\ e_1 \mapsto b,\ e_2 \mapsto b,\ e_3 \mapsto c$

$\sigma_8^3 : e_0 \mapsto a,\ e_1 \mapsto c,\ e_2 \mapsto c,\ e_3 \mapsto c$

$\sigma_9^3 : e_0 \mapsto b,\ e_1 \mapsto c,\ e_2 \mapsto c,\ e_3 \mapsto c$

$\sigma_{10}^3 : e_0 \mapsto a,\ e_1 \mapsto a,\ e_2 \mapsto b,\ e_3 \mapsto c$

$\sigma_{11}^3 : e_0 \mapsto a,\ e_1 \mapsto b,\ e_2 \mapsto b,\ e_3 \mapsto c$

$\sigma_{12}^3 : e_0 \mapsto a,\ e_1 \mapsto b,\ e_2 \mapsto c,\ e_3 \mapsto c$

$\sigma_{13}^3 : e_0 \mapsto a,\ e_1 \mapsto a,\ e_2 \mapsto b,\ e_3 \mapsto b$

$\sigma_{14}^3 : e_0 \mapsto a,\ e_1 \mapsto a,\ e_2 \mapsto c,\ e_3 \mapsto c$

$\sigma_{15}^3 : e_0 \mapsto b,\ e_1 \mapsto b,\ e_2 \mapsto c,\ e_3 \mapsto c.$

It's easy to see that

$$S_0(X) \cong \mathbb{Z}^3, \quad S_1(X) \cong \mathbb{Z}^6, \quad S_2(X) \cong \mathbb{Z}^{10}, \quad S_3(X) \cong \mathbb{Z}^{15}.$$

Now we'll determine the cycles and the boundaries of each digital singular n-chains:

$$\partial_1 : S_1(X) \to S_0(X).$$

For $\sigma_i^1 \in S_1(X)$, we have a differential map

$$\partial_1(\sigma_i^1) = \sigma_i^1(e_1) - \sigma_i^1(e_0) \qquad \text{for } i = 1,...,6.$$

Indeed we see that

- $\partial_1(\sigma_1^1) = 0$
- $\partial_1(\sigma_2^1) = 0$
- $\partial_1(\sigma_3^1) = 0$
- $\partial_1(\sigma_4^1) = \sigma_2^0 - \sigma_1^0$
- $\partial_1(\sigma_5^1) = \sigma_3^0 - \sigma_1^0$
- $\partial_1(\sigma_6^1) = \sigma_3^0 - \sigma_2^0$.

Then we get Image $\partial_1 \cong \mathbb{Z}^2$.
To determine the kernel of ∂_1, let

$$\partial_1(\sum_{i=1}^{5} s_i\sigma_i^1) = 0$$

where $s_i \in \mathbb{Z},\ i = 1,...,6$. Since ∂_1 is linear,

$$\sum_{i=1}^{6} s_i\partial_1(\sigma_i^1) = 0.$$

Solving the equation

$$s_4(\sigma_2^0 - \sigma_1^0) + s_5(\sigma_3^0 - \sigma_1^0) + s_6(\sigma_3^0 - \sigma_2^0) = 0,$$

we obtain

$$s_4 = -s_5 = s_6.$$

Thus we conclude that Kernel $\partial_1 \cong \mathbb{Z}^4$.
Now consider

$$\partial_2 : S_2(X) \to S_1(X).$$

For $\sigma_i^2 \in S_2(X)$ we have a differential map

$$\partial_2(\sigma_i^2) = \sigma_i^2([e_1,e_2]) - \sigma_i^2([e_0,e_2]) + \sigma_i^2([e_0,e_1])$$

for $i = 1,...,10$. The following are observed:

- $\partial_2(\sigma_1^2) = \sigma_1^1 - \sigma_1^1 + \sigma_1^1 = \sigma_1^1$
- $\partial_2(\sigma_2^2) = \sigma_2^1 - \sigma_2^1 + \sigma_2^1 = \sigma_2^1$
- $\partial_2(\sigma_3^2) = \sigma_3^1 - \sigma_3^1 + \sigma_3^1 = \sigma_3^1$
- $\partial_2(\sigma_4^2) = \sigma_6^1 - \sigma_5^1 + \sigma_3^1$
- $\partial_2(\sigma_5^2) = \sigma_4^1 - \sigma_4^1 + \sigma_1^1 = \sigma_3^1$
- $\partial_2(\sigma_6^2) = \sigma_2^1 - \sigma_4^1 + \sigma_4^1 = \sigma_2^1$
- $\partial_2(\sigma_7^2) = \sigma_5^1 - \sigma_5^1 + \sigma_1^1 = \sigma_1^1$
- $\partial_2(\sigma_8^2) = \sigma_3^1 - \sigma_5^1 + \sigma_5^1 = \sigma_3^1$
- $\partial_2(\sigma_9^2) = \sigma_6^1 - \sigma_6^1 + \sigma_2^1 = \sigma_2^1$
- $\partial_2(\sigma_{10}^2) = \sigma_3^1 - \sigma_6^1 + \sigma_6^1 = \sigma_3^1$.

Then we get Image $\partial_2 \cong \mathbb{Z}^4$.
To determine the kernel of ∂_2 we have

$$\partial_2(\sum_{i=1}^{10} s_i\sigma_i^2) = 0$$

where $s_i \in \mathbb{Z},\ i = 1,...,10$. Since ∂_2 is linear, we have

$$\sum_{i=1}^{10} s_i\partial_2(\sigma_i^2) = 0.$$

Solving the equation

$$\sigma_1^1(s_1+s_5+s_7) + \sigma_2^1(s_2+s_6+s_9) + \sigma_3^1(s_3+s_8+s_{10}) + \sigma_4^1 s_4 + \sigma_5^1(-s_4) + \sigma_6^1(s_4) = 0,$$

we have

$$\left.\begin{aligned} s_1+s_5+s_7 &= 0 \\ s_2+s_6+s_9 &= 0 \\ s_3+s_8+s_{10} &= 0 \\ s_4 &= 0 \end{aligned}\right\}$$

So, Kernel $\partial_2 \cong \mathbb{Z}^6$.
Now consider

$$\partial_3 : S_3(X) \to S_2(X).$$

For $\sigma_i^3 \in S_3(X)$ we have a differential map

$$\partial_3(\sigma_i^3) = \sigma_i^3([e_1,e_2,e_3]) - \sigma_i^3([e_0,e_2,e_3]) + \sigma_i^3([e_0,e_1,e_3]) - \sigma_i^3([e_0,e_1,e_2])$$

for $i = 1,...,15$. It is seen that

- $\partial_3(\sigma_1^3) = \sigma_1^2 - \sigma_1^2 + \sigma_1^2 - \sigma_1^2 = 0$
- $\partial_3(\sigma_2^3) = \sigma_2^2 - \sigma_2^2 + \sigma_2^2 - \sigma_2^2 = 0$
- $\partial_3(\sigma_3^3) = \sigma_3^2 - \sigma_3^2 + \sigma_3^2 - \sigma_3^2 = 0$
- $\partial_3(\sigma_4^3) = \sigma_5^2 - \sigma_5^2 + \sigma_5^2 - \sigma_1^2 = \sigma_5^2 - \sigma_1^2$
- $\partial_3(\sigma_5^3) = \sigma_7^2 - \sigma_7^2 + \sigma_7^2 - \sigma_1^2 = \sigma_7^2 - \sigma_1^2$
- $\partial_3(\sigma_6^3) = \sigma_2^2 - \sigma_6^2 + \sigma_6^2 - \sigma_6^2 = \sigma_2^2 - \sigma_6^2$
- $\partial_3(\sigma_7^3) = \sigma_9^2 - \sigma_9^2 + \sigma_9^2 - \sigma_2^2 = \sigma_9^2 - \sigma_2^2$
- $\partial_3(\sigma_8^3) = \sigma_3^2 - \sigma_8^2 + \sigma_8^2 - \sigma_8^2 = \sigma_3^2 - \sigma_8^2$
- $\partial_3(\sigma_9^3) = \sigma_3^2 - \sigma_{10}^2 + \sigma_{10}^2 - \sigma_{10}^2 = \sigma_3 - \sigma_{10}^2$
- $\partial_3(\sigma_{10}^3) = \sigma_4^2 - \sigma_4^2 + \sigma_7^2 - \sigma_5^2 = \sigma_7^2 - \sigma_5^2$
- $\partial_3(\sigma_{11}^3) = \sigma_9^2 - \sigma_4^2 + \sigma_4^2 - \sigma_6^2 = \sigma_9^2 - \sigma_6^2$
- $\partial_3(\sigma_{12}^3) = \sigma_{10}^2 - \sigma_8^2 + \sigma_4^2 - \sigma_4^2 = \sigma_{10}^2 - \sigma_8^2$
- $\partial_3(\sigma_{13}^3) = \sigma_6^2 - \sigma_6^2 + \sigma_5^2 - \sigma_5^2 = 0$
- $\partial_3(\sigma_{14}^3) = \sigma_8^2 - \sigma_8^2 + \sigma_7^2 - \sigma_7^2 = 0$
- $\partial_3(\sigma_{15}^3) = \sigma_{10}^2 - \sigma_{10}^2 + \sigma_9^2 - \sigma_9^2 = 0$.

Then one can get Image $\partial_3 \cong \mathbb{Z}^6$. Hence the digital singular homology groups of X are

$$H_0(X) \cong \mathbb{Z} \qquad H_1(X) = 0 \qquad H_2(X) = 0. \qquad \square$$

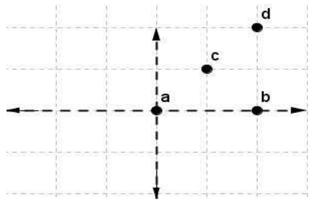
Theorem 3.10. Let

$$X = \{a = (0,0), b = (2,0), c = (1,1), d = (2,2)\}$$

be a subset in $\mathbb{Z}^2$. Then the digital homology groups of X up to the dimension 2 are as follows:

$$H_0(X) \cong \mathbb{Z}, \quad H_1(X) \cong \mathbb{Z}, \quad H_2(X) = 0.$$

Proof. The Khalimsky topology on X is



$\tau_X = \{\emptyset, X, \{c\}, \{a,c\}, \{c,b\}, \{c,d\}, \{a,b,c\}, \{a,c,d\}, \{b,c,d\}\}.$

$S_0(X)$ has for a basis

$\sigma_1^0 : e_0 \mapsto a$

$\sigma_2^0 : e_0 \mapsto b$

$\sigma_3^0 : e_0 \mapsto c$

$\sigma_4^0 : e_0 \mapsto d.$

$S_1(X)$ has for a basis

$\sigma_1^1 : e_0 \mapsto a$, $e_1 \mapsto a$

$\sigma_2^1 : e_0 \mapsto b$, $e_1 \mapsto b$

$\sigma_3^1 : e_0 \mapsto c$, $e_1 \mapsto c$

$\sigma_4^1 : e_0 \mapsto d$, $e_1 \mapsto d$

$\sigma_5^1 : e_0 \mapsto a$, $e_1 \mapsto c$

$\sigma_6^1 : e_0 \mapsto b$, $e_1 \mapsto c$

$\sigma_7^1 : e_0 \mapsto d$, $e_1 \mapsto c.$

$S_2(X)$ has for a basis

$\sigma_1^2 : e_0 \mapsto a$, $e_1 \mapsto a$, $e_2 \mapsto a$

$\sigma_2^2 : e_0 \mapsto b$, $e_1 \mapsto b$, $e_2 \mapsto b$

$\sigma_3^2 : e_0 \mapsto c$, $e_1 \mapsto c$, $e_2 \mapsto c$

$\sigma_4^2 : e_0 \mapsto d$, $e_1 \mapsto d$, $e_2 \mapsto d$

$\sigma_5^2 : e_0 \mapsto a$, $e_2 \mapsto a$, $e_1 \mapsto c$

$\sigma_6^2 : e_0 \mapsto b$, $e_1 \mapsto b$, $e_2 \mapsto c$

$\sigma_7^2 : e_0 \mapsto a$, $e_1 \mapsto c$, $e_2 \mapsto c$

$\sigma_8^2 : e_0 \mapsto b$, $e_1 \mapsto c$, $e_2 \mapsto c$

$\sigma_9^2 : e_0 \mapsto d$, $e_1 \mapsto c$, $e_2 \mapsto c$

$\sigma_{10}^2 : e_0 \mapsto d$, $e_1 \mapsto d$, $e_2 \mapsto c.$

$S_3(X)$ has for a basis

$\sigma_1^3 : e_0 \mapsto a$, $e_1 \mapsto a$, $e_2 \mapsto a$, $e_3 \mapsto a$

$\sigma_2^3 : e_0 \mapsto b$, $e_1 \mapsto b$, $e_2 \mapsto b$, $e_3 \mapsto b$

$\sigma_3^3 : e_0 \mapsto c$, $e_1 \mapsto c$, $e_2 \mapsto c$, $e_3 \mapsto c$

$\sigma_4^3 : e_0 \mapsto d$, $e_1 \mapsto d$, $e_2 \mapsto d$, $e_3 \mapsto d$

$\sigma_5^3 : e_0 \mapsto a$, $e_2 \mapsto a$, $e_1 \mapsto a$, $e_3 \mapsto c$

$\sigma_6^3 : e_0 \mapsto b$, $e_1 \mapsto b$, $e_2 \mapsto b$, $e_3 \mapsto c$

$\sigma_7^3 : e_0 \mapsto a$, $e_1 \mapsto c$, $e_2 \mapsto c$, $e_3 \mapsto c$

$\sigma_8^3 : e_0 \mapsto b$, $e_1 \mapsto c$, $e_2 \mapsto c$, $e_3 \mapsto c$

$\sigma_9^3 : e_0 \mapsto d$, $e_1 \mapsto c$, $e_2 \mapsto c$, $e_3 \mapsto c$

$\sigma_{10}^3 : e_0 \mapsto d$, $e_1 \mapsto d$, $e_2 \mapsto d$, $e_3 \mapsto c$

$\sigma_{11}^3 : e_0 \mapsto a$, $e_1 \mapsto a$, $e_2 \mapsto c$, $e_3 \mapsto c$

$\sigma_{12}^3 : e_0 \mapsto b$, $e_1 \mapsto b$, $e_2 \mapsto c$, $e_3 \mapsto c$

$\sigma_{13}^3 : e_0 \mapsto d$, $e_1 \mapsto d$, $e_2 \mapsto c$, $e_3 \mapsto c.$

It's clear that

$$S_0(X) \cong \mathbb{Z}^4, \quad S_1(X) \cong \mathbb{Z}^7, \quad S_2(X) \cong \mathbb{Z}^{10}, \quad S_3(X) \cong \mathbb{Z}^{13}.$$

Now we'll determine the cycles and boundaries of each singular digital singular n-chains:

$$\partial_1 : S_1(X) \to S_0(X).$$

For $\sigma_i^1 \in S_1(X)$ we have a differential map

$$\partial_1(\sigma_i^1) = \sigma_i^1(e_1) - \sigma_i^1(e_0) \qquad \text{for} i = 1, \ldots, 7.$$

The following are hold:

- $\partial_1(\sigma_1^1) = \sigma_1^0 - \sigma_1^0 = 0$
- $\partial_1(\sigma_2^1) = \sigma_2^0 - \sigma_2^0 = 0$
- $\partial_1(\sigma_3^1) = \sigma_3^0 - \sigma_3^0 = 0$
- $\partial_1(\sigma_4^1) = \sigma_4^0 - \sigma_4^0 = 0$
- $\partial_1(\sigma_5^1) = \sigma_3^0 - \sigma_1^0$
- $\partial_1(\sigma_6^1) = \sigma_3^0 - \sigma_2^0$
- $\partial_1(\sigma_7^1) = \sigma_3^0 - \sigma_4^0.$

Then we get Image $\partial_1 \cong \mathbb{Z}^3$.

To determine the kernel of ∂_1, let

$$\partial_1(\sum_{i=1}^{7} s_i\sigma_i^1) = 0$$

where $s_i \in \mathbb{Z},\ i = 1,\ldots,7$. Since ∂_1 is linear, then

$$\sum_{i=1}^{7} s_i\partial_1(\sigma_i^1) = 0.$$

Solving the equation

$$\sigma_1^0(-s_5) + \sigma_3^0(s_5+s_6+s_7) + \sigma_4^0(-s_7) = 0,$$

we get

$$s_5 = s_6 = s_7 = 0,$$

and hence

$$\text{Kernel } \partial_1 \cong \mathbb{Z}^4.$$

Consider

$$\partial_2 : S_2(X) \to S_1(X).$$

For $\sigma_i^2 \in S_2(X)$ we have a differential map

$$\partial_2(\sigma_i^2) = \sigma_i^2([e_1,e_2]) - \sigma_i^2([e_0,e_2]) + \sigma_i^2([e_0,e_1])$$

for $k = 1,\ldots,10$. The following are observed:

- $\partial_2(\sigma_1^2) = \sigma_1^1 - \sigma_1^1 + \sigma_1^1 = \sigma_1^1$
- $\partial_2(\sigma_2^2) = \sigma_2^1 - \sigma_2^1 + \sigma_2^1 = \sigma_2^1$
- $\partial_2(\sigma_3^2) = \sigma_3^1 - \sigma_3^1 + \sigma_3^1 = \sigma_3^1$
- $\partial_2(\sigma_4^2) = \sigma_4^1 - \sigma_4^1 + \sigma_4^1 = \sigma_4^1$
- $\partial_2(\sigma_5^2) = \sigma_5^1 - \sigma_5^1 + \sigma_1^1 = \sigma_1^1$
- $\partial_2(\sigma_6^2) = \sigma_6^1 - \sigma_6^1 + \sigma_2^1 = \sigma_2^1$
- $\partial_2(\sigma_7^2) = \sigma_3^1 - \sigma_5^1 + \sigma_5^1 = \sigma_3^1$
- $\partial_2(\sigma_8^2) = \sigma_3^1 - \sigma_6^1 + \sigma_6^1 = \sigma_3^1$
- $\partial_2(\sigma_9^2) = \sigma_3^1 - \sigma_7^1 + \sigma_7^1 = \sigma_3^1$
- $\partial_2(\sigma_{10}^2) = \sigma_7^1 - \sigma_7^1 + \sigma_4^1 = \sigma_4^1$.

From this observation, we get

$$\text{Image } \partial_2 \cong \mathbb{Z}^4.$$

To determine the kernel of ∂_2 we have

$$\partial_2(\sum_{i=1}^{10} s_i\sigma_i^2) = 0$$

where $s_i \in \mathbb{Z},\ i = 1,\ldots,10$. Since ∂_2 is linear, we have

$$\sum_{i=1}^{10} s_i\partial_2(\sigma_i^2) = 0.$$

Solving the equation

$$\sigma_1^1(s_1+s_5) + \sigma_2^1(s_2+s_6) + \sigma_3^1(s_3+s_7+s_8+s_9) + \sigma_4^1(s_4+s_{10}) = 0,$$

we obtain

$$\left.\begin{aligned} s_1 &= -s_5 \\ s_2 &= -s_6 \\ s_4 &= -s_{10} \\ s_3+s_7+s_8+s_9 &= 0 \end{aligned}\right\}$$

and hence

$$\text{Kernel } \partial_2 \cong \mathbb{Z}^6.$$

Consider

$$\partial_3 : S_3(X) \to S_2(X)$$

For $\sigma_i^3 \in S_3(X)$ we have a differential map

$$\partial_3(\sigma_i^3) = \sigma_i^3([e_1,e_2,e_3]) - \sigma_i^3([e_0,e_2,e_3]) + \sigma_i^3([e_0,e_1,e_3]) - \sigma_i^3([e_0,e_1,e_2])$$

for $\ell = 1,\ldots,15$.

It's clear that

- $\partial_3(\sigma_1^3) = \sigma_1^2 - \sigma_1^2 + \sigma_1^2 - \sigma_1^2 = 0$
- $\partial_3(\sigma_2^3) = \sigma_2^2 - \sigma_2^2 + \sigma_2^2 - \sigma_2^2 = 0$
- $\partial_3(\sigma_3^3) = \sigma_3^2 - \sigma_3^2 + \sigma_3^2 - \sigma_3^2 = 0$
- $\partial_3(\sigma_4^3) = \sigma_4^2 - \sigma_4^2 + \sigma_4^2 - \sigma_4^2 = 0$
- $\partial_3(\sigma_5^3) = \sigma_5^2 - \sigma_5^2 + \sigma_5^2 - \sigma_1^2 = \sigma_5^2 - \sigma_1^2$
- $\partial_3(\sigma_6^3) = \sigma_6^2 - \sigma_6^2 + \sigma_6^2 - \sigma_2^2 = \sigma_6^2 - \sigma_2^2$
- $\partial_3(\sigma_7^3) = \sigma_3^2 - \sigma_7^2 + \sigma_7^2 - \sigma_7^2 = \sigma_3^2 - \sigma_7^2$
- $\partial_3(\sigma_8^3) = \sigma_3^2 - \sigma_8^2 + \sigma_8^2 - \sigma_8^2 = \sigma_3^2 - \sigma_8^2$
- $\partial_3(\sigma_9^3) = \sigma_3^2 - \sigma_9^2 + \sigma_9^2 - \sigma_9^2 = \sigma_3^2 - \sigma_9^2$
- $\partial_3(\sigma_{10}^3) = \sigma_{10}^2 - \sigma_{10}^2 + \sigma_{10}^2 - \sigma_4^2 = \sigma_{10}^2 - \sigma_4^2$
- $\partial_3(\sigma_{11}^3) = \sigma_7^2 - \sigma_7^2 + \sigma_5^2 - \sigma_5^2 = 0$
- $\partial_3(\sigma_{12}^3) = \sigma_8^2 - \sigma_8^2 + \sigma_6^2 - \sigma_6^2 = 0$
- $\partial_3(\sigma_{13}^3) = \sigma_9^2 - \sigma_9^2 + \sigma_{10}^2 - \sigma_{10}^2 = 0$.

Thus we have

$$\text{Image } \partial_3 \cong \mathbb{Z}^6.$$

Then the digital singular homology groups of X are as follows:

$$H_0(X) \cong \mathbb{Z} \qquad H_1(X) \cong \mathbb{Z} \qquad H_2(X) = 0. \ \square$$

4 Functorial Property of H_n

Let $(X_{m_1,\kappa_1}, \tau_X)$ and $(Y_{m_2,\kappa_2}, \tau_Y)$ be two spaces. If

$$f : (X_{m_1,\kappa_1}, \tau_X) \to (Y_{m_2,\kappa_1}, \tau_Y)$$

is a KD-(κ_1,κ_2)-continuous map and

$$\sigma^n : \Delta^n \to X$$

is a singular digital n-simplex in X, then

$$f \circ \sigma^n : \Delta^n \to Y$$

is also an n-simplex in Y. If we extend f by linearity of singular digital n-simplexes in X, we have a homomorphism

$$f_\sharp : S_n(X) \to S_n(Y), \quad f_\sharp(\sum s_{\sigma^n}\sigma^n) = \sum s_{\sigma^n}(f\circ\sigma^n)$$

where $s_{\sigma^n} \in \mathbb{Z}$.

Theorem 4.1. Let $(X_{m_1,\kappa_1},\tau_X)$ and $(Y_{m_2,\kappa_2},\tau_Y)$ be two spaces. If

$$f : (X_{m_1,\kappa_1},\tau_X) \to (Y_{m_2,\kappa_1},\tau_Y)$$

is KD-(κ_1,κ_2)-continuous map, then for every $n \geq 0$, then the following diagram commutes:

$$\begin{array}{ccc} S_n(X) & \xrightarrow{\partial_n} & S_{n-1}(X) \\ \downarrow f_\sharp & & \downarrow f_\sharp \\ S_n(Y) & \xrightarrow[\partial_n]{} & S_{n-1}(Y) \end{array}$$

Proof. Since ∂_n is linear, it's enough to show the evaluation of each composition on a generator $\sigma^n \in S_n(X)$.

$$\begin{aligned} f_\sharp\partial_n(\sigma^n) &= f_\sharp(\sum_{i=1}^{n}(-1)^i\sigma^n\circ\varepsilon_i) \\ &= \sum_{i=1}^{n}(-1)^i f_\sharp(\sigma^n\circ\varepsilon_i) \\ &= \sum_{i=1}^{n}(-1)^i(f_\sharp\circ\sigma^n)\circ\varepsilon_i \\ &= \partial_n(f\circ\sigma^n) \\ &= \partial_n f_\sharp(\sigma^n). \end{aligned}$$

□

Theorem 4.2. Let $(X_{m_1,\kappa_1},\tau_X)$ and $(Y_{m_2,\kappa_2},\tau_Y)$ be two digital spaces. If

$$f : (X_{m_1,\kappa_1},\tau_X) \to (Y_{m_2,\kappa_2},\tau_Y)$$

is KD-(κ_1,κ_2)-continuous map, then for every $n \geq 0$

$$f_\sharp(Z_n(X)) \subset Z_n(Y) \quad \text{and} \quad f_\sharp(B_n(X)) \subset B_n(Y).$$

Proof. Let $z_n \in Z_n(X)$, then $\partial_n(z_n) = 0$. By the previous theorem, we have

$$\partial_n f_\sharp(z_n) = f_\sharp\partial_n(z_n) = f_\sharp(0) = 0$$

so that

$$f_\sharp\partial_n \in \text{Ker }\partial_n = Z_n(Y).$$

Now let $\beta \in B_n(X)$, then there exists $\alpha \in S_{n+1}(X)$ such that

$$\beta = \partial_{n+1}(\alpha).$$

Again, by Theorem 4, we have

$$f_\sharp(\beta) = f_\sharp\partial_{n+1}(\alpha) = \partial_{n+1} f_\sharp(\alpha) \in \text{Im }\partial_{n+1} = B_n(Y). \ \square$$

Consider the KD-topological category KDTC, where

- The objects are $(X_{n,\kappa},\tau_X)$; and
- the morphisms are KD-(κ_1,κ_2)-continuous functions.

Theorem 4.3. For each $n \geq 0$, H_n : KDTC $\to$ Ab is a functor.

Proof. Let $(X_{m_1,\kappa_1},\tau_X)$ and $(Y_{m_2,\kappa_2},\tau_Y)$ be two spaces and let

$$f : (X_{m_1,\kappa_1},\tau_X) \to (Y_{m_2,\kappa_2},\tau_Y)$$

be KD-(κ_1,κ_2)-continuous map, be a continuous function. Define

$$H_n(f) : H_n(X) \to H_n(Y)$$

by

$$\overline{z} = z_n + B_n(X) \mapsto f_\sharp(z_n) + B_n(Y)$$

where $z_n \in Z_n(X)$. Note that, z_n being an n-cycle implies that $f_\sharp(z_n)$ is an n-cycle in Y. Also this definition is well defined, that is, independent of the choice of representative since

$$f_\sharp(B_n(X)) \subset B_n(Y).$$

- $H_n(f)$ is a homomorphism:
  For all $\overline{z_n},\overline{z'_n}$ in $H_n(X)$, we have

  $$\begin{aligned} H_n(f)(\overline{z_n}+\overline{z'_n}) &= f_\sharp(z_n+z'_n)+B_n(Y) \\ &= (f_\sharp(z_n)+f_\sharp(z'_n))+B_n(Y) \\ &= f_\sharp(z_n)+B_n(Y)+f_\sharp(z'_n)+B_n(Y) \\ &= H_n(\overline{z_n})+H_n(\overline{z'_n}). \end{aligned}$$

  It's also clear that H_n sends the identity function to identity homomorphism.

- H_n preserves the composition: For the spaces $(X_{m_1,\kappa_1},\tau_X)$, $(Y_{m_2,\kappa_2},\tau_Y)$ and $(Z_{m_3,\kappa_3},\tau_Z)$, let

  $$f : (X_{m_1,\kappa_1},\tau_X) \to (Y_{m_2,\kappa_2},\tau_Y)$$

  and

  $$g : (Y_{m_2,\kappa_2},\tau_Y) \to (Z_{m_3,\kappa_3},\tau_Z)$$

  be two KD-(κ_1,κ_2) and KD-(κ_2,κ_3) continuous maps respectively. Then

  $$\begin{aligned} H_n(g\circ f)(\overline{z_n}) &= (g\circ f)_\sharp(z_n)+B_n(X) = g_\sharp(f_\sharp(z_n)+B_n(X)) \\ &= (H_n(g)\circ H_n(f))(\overline{z}). \end{aligned}$$

So this shows that H_n is a functor. □

Corollary 4.4. If $(X_{m_1,\kappa_1},\tau_X)$ and $(Y_{m_2,\kappa_2},\tau_Y)$ are KD-(κ_1,κ_2)-isomorphic, then

$$H_n(X) \cong H_n(Y)$$

for all $n \geq 0$.

Proof. It's a consequence of H_n being a functor. □

5 Conclusion

In this paper, we define the digital singular homology and compute the homology groups of some certain digital spaces. We have seen that the digital singular homology is a functorial property so that it can be used to distinguish and classify the digital spaces. The next work based on this paper is to investigate whether the homology axioms are valid or not in the digital singular homology theory, define the digital singular cohomology and compute the cohomology groups of some digital spaces, to compare the digital singular homology groups with the digital simplicial groups.

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