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On S-Quasinormal Subgroups and some Applications

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Abstract: A subgroup H of a group G is called S-quasinormal in G if it permutes with every Sylow subgroup of G.The structure of the group G has been studied earlier by many authors under the assumption that the maximal or the minimal subgroups of the Sylow subgroups are well situated in G. In the present paper we are cincerned with the study of the structure of a finite group under the assumption that some subgroups of G are S-quasinormal in G, and we discuss some methods and applications.

Keywords: Finite group, saturated formation; S-quasinormal subgroup, Sylow subgroup, supersoluble group, representations of groups

1 Introduction

Throughout this paper, all groups are finite. Recall that two subgroups A and B of a group G are said to permute if $AB = BA$. A subgroup A of the group G is called S-quasinormal if it permutes with all Sylow subgroups of G. Recall that a formation is a hypomorph $\mathscr{F}$ of groups such that each group G has the smallest normal subgroup (denoted by $G^{\mathscr{F}}$) whose quotient is still in $\mathscr{F}$. A formation $\mathscr{F}$ is said to be saturated if it contains each group G with $G/\Phi(G) \in \mathscr{F}$. In this paper we use $\mathscr{U}$ to denote the class of the supersoluble groups.

The structure of the group G has been investigated by several authors under the assumption that the maximal or the minimal subgroups of the Sylow subgroups in G are well situated in G. Buckly [1] proved that a group of odd order is supersoluble if all its minimal subgroups are normal. Later on, Srinivasan [2] showed that the group G is supersoluble if it has a normal subgroup N with supersoluble quotient G/N such that all maximal subgroups of the Sylow subgroups of N are normal in G. Ramadan [9] proved: *If G is a soluble group and all maximal subgroups of any Sylow subgroup of $F(G)$ are normal in G, then G is supersoluble.* Some later several authors were stadying G groups in which the maximal or the minimal subgroups of the Sylow subgroups in G are S-quasinormal in G (see, for example, [1,2,3,4,5,6,7,8, 9,10,11,12,13] The most general results in this trend were obtained in [9,10] where the following two nice theorems were proved:

Theorem A. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and G be a group with a normal subgroup N such that $G/N \in \mathscr{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F^*(N)$ are S-quasinormal in G, then $G \in \mathscr{F}$* (see [10, Theorem 3.1).)

Theorem B *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and G be a group with a normal subgroup E such that $G/E \in \mathscr{F}$. If all maximal subgroups of the Sylow subgroups of $F^*(E)$ are S-quasinormal in G, then $G \in \mathscr{F}$* (see [9, Theorem 3.1]).

In the connection with Theorems A, B the following natural question arises: Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and G be a group with a normal soluble subgroup E such that $G/E \in \mathscr{F}$. Is the group G in $\mathscr{F}$ if for every Sylow subgroup P of $F(G)$ at least one of the following conditions holds:

(1) The maximal subgroups of P are S-quasinormal in G;

(2) The minimal subgroups of P and all its cyclic subgroups with order 4 are S-quasinormal in G?

We prove the following theorem which gives the positive answer to this question.

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Theorem 1. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and G be a group with a soluble normal subgroup E such that $G/E \in \mathscr{F}$. Suppose that every Sylow subgroup P of $F(E)$ there is a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and order $2|D|$ (if P is a non-abelian 2-group) are S-quasinormal in G. Then $G \in \mathscr{F}$.*

One of the main steps in the proof of Theorem 1 is the following result.

Theorem 2. *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and G be a group with a normal subgroup E such that $G/E \in \mathscr{F}$. Suppose that every Sylow subgroup P of E there is a subgroup D such that $1 < |D| < |P|$ and all subgroups H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group) not having a supersoluble supplement in G are S-quasinormal in G. Then $G \in \mathscr{F}$.*

Remark that some results of the papers [1,2,3,4,5,6,7,8,11] may be obtained as special cases of these two theorems (see Section 5).

Finally, note that study of S-quasinormal and supersoluble subgroups can use some related results from the theory of integral representations of finite groups, see [23,24].

2 Preliminaries

The reader is referred to [17,18] for the necessary background. For convenience we summarize in this section some basic statements.

Lemma 2.1 [12, Lemma 2.2]. *Let G be a group and $P = P_1 \times \ldots \times P_t$ be a p-subgroup of G where $t > 1$ and $P_1, \ldots, P_t$ are minimal normal subgroups of G. Assume that P has a subgroup D such that $1 < |D| < |P|$ and a product or an intersection of subgroups of order $|D|$ is normal in G. Then the order of P_i is prime.*

The following known results about subnormal subgroups will be used in the paper several times.

Lemma 2.2. *Let G be a group and $A \leq K \leq G$, $B \leq G$. Then:*

(1) *If A is a subnormal Hall subgroup of G, then A is normal in G* [14].

(2) *If A is subnormal in G and A is a π-subgroup of G, then $A \leq O_\pi(G)$* [14].

(3) *If A is a subnormal soluble (nilpotent) subgroup of G, then A is contained in some soluble (respectively in some nilpotent) normal subgroup of G* [14].

We shall need also in our proofs the following facts about S-quasinormal subgroups.

Lemma 2.3 [15]. *Let G be a group and $H \leq K \leq G, T \leq G$. Then*

(1) If H is S-quasinormal in G, then H is S-quasinormal in K.

(2) Suppose that H is normal in G. Then K/H is S-quasinormal in G if and only if K is S-quasinormal in G.

(3) If H S-quasinormal is in G, then H is subnormal in G.

(4) If H and T are S-quasinormal in G, then $< H, T >$ does.

The following observation is well known (see, for example, [16, Lemma A]).

Lemma 2.4. *If H is a S-quasinormal subgroup of the group G and H is a p-group for some prime p, then $O^p(G) \leq N_G(H)$.*

Lemma 2.5. *Let N be an elementary abelian normal p-subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is S-quasinormal in G. Then some maximal subgroup of N is normal in G.*

Proof. Assume that this lemma is false and G is a counterexample of minimal order. Let M be a maximal subgroup of N. Then $N \leq N_G(M) \neq G$ and by Lemma 2.3, M is S-quasinormal in G, as M is the product of some S-quasinormal in G subgroups. By Lemma 2.4, $O^p(G) \leq N_G(M)$ and so $|G : N_G(M)| = p^n$ for some natural $n > 0$. Thus for the set Σ of all maximal subgroups of N we have $p||\Sigma|$, which contradicts [17; III, Lemma 8.5(d)].

Lemma 2.6. *Let $\mathscr{F}$ be a saturated formation containing all nilpotent groups and let G be a group with the soluble $\mathscr{F}$-residual $P = G^{\mathscr{F}}$. Suppose that every maximal subgroup of G not containing P belongs to $\mathscr{F}$. Then $P = G^{\mathscr{F}}$ is a p-group for some prime p and if every cyclic subgroup of P with prime order and order 4 (in the case when $p = 2$ and P is non-abelian) not having a supersoluble supplement in G is S-quasinormal in G, then $|P/\Phi(P)| = p$.*

Proof. By [18; VI, Theorem 24.2], $P = G^{\mathscr{F}}$ is a p-group for some prime p and the following hold:

(1) $P/\Phi(P)$ is a G-chief factor of P;

(2) P is a group of exponent p or exponent 4 (if $p = 2$ and P is non-abelian).

Assume that every cyclic subgroup of P with prime order and order 4 (if $p = 2$ and P is non-abelian) not having a supersoluble supplement in G is S-quasinormal in G. Let $\Phi = \Phi(P)$, X/Φ is a subgroup of P/Φ with prime order, $x \in X \setminus \Phi$ and $L = \langle x \rangle$. Then $|L| = p$ or $|L| = 4$ and so either L has a supersoluble supplement T in G or it is S-quasinormal in G. In the former case we may assume that $T \neq G$ and so $T\Phi \neq G$, since $\Phi \leq \Phi(G)$. On the other hand, $LT = G$ and so $(T\Phi/\Phi)(L\Phi/\Phi) = (T\Phi/\Phi)(X/\Phi) = G/\Phi$. Hence $|G/\Phi : T\Phi/\Phi| = p$ and so $|P/\Phi(P)| = p$, since $G/\Phi = (P/\Phi)(T\Phi/\Phi)$. Now suppose that L is S-quasinormal in G. Then by Lemma 2.3, $L\Phi(P)/\Phi(P) = X/\Phi(P)$ is S-quasinormal in $G/\Phi(P)$. Now by Lemma 2.5 we have to conclude that $|P/\Phi(P)| = p$.

Lemma 2.7 [17; II, Lemma 7.9]. *Let P be a nilpotent normal subgroup of a group G. If $P \cap \Phi(G) = 1$, then P is the direct product of some minimal normal subgroup of G.*

Lemma 2.8 [17; III, Theorem 3.5]. *Let A,B be normal subgroups of a group G and $A \leq \Phi(G)$. Suppose that $A \leq B$ and B/A is nilpotent. Then B is nilpotent.*

Let p be a prime. A group G is said to be p-closed if a Sylow p-subgroup of G is normal.

Lemma 2.9 [18, I, p.34]. *Let p be a prime. Then the class of all p-closed groups is a saturated formation.*

Lemma 2.10 [12, Lemma 2.10**].** *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and G be a group with a normal subgroup E such that $G/E \in \mathscr{F}$. If E is cyclic, then $G \in \mathscr{F}$.*

Lemma 2.11 [19, Theorem 1]. *Let A be a p'-group of automorphisms of the p-group P of odd order. Assume that every subgroup of P of a prime order is A-invariant. Then A is cyclic.*

Lemma 2.12 [20, Lemma 2.24**.** *Let G be a group, p,q be different prime divisors of $|G|$, P be a non-cyclic Sylow p-subgroup of G and Q be a Sylow q-subgroup of G. If all maximal subgroups of P (except one) has a q-closed supplement in G, then Q is normal in G.*

3 The proof of Theorem 2

Proof. Suppose that this theorem is false and consider a counterexample for which $|G|+|E|$ is minimal. Let p be the smallest prime dividing E and P a Sylow p-subgroup of E. We now prove the theorem via the following steps.

(1) *Let X be a Hall subgroup of E. Then the hypothesis is still true for X and for G/X if X is normal in G.*

The first statement is evident. Now assume that X is normal in G. Then $(G/X)/(E/X) \simeq G/X \in \mathscr{F}$. Let P^*/X be a non-cyclic Sylow p-subgroup of E/X where $p \mid |G/X|$, P be a Sylow p-subgroup of E such that $P^* = PX$. Then P is a non-cyclic Sylow subgroup of E and so by hypothesis P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ and with order $2|D|$ (if P is a non-abelian 2-group) either has a supersoluble supplement T in G or is S-quasinormal in G. Let H^*/X be a subgroup of P^*/X with order $|H^*/X| = |D|$. Then $H^* = [X]H$ where H is a Sylow p-subgroup of H^*. Clearly, $|H| = |D|$ and so either $H^*/X = HX/X$ has a supersoluble supplement $TX/X \simeq T/T\cap X$ in G/X or it is S-quasinormal in G/X, by Lemma 2.3. Thus the hypothesis is still true for G/X (respectively E/X).

(2) *If X is a non-identity normal Hall subgroup of E, then $X = E$.*

Since X is a characteristic subgroup of E, it is normal in G and so by (1) the hypothesis is still true for G/X. Hence $G/X \in \mathscr{F}$, by the choice of G. Thus the hypothesis is still true for G respectively X and so $X = E$, by the choice of (G,E).

(3) *P is not cyclic.*

Indeed, if P is cyclic, then by [17, IV, Theorem 2.8], E is p-nilpotent and so by (2), $P = E$. But then $G \in \mathscr{F}$, by Lemma 2.10, a contradiction.

(4) *If either $E = G$ or $E = P$, then $|D| > p$.*

Indeed, if $E = G$, then by (2), G is not p-nilpotent and so it has a p-closed Schmidt group $H = [H_p]H_q$ [17, IV, Theorem 5.4]. If $|D| = p$, then by Lemma 2.6 $|H_p/\Phi(H_p)| = p$, a contradiction, since p is the smallest prime divisor of $|G|$. So in this case we have $|D| > p$.

Now let $E = P$. Consider a maximal subgroup M of G not containing E. Then $G/E \simeq M/M\cap E \in \mathscr{F}$. Assume that $|D| = p$. Let $L = G^{\mathscr{F}}$ and $\Phi = \Phi(L)$. Then $L \leq E$ and so the hypothesis is still true for G (respectively L). Hence $L = E$ and $|L/\Phi| = p$, by Lemma 2.6. So $G/\Phi \in \mathscr{F}$, by Lemma 2.10. But then $P \leq \Phi$ and hence $P = \Phi$, a contradiction. Thus $|D| > p$.

(5) *$|L| \leq |D|$ for any minimal normal subgroup L of G contained in P.*

Assume that $|D| < |L|$. If some subgroup H of L with order $|H| = |D|$ has a supersoluble supplement T in G, then $TL = G$ and $T \neq G$, by the choice of G. Hence $L\cap T$ is a proper non-identity subgroup of L, because $L = L\cap HT = H(L\cap T)$. But evidently $L\cap T$ is normal in G, which contradicts the minimality of L. Hence every subgroup H of L with order $|H| = |D|$ is S-quasinormal in G and so by Lemma 2.5 some maximal subgroup of L is normal in G. Then $|L| = p$ and so $|D| = 1$, a contradiction. Thus we have (5).

(6) *If either $E = G$ or $E = P$ and N is an abelian minimal normal subgroup of G contained in E, then the hypothesis is still true for G/N.*

Let $E = P$. Since $(G/N)/(E/N) \simeq G/E$, it is clear that the hypothesis is still true for G/N respectively E/N if either $p > 2$ or $|P:D| = p$ or $|N| < |D|$. From (5) we have $|N| \leq |D|$. So we need only to consider the case when P is a 2-group, $|P:N| > 2$ and $|N| = |D|$. By (5) every subgroup H of P with order $|H| = |D|$ not having a supersoluble supplement in G is S-quasinormal in G. By (4), N is non-cyclic and hence every subgroup of G containing N is not cyclic. Let $N \leq K \leq P$ where $|K:N| = 2$. Since K is non-cyclic, it has a maximal subgroup $L \neq N$. If at least one the subgroups N, L has a supersoluble supplement in G, then K has a supersoluble supplement in G. If L is S-quasinormal in G, then $K = LN$ is S-quasinormal in G. Thus if P/N is abelian, the hypothesis is true for G/N. Next suppose that P/N is non-abelian. Then P is non-abelian and every subgroup of P with order $2|D|$ not having a supersoluble supplement in G is S-quasinormal in G. In this case one can show as above that every subgroup X of P containing N and such that $|X:N| = 4$ either has a supersoluble supplement in G or is normal in G. Thus again the hypothesis is still true for G/N respectively E/N. Analogously one can prove this statement in the case when $G = E$.

(7) *If $E = G$, then at least one of the maximal subgroups of P, P_1 say, has no a supersoluble supplement in G* (this directly follows from (2) and Lemma 2.12).

(8) *E is soluble.*

By (1) and the choice of G we have only to consider the case when $E = G$. Besides, by (6) we need only to show that $O_p(G) \neq 1$.

Let $H \leq P_1$ where $|H| = |D|$. Then H is S-quasinormal in G and so it is subnormal in G by Lemma 2.3. From Lemma 2.2 it follows that $H \leq O_p(G)$ and so $O_p(E) \neq 1$. So we have (8).

(9) *E is q-closed where q is the largest prime divisor of $|E|$.*

By (1) we have only to consider the case $E = G$. Moreover, since by (8), E is soluble and by (1) the hypothesis is still true for any Hall subgroup X of G , we may suppose that $|G| = p^a q^b$ for some $a, b \in \mathbb{N}$. Assume that G is not q-closed. By (6) and the choice of G for every minimal normal subgroup N of G contained in P the quotient G/N is supersoluble. Thus $N \not\subseteq \Phi(G)$ and N is the only minimal normal subgroup of G contained in P. We show that $N = O_p(G)$. Indeed, let M be a maximal subgroup of G such that $G = [N]M$. Then $O_p(G) = O_p(G) \cap NM = N(O_p(G) \cap M)$. Since $O_p(G) \leq F(G) \leq C_G(N)$, it follows that $O_p(G) \cap M$ is normal in G and so $O_p(G) \cap M = 1$. Hence $N = O_p(G)$. Assume that $|P : D| = p$. For every maximal subgroup A of P containing N we have $AM = G$, so $M \simeq G/N$ is a supersoluble supplement of A in G. Hence by (7) some maximal subgroup V of P neither contains N nor has a supersoluble supplement in G. Hence by hypothesis V is S-quasinormal in G. By Lemma 2.3, V is subnormal in G and so $V \leq O_p(G) = N$. But then $N = P$ and so $E = P$, by (2).

Therefore we may assume that $|P : D| > p$. Then by hypothesis every subgroup H of P satisfying $|H| = |D|$ and not having a supersoluble supplement in G is S-quasinormal. Since every S-quasinormal subgroup of G is contained in $O_p(G) = N$, it follows that every different from N subgroup H of P satisfying $|H| = |D|$ has a supersoluble supplement in G. Therefore every maximal subgroup of P has a supersoluble supplement in G, which contradicts (7). Thus we have (9).

(10) $E = P$.

Indeed, let q be the largest prime divisor of $|E|$ and Q be a Sylow q-subgroup of E. Then by (9), Q is normal in E and so $Q = E = P$, by (2).

Final contradiction.

Let N be a minimal normal subgroup of G contained in P. Then by (6) and (10), N is the only minimal normal subgroup of G contained in P and so $N = O_p(G) = P$. But by Lemma 2.5 it is impossible, because P is a minimal normal subgroup of G. This contradiction completes the proof of this theorem.

4 The proof of Theorem 1

Proof. Assume that this theorem is false and let G be a counterexample with minimal $|G| + |E|$.

Let $F = F(E)$ and p the smallest prime divisor of $|F|$. Let P be the Sylow p-subgroup of F and $F_0/P = F(E/P)$. We divide the proof into the following steps:

((1) $F \neq E$.

Indeed, if $F = E$, then $G \in \mathscr{F}$, by Theorem 2, which contradicts the choice of (G, E). Hence $F \neq E$.

(2) *Let Q be a Sylow q-subgroup of F_0 where q divides $|F_0/P|$. Then $q \neq p$ and either $Q \leq F$ or $p > q$ and $C_Q(P) = 1$.*

Consider the group $D = PQ$. Let $C = C_D(P)$. The hypothesis of Theorem 2 is true for D and so D is supersoluble. Suppose that $q > p$. Then Q *charD*. But D, clearly, is normal in E and so $Q \leq F$. Now, let $p > q$. Then q does not divide $|F|$. It follows that $O_q(D) = 1$ and so $F(D) = P$. But then $C \leq P$ and hence $C_Q(P) = 1$.

(3) $p > 2$.

Assume that $p = 2$. In this case by (2) we have $F/P = F(E/P)$. Thus by Lemma 2.3 the hypothesis is still true for G/P respectively E/P, since $G/E \simeq (G/P)/(E/P) \in \mathscr{F}$. Therefore $G/P \in \mathscr{F}$ and so $G \in \mathscr{F}$, by Theorem 2. This contradiction shows that we have (3).

(4) *Some minimal subgroup of P is not S-quasinormal in G.*

Suppose that every minimal subgroup of P is S-quasinormal in G. Let $F_0/P = F(E/P)$ and Q be a Sylow q-subgroup of V where q divides $|F_0/P|$. Then by (4), either $Q \leq F$ or $C_Q(P) = 1$. In the second case, Q is cyclic, by (3) and Lemma 2.11. Thus by Lemma 2.3 the hypothesis is still true for G/P (respectively E/P) and so $G/P \in \mathscr{F}$, by the choice of (G, E). But then $G \in \mathscr{F}$, by Theorem 2. This contradiction completes the proof of (4).

(5) *P is not cyclic* (this directly follows from (4)).

By (4), P is not cyclic and so by hypothesis P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with $|H| = |D|$ is S-quasinormal in G.

(6) $|D| > p$ (this follows from hypothesis and from (4)).

(7) *If L is a minimal normal subgroup of G and $L \leq P$, then $|L| > p$.*

Assume that $|L| = p$. Let $C_0 = C_E(L)$. Then the hypothesis is true for G/L (respectively C_0/L). Indeed, clearly, $G/C_0 = G/E \cap C_G(L) \in \mathscr{F}$. Besides, since $L \leq Z(C_0)$ and evidently $F \leq C_0$ and $L \leq Z(F)$, we have $F(C_0/L) = F/L$. On the other hand, if H/L is a subgroup of G/L such that $|H| = |D|$, we have $1 < |H/L| < |P/L|$, by (6). Besides, H/L is S-quasinormal in G/L, by Lemma 2.3. Hence the hypothesis is still true for G/L. Hence $G/L \in \mathscr{F}$ and so $G \in \mathscr{F}$, by Lemma 2.10, a contradiction.

(8) $\Phi(G) \cap P \neq 1$.

Suppose that $\Phi(G) \cap P = 1$. Then P is the direct product of some minimal normal subgroups of G, by Lemma 2.7. Hence by Lemma 2.5, P has a maximal subgroup M such which is normal in G. Now by [13, A, Theorem (9.13)] for some minimal normal subgroup L of G contained in P we have $|L| = p$, which contradicts (7). Thus $\Phi(G) \cap P \neq 1$.

Final contradiction.

Let $L \leq \Phi(G) \cap P$ where L is some minimal normal subgroup of G. We show that the hypothesis is still true for G/L (respectively E/L). By Lemma 2.8 we have $F(E/L) = F/L$. By Lemma 2.5, $|L| \leq |D|$ and so the hypothesis is true for G/L in the case $|P : D| = p$.

Besides, by (3) the hypothesis is true for G/L in the case $|L| < |D|$. So let $|P : D| > p$ and $|L| = |D|$. By (7), L is non-cyclic and so every subgroup of G containing L is non-cyclic. Let $L \leq K$, $M \leq K$ where $M \neq L$ and L, M are maximal subgroups of $K \leq P$. Then $K = LM$ and so K is S-quasinormal in G. Thus the hypothesis is true for G/L and $G/L \in \mathscr{F}$, by the choice of (G,E). But then $G \in \mathscr{F}$, since $L \leq \Phi(G)$ and the formation $\mathscr{F}$ is saturated, by hypothesis. This contradiction completes the proof of this theorem.

5 Some applications

Finally, consider some applications of Theorems 1, 2.

Corollary 5.1 (Buckley [1]). *Let G be a group of odd order. If all subgroups of G of prime order are normal in G, then G is supersoluble.*

Corollary 5.2 (Guo W., Shum K.P. and Skiba A.N. [21]). *If the maximal subgroups of the Sylow subgroups of G not having supersoluble supplement in G are normal in G, then G is supersoluble.*

Corollary 5.3 (Srinivasan [2]). *If the maximal subgroups of the Sylow subgroups of G are S-quasinormal in G, then G is supersoluble.*

Corollary 5.4 (Shaalan A. [4]). *Let G be a group and E a normal subgroup of G with supersoluble quotient. Suppose that all minimal subgroups of E and all its cyclic subgroups with order 4 are S-quasinormal in G. Then G is supersoluble.*

Corollary 5.5 (Ballester-Bolinches A., Pedraza-Aguilera M.C. [11]). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and G a group with normal subgroup E such that $G/E \in \mathscr{F}$. Assume that a Sylow 2-subgroup of G is abelian. If all minimal subgroups of E are permutable in G, then $G \in \mathscr{F}$.*

Corollary 5.6 (Ballester-Bolinches A., Pedraza-Aguilera M.C. [11]). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and G a group with a soluble normal subgroup E such that $G/E \in \mathscr{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of E are permutable in G, then $G \in \mathscr{F}$.*

Corollary 5.7 (Ramadan M. [3]). *Let G be a soluble group. If all maximal subgroups of the Sylow subgroups of $F(E)$ are normal in G, then G is supersoluble.*

Corollary 5.8 (Asaad M., Ramadan M. and Shaalan A. [5]). *Let G be a group and E a soluble normal subgroup of G with supersoluble quotient G/E. Suppose that all maximal subgroups of any Sylow subgroup of $F(E)$ are S-quasinormal in G. Then G is supersoluble.*

Corollary 5.9 (Asaad M., Csorgo P. [7]). *Let $\mathscr{F}$ be a saturated formation containing $\mathscr{U}$ and G be a group with a soluble normal subgroup E such that $G/E \in \mathscr{F}$. If all minimal subgroups and all cyclic subgroups with order 4 of $F(E)$ are S-quasinormal in G, then $G \in \mathscr{F}$.*

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