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Bayes Prediction under Progressive Type - II Censored Rayleigh Data

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Abstract: In present article, we discussed about Two - Sample Bayes prediction scenario under progressive Type - II right censoring scheme for two - parameter Rayleigh model. The Bayes prediction length of bounds and Bayes predictive estimator for r^{th} order statistic drawn from a future random sample of parent population, independently and with an arbitrary progressive censoring scheme are obtained. The properties of Highest Posterior Density intervals are also studied

Keywords: Bayes prediction bounds, Bayes predictive estimator, Progressive Type - II right censoring scheme.

1 Introduction

Prediction based on censored data is an important topic in many fields like as medical and engineering sciences. An important objective of a life - testing experiment is to predict the nature of future sample based on current sample. Prediction of mean, smallest or largest observation in a future sample has a topic of interest and importance in the context of quality and reliability analysis.

The objective of present paper is to predict about the nature of future behavior of an observation when sufficient information regarding the past and present behavior of an event or an observation is known or given.

A good deal of literature is available on predictive inference for Rayleigh distribution under different criterion. [7] presented highest posterior density prediction intervals for k^{th} order statistic of a future sample. The Bayes prediction for independent future sample based on Type - II doubly censored Rayleigh data have discussed by [5]. [15], based on doubly Rayleigh censored samples, derived estimation of the predictive distribution for total time on test up to a certain failure in a future sample. [13] have discussed about some Bayes prediction intervals for Rayleigh model. Some Bayes estimators for inverse Rayleigh model under different criterion have discussed by [11]. Recently, [12] presents a comparative study based on two different asymmetric loss functions for Progressive censored two - parameter Rayleigh distribution.

The prediction problems of lifetime distribution are important and have been studied by many authors. Few of those who have been extensively studied predictive inference for future observations are [6], [10], [1], [9], [16], [17], [18], [8], [2] and [14].

The probability density function and distribution function of the considered two - parameter Rayleigh distribution is

$$f(x;\theta,\sigma) = \frac{x-\sigma}{\theta^2} exp\left(-\frac{(x-\sigma)^2}{2\theta^2}\right) ; x > \sigma > 0, \theta > 0. \tag{1}$$

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and

$$F(x;\theta,\sigma) = 1 - exp\left(-\frac{(x-\sigma)^2}{2\theta^2}\right); x > \sigma > 0, \theta > 0. \quad (2)$$

Here, parameters θ and σ are known as scale and location parameter respectively ([3]).

In the present paper, our focus is on Bayes prediction length of bounds and Bayes predictive estimators for r^{th} order statistic in a future random sample drawn from parent population independently and with arbitrary progressive censoring schemes based on Two - sample plan. Both known and unknown cases of location parameter are considered here for the prediction. For evaluation of performances of the proposed procedures, a simulation study carries out also. The properties of the HPD intervals are studied in last section also.

2 The Progressive Type - II Right Censoring

The time and cost restrictions censoring are useful in life testing experiments. The censoring arises when exact lifetimes are only partially known. The progressive censoring appears to be a great importance in planned duration experiments in reliability studies. In many industrial experiments involving lifetimes of machines or units, experiments have to be terminated early and the number of failures must be limited for various reasons. In addition, some life tests require removal of functioning test specimens to collect degradation related information to failure time data. The samples that arise from such experiments are called censored samples.

The planning of experiments with aim of reducing total duration of experiment or the number of failures leads naturally to the Type - I & Type - II censoring scheme. The main disadvantage of Type - I & Type - II censoring schemes is that they do not allow removal of units at points other than the termination point of an experiment. Progressively Type - II censored sampling is an important method of obtaining data in such lifetime studies.

The Progressive Type - II right censoring scheme is describes as follows:

Let us suppose an experiment in which n independent and identical units $X_1, X_2, ..., X_n$ are placed on a life test at the beginning time and first $m; (1 \leq m \leq n)$ failure times are observed. At the time of each failure occurring prior to the termination point, one or more surviving units are removed from the test. The experiment is terminated at the time of m^{th} failure, and all remaining surviving units are removed from the test.

Let $x_{(1)} \leq x_{(2)} \leq ... \leq x_{(m)}$ are the lifetimes of completely observed units to fail and $R_1, R_2, ..., R_m; (m \leq n)$ are the numbers of units withdrawn at these failure times. Here, $R_1, R_2, ..., R_m; (m \leq n)$ all are predefined integers follows the relation

$$\sum_{j=1}^{m} R_j = n - m.$$

At first failure time $x_{(1)}$, withdraw R_1 items randomly from remaining $n-1$ surviving units. Immediately after the second observed failure time $x_{(2)}, R_2$ items are withdrawn from remaining $n-2-R_1$ surviving units at random, and so on. The experiments continue until at m^{th} failure time $x_{(m)}$, the remaining items $R_m = n - m - \sum_{j=1}^{m-1} R_j$ are withdrawn. Here, $X_{1:m:n}^{(R_1,R_2,...,R_m)}, X_{2:m:n}^{(R_1,R_2,...,R_m)}, ..., X_{m:m:n}^{(R_1,R_2,...,R_m)}$ be m ordered failure times and $(R_1, R_2, ..., R_m)$ be the progressive censoring scheme (See for details [4]).

The resulting m ordered values, which are obtained as a consequence of this type of censoring, are appropriately referred to as progressively Type - II right censored order statistics.

Further, it is noted that if

$$R_i = 0\, \forall\, i = 1, 2, ..., m-1 \Rightarrow R_m = n - m$$

Progressively Type - II right censoring scheme reduces to the conventional Type - II censoring scheme. Also, noted that if

$$R_i = 0\, \forall\, i = 1, 2, ..., m \Rightarrow n = m$$

above censoring scheme reduces to the complete sample case.

Based on progressively Type - II censoring scheme the joint probability density function of order statistics $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}, X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$ is defined as

$$f_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}(\theta,\sigma|\underline{x}) = C_p \prod_{i=1}^{m} f\left(x_{(i)};\theta,\sigma\right)\left(1-F\left(x_{(i)};\theta,\sigma\right)\right)^{R_i}; \tag{3}$$

where $f(\cdot)$ and $F(\cdot)$ are given respectively by (1) and (2) and C_p is a progressive normalizing constant defined as $C_p = n(n-R_1-1)(n-R_1-R_2-2)\ldots\left(n+1-\sum_{j=1}^{m-1} R_j - m\right)$. The progressive Type - II censored sample is denoted by $\underline{x} \equiv \left(x_{(1)}, x_{(2)}, \ldots, x_{(m)}\right)$ and $(R_1, R_2, \ldots, R_m)$ being progressive censoring scheme for considered Rayleigh model.

Subtitling (1) and (2) in (3), the joint probability density function is obtain as:

$$f_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}(\theta,\sigma|\underline{x}) = C_p A_m(\underline{x},\sigma)\ \theta^{-2m}\, exp\left(-\frac{T_p}{2\theta^2}\right); \tag{4}$$

where $A_m(\underline{x},\sigma) = \prod_{i=1}^{m}\left(x_{(i)}-\sigma\right)$ and $T_p = \sum_{i=1}^{m}\left(x_{(i)}-\sigma\right)^2(R_i+1)$.

3 Bayes Prediction Length of Bounds when Location Parameter is Known

Assuming the location parameter σ is known and scale parameter θ is a realization of a random variable. A conjugate family of prior density for parameter θ is taken as an inverted Gamma with probability density function

$$g_1(\theta) \propto \left(\frac{1}{\theta}\right)^{2\alpha+1} \exp\left(-\frac{1}{2\theta^2}\right); \alpha > 0, \theta > 0. \tag{5}$$

There is clearly no way in which one can say that one prior is better than other. It is more frequently the case that, we select to restrict attention to a given flexible family of priors, and we choose one from that family, which seems to match best with our personal beliefs. The prior (5) has advantages over many other distributions because of its analytical tractability, richness and easy interpretability.

Based on Bayes theorem, the posterior density is defined as

$$\pi(\theta|\underline{x},\sigma) = \frac{f_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}(\theta,\sigma|\underline{x})\cdot g_1(\theta)}{\int_\theta f_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}(\theta,\sigma|\underline{x})\cdot g_1(\theta)\, d\theta}. \tag{6}$$

Using (4) and (5) in (6), the posterior density is obtain as

$$\pi(\theta|\underline{x},\sigma) = \frac{C_p A_m(\underline{x},\sigma)\ \theta^{-2m}\, exp\left(-\frac{T_p}{2\theta^2}\right)\cdot\theta^{-2\alpha-1}\exp\left(-\frac{1}{2\theta^2}\right)}{\int_\theta C_p A_m(\underline{x},\sigma)\ \theta^{-2m}\, exp\left(-\frac{T_p}{2\theta^2}\right)\cdot\theta^{-2\alpha-1}\exp\left(-\frac{1}{2\theta^2}\right) d\theta}$$

$$\Rightarrow \pi(\theta|\underline{x},\sigma) = \eta^{*} exp\left(-\frac{\hat{T}_p}{2\theta^2}\right)\theta^{-2(m+\alpha)-1} \tag{7}$$

where $\eta^{*} = \frac{\left(\hat{T}_p\right)^{m+\alpha}}{\Gamma(m+\alpha)\,2^{m+\alpha-1}}$ and $\hat{T}_p = T_p + 1$.

We have $X_{1:m:n}^{(R_1,R_2,\ldots,R_m)}, X_{2:m:n}^{(R_1,R_2,\ldots,R_m)}, \ldots, X_{m:m:n}^{(R_1,R_2,\ldots,R_m)}$ be the progressive Type - II censored ordered statistics of size m from a sample of size n with progressive censoring scheme $(R_1, R_2, \ldots, R_m)$ from considered model(1). Now we assume that $Y_{1:M:N}^{(S_1,S_2,\ldots,S_M)}, Y_{2:M:N}^{(S_1,S_2,\ldots,S_M)}, \ldots, Y_{M:M:N}^{(S_1,S_2,\ldots,S_M)}$ is another (unobserved) independent progressively Type - II right censored ordered statistics of size M from another sample, of size N with progressive censoring scheme $(S_1, S_2, \ldots, S_M)$ from same model (1).

The first sample is considered as "informative" (past) sample, whereas the second sample is considered as the "future" sample. Now, let us take that $Y_{(r)}$ be the r^{th} ; $1 \leq r \leq M$ order statistic from the future sample of size M. Based on informative progressive Type - II right censored sample, the prediction of r^{th} order statistic from future sample is the objective of the present article.

Following [4], the probability density function of $Y_{(r)}$ (r^{th} order statistic) is obtain as

$$h\left(y_{(r)}|\theta,\sigma\right) = \phi f\left(y_{(r)}|\theta,\sigma\right)\sum_{i=1}^{r}\lambda_i^*\left(1-F\left(y_{(r)}|\theta,\sigma\right)\right)^{\lambda_i-1}$$

$$= \phi\frac{\left(y_{(r)}-\sigma\right)}{\theta^2}\sum_{i=1}^{r}\lambda_i^*\,exp\left(-\lambda_i\frac{\left(y_{(r)}-\sigma\right)^2}{2\theta^2}\right); \tag{8}$$

where $\lambda_i = \sum_{j=i}^{M}(S_j+1) = N - \sum_{j=1}^{i-1}(S_j+1)$, $\phi = \prod_{i=1}^{r}\lambda_i$ and $\lambda_i^* = \prod_{j=i}^{r}\frac{1}{(\lambda_j-\lambda_i)}$, $\forall\, i \neq j, r > 1, \lambda_1^* = 1$ for $r = 1$.

The Bayes predictive density function for r^{th} order statistic $Y_{(r)}$ is thus defined as

$$h^*\left(y_{(r)}|\underline{x},\theta,\sigma\right) = \int_\theta h\left(y_{(r)}|\theta,\sigma\right)\cdot\pi\left(\theta|\underline{x},\sigma\right)d\theta. \tag{9}$$

Applying (7) and (8) in (9), we get

$$h^*\left(y_{(r)}|\underline{x},\theta,\sigma\right) = \phi\,\eta^*\left(y_{(r)}-\sigma\right)\sum_{i=1}^{r}\lambda_i^*\int_\theta exp\left(-\frac{\lambda_i\left(y_{(r)}-\sigma\right)^2+\hat{T}_p}{2\theta^2}\right)\cdot\theta^{-2(m+\alpha)-3}\,d\theta$$

$$\Rightarrow h^*\left(y_{(r)}|\underline{x},\theta,\sigma\right) = 2\phi(m+\alpha)\sum_{i=1}^{r}\lambda_i^*\frac{\left(\lambda_i\left(y_{(r)}-\sigma\right)^2+\hat{T}_p\right)^{-(m+\alpha+1)}}{\left(y_{(r)}-\sigma\right)\left(\hat{T}_p\right)^{m+\alpha}}. \tag{10}$$

The Bayes prediction bounds for $Y_{(r)}; 1 \leq r \leq M$ are obtained by evaluating $Pr\left(Y_{(r)} \geq \varepsilon|\underline{x}\right)$, for some given value of ε. We have from (10)

$$Pr\left(Y_{(r)} \geq \varepsilon|\underline{x}\right) = \int_\varepsilon^\infty h^*\left(y_{(r)}|\underline{x},\theta,\sigma\right)d\,y_{(r)}$$

$$Pr\left(Y_{(r)} \geq \varepsilon|\underline{x}\right) = 2\phi(m+\alpha)\left(\hat{T}_p\right)^{m+\alpha}\int_\varepsilon^\infty\left(y_{(r)}-\sigma\right)\sum_{i=1}^{r}\lambda_i^*\left(\lambda_i\left(y_{(r)}-\sigma\right)^2+\hat{T}_p\right)^{-(m+\alpha+1)}dy_{(r)}$$

$$\Rightarrow Pr\left(Y_{(r)} \geq \varepsilon|\underline{x}\right) = \phi\sum_{i=1}^{r}\left(\frac{\lambda_i^*}{\lambda_i}\right)\left(\lambda_i\frac{(\varepsilon-\sigma)^2}{\hat{T}_p}+1\right)^{-m-\alpha}. \tag{11}$$

Now, One - Sided Bayes prediction bounds are obtain by solving following equality

$$Pr\left(Y_{(r)} \geq l_1|\underline{x}\right) = 1-\frac{\tau}{2} \tag{12}$$

and

$$Pr\left(Y_{(r)} \geq l_2|\underline{x}\right) = \frac{\tau}{2}. \tag{13}$$

Here, l_1 and l_2 are the lower and upper Bayes prediction bounds for the random variable $Y_{(r)}$ and $1-\tau$ is the confidence prediction coefficient.

Using (11) in (12) and (13) we get the lower and upper Bayes prediction bounds as:

$$\phi^{-1}\left(1-\frac{\tau}{2}\right) = \sum_{i=1}^{r}\left(\frac{\lambda_i^*}{\lambda_i}\right)\left(\lambda_i\frac{(l_1-\sigma)^2}{\hat{T}_p}+1\right)^{-m-\alpha}$$

and

$$\phi^{-1}\left(\frac{\tau}{2}\right)=\sum_{i=1}^{r}\left(\frac{\lambda_i^*}{\lambda_i}\right)\left(\lambda_i\frac{(l_2-\sigma)^2}{\hat{T}_p}+1\right)^{-m-\alpha}. \tag{14}$$

Further simplification of the equalities (14) does not possible. A numerical technique is applied here for obtaining the values of l_1 and l_2 for some τ.

For a particular case, substituting $r=1$ in the predictive survival function (11), for predicting the first item $Y_{(1)}$ of next item to fail, and is obtain as

$$Pr\left(Y_{(1)}\geq\varepsilon|\underline{x}\right)=\left(N\frac{(\varepsilon-\sigma)^2}{\hat{T}_p}+1\right)^{-m-\alpha}. \tag{15}$$

The, One - Sided Bayes prediction lower and upper bounds for $Y_{(1)}$ are now obtain as

$$l_1=\sigma+\sqrt{\tau^*\frac{\hat{T}_p}{N}};\ \tau^*=\left\{\left(1-\frac{\tau}{2}\right)^{-1/(\alpha+m)}-1\right\}$$

and

$$l_2=\sigma+\sqrt{\tau^{**}\frac{\hat{T}_p}{N}};\ \tau^{**}=\left\{\left(\frac{\tau}{2}\right)^{-1/(\alpha+m)}-1\right\}. \tag{16}$$

Hence, the Bayes prediction length of bounds for $Y_{(1)}$ is

$$L=l_2-l_1. \tag{17}$$

Now, the Bayes predictive estimator for r^{th} order statistic $Y_{(r)}; 1\leq r\leq M$ under squared error loss function is obtained as

$$\hat{y}_{(r)}=E\left(Y_{(r)}|\underline{x}\right)=\int_{\varepsilon}Pr\left(Y_{(r)}\geq\varepsilon|\underline{x}\right)d\varepsilon=\phi\frac{\sqrt{\hat{T}_p}}{2}\sum_{i=1}^{r}\left(\frac{\lambda_i^*}{\lambda_i^{3/2}}\right)\Phi(z); \tag{18}$$

where $\Phi(z)=\int_{v}^{\infty}\frac{(1+z)^{-m-\alpha}}{\sqrt{z}}dz$ and $v=\frac{\lambda_i\sigma^2}{\hat{T}_p}$.

4 Bayes Prediction Length of Bounds when Location Parameter is Unknown

When, scale and location both parameters are considered as random variable, the joint probability density function under progressive Type - II censoring criterion is given by

$$f_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}\left(\theta,\sigma|\underline{x}\right)=C_pA_m\left(\underline{x},\sigma\right)\frac{1}{\theta^{2m}}exp\left(-\frac{T_p}{2\theta^2}\right). \tag{19}$$

It is clear from (19) that, the function T_p depends on the location parameter σ. Hence, in present case when both parameters are consider being random variable, the joint prior density for parameter θ and σ is considered as

$$g\left(\theta,\sigma\right)=g_2\left(\theta|\sigma\right)\cdot g_3\left(\sigma\right). \tag{20}$$

Here $g_2\left(\theta|\sigma\right)$ and $g_3\left(\sigma\right)$ both are inverted gamma densities and defined as

$$g_2\left(\theta|\sigma\right)=\frac{\theta^{-2\sigma-1}e^{-1/2\theta^2}}{\Gamma(\sigma)2^{\sigma-1}};\ \theta>0,\sigma>0 \tag{21}$$

and

$$g_3\left(\sigma\right)=\frac{\sigma^{-2\beta-1}e^{-1/2\sigma^2}}{\Gamma(\beta)2^{\beta-1}};\ \beta>0,\sigma>0. \tag{22}$$

The joint posterior density function is now obtained as

$$\pi^*(\theta,\sigma|\underline{x}) = \frac{f_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}(\theta,\sigma|\underline{x})\cdot g(\theta,\sigma)}{\int_\sigma\int_\theta f_{X_{1:m:n},X_{2:m:n},\ldots,X_{m:m:n}}(\theta,\sigma|\underline{x})\cdot g(\theta,\sigma)\,d\theta d\sigma}$$

$$= \frac{\left(\theta^{-2(\sigma+m)-1}exp\left(-\frac{\hat{T}_p}{2\theta^2}\right)\right)\left(\frac{A_m(\underline{x},\sigma)}{\Gamma(\sigma)2^\sigma}\sigma^{-2\beta-1}exp\left(-\frac{1}{2\sigma^2}\right)\right)}{\int_\sigma\frac{A_m(\underline{x},\sigma)}{\Gamma(\sigma)2^\sigma}\sigma^{-2\beta-1}exp\left(-\frac{1}{2\sigma^2}\right)\int_\theta\theta^{-2(\sigma+m)-1}exp\left(-\frac{\hat{T}_p}{2\theta^2}\right)d\theta d\sigma}$$

$$\Rightarrow \pi^*(\theta,\sigma|\underline{x}) = \bar{\bar{\sigma}}\left(\theta^{-2(\sigma+m)-1}exp\left(-\frac{\hat{T}_p}{2\theta^2}\right)\right)\left(\frac{A_m(\underline{x},\sigma)}{\Gamma(\sigma)2^\sigma}\sigma^{-2\beta-1}exp\left(-\frac{1}{2\sigma^2}\right)\right); \quad (23)$$

where $\bar{\bar{\sigma}} = \frac{1}{2^{m-1}\bar{\sigma}}$ and $\bar{\sigma} = \int_\sigma\frac{\Gamma(m+\sigma)}{\Gamma(\sigma)}\frac{A_m(\underline{x},\sigma)}{(\hat{T}_p)^{m+\alpha}}\sigma^{-2\beta-1}exp\left(-\frac{1}{2\sigma^2}\right)d\sigma.$

On similar line, the Bayes predictive density function for $Y_{(r)}; 1\le r\le M$ is obtained as

$$h^{**}\left(y_{(r)}|\underline{x},\theta,\sigma\right) = \int_\sigma\int_\theta h\left(y_{(r)}|\theta,\sigma\right)\cdot\pi^*(\theta,\sigma|\underline{x})\,d\theta d\sigma$$

$$= \phi\,\bar{\bar{\sigma}}\int_\sigma\left(y_{(r)}-\sigma\right)\sum_{i=1}^{r}\lambda_i^*\frac{A_m(\underline{x},\sigma)}{\Gamma(\sigma)2^\sigma}\sigma^{-2\beta-1}exp\left(-\frac{1}{2\sigma^2}\right)$$
$$\cdot\int_\theta\theta^{-2(m+\sigma+1)-1}exp\left(-\frac{\lambda_i\left(y_{(r)}-\sigma\right)^2+\hat{T}_p}{2\theta^2}\right)d\theta d\sigma$$

$$\Rightarrow h^{**}\left(y_{(r)}|\underline{x},\theta,\sigma\right) = \frac{2\phi}{\bar{\bar{\sigma}}}\sum_{i=1}^{r}\lambda_i^*\int_\sigma\frac{A_m(\underline{x},\sigma)\Gamma(m+\sigma+1)}{\Gamma(\sigma)}\sigma^{-2\beta-1}$$

$$exp\left(-\frac{1}{2\sigma^2}\right)\left(y_{(r)}-\sigma\right)\left(\lambda_i\left(y_{(r)}-\sigma\right)^2+\hat{T}_p\right)^{-m-\sigma-1}d\sigma \quad (24)$$

Hence, the Bayes prediction bounds for $Y_{(r)}; 1\le r\le M$ are obtained similarly by evaluating $Pr\left(y_{(r)}\ge\varepsilon|\underline{x}\right)$, for some given value of ε. We have from (24)

$$Pr\left(Y_{(r)}\ge\varepsilon|\underline{x}\right) = \int_\varepsilon^\infty h^{**}\left(y_{(r)}|\underline{x}\right)dy_{(r)}$$

$$= \frac{2\phi}{\bar{\sigma}}\sum_{i=1}^{r}\lambda_i^*\int_\sigma\frac{A_m(\underline{x},\sigma)\,\Gamma(m+\sigma+1)}{\Gamma(\sigma)}\sigma^{-2\beta-1}exp\left(-\frac{1}{2\sigma^2}\right)$$
$$\cdot\int_{y_{(r)}=\varepsilon}^\infty\left(y_{(r)}-\sigma\right)\left(\lambda_i\left(y_{(r)}-\sigma\right)^2+\hat{T}_p\right)^{-m-\sigma-1}dy_{(r)}d\sigma$$

$$\Rightarrow Pr\left(Y_{(r)}\ge\varepsilon|\underline{x}\right) = \frac{\phi}{\bar{\bar{\sigma}}}\sum_{i=1}^{r}\left(\frac{\lambda_i^*}{\lambda_i}\right)\int_\sigma\Omega_p(\sigma)\,d\sigma \quad (25)$$

where $\Omega_p(\sigma) = \frac{A_m(\underline{x},\sigma)\Gamma(m+\sigma+1)}{(m+\sigma)\Gamma(\sigma)}\sigma^{-2\beta-1}exp\left(-\frac{1}{2\sigma^2}\right)\left(\lambda_i(\varepsilon-\sigma)^2+\hat{T}_p\right)^{-m-\sigma}.$

Using (25) in (12) and (13) we get One - Sided lower and upper Bayes prediction bounds as

$$\left(\frac{\bar{\sigma}}{\phi}\right)\left(1-\frac{\tau}{2}\right)=\sum_{i=1}^{r}\left(\frac{\lambda_i^*}{\lambda_i}\right)\int_{\sigma}\Omega_{p_{l_1}}(\sigma)\,d\sigma$$

and

$$\left(\frac{\bar{\sigma}}{\phi}\right)\left(\frac{\tau}{2}\right)=\sum_{i=1}^{r}\left(\frac{\lambda_i^*}{\lambda_i}\right)\int_{\sigma}\Omega_{p_{l_2}}(\sigma)\,d\sigma \tag{26}$$

where $\Omega_{p_{l_j}}(\sigma)=\frac{A_m(\underline{x},\sigma)\Gamma(m+\sigma+1)}{(m+\sigma)\Gamma(\sigma)}\,\sigma^{-2\beta-1}\,exp\left(-\frac{1}{2\sigma^2}\right)\left(\lambda_i\,(l_j-\sigma)^2+\hat{T}_p\right)^{-m-\sigma}$, $j=1,2.$

Further simplifications of equalities (26) do not possible. A numerical technique is applied here for obtaining the values of l_1 and l_2 for some τ.

Similarly, the Bayes predictive estimator for r^{th} order statistic $Y_{(r)}; 1\leq r\leq M$ under squared error loss function in present case is

$$\hat{\hat{y}}_{(r)}=E\left(Y_{(r)}|\underline{x}\right)=\int_{\varepsilon}Pr\left(Y_{(r)}\geq\varepsilon|\underline{x}\right)d\varepsilon=\frac{\phi}{2\bar{\sigma}}\sum_{i=1}^{r}\left(\frac{\lambda_i^*}{\lambda_i^{3/2}}\right)\int_{\sigma}\bar{\Omega}_p(\sigma)\,d\sigma; \tag{27}$$

where $\bar{\Omega}_p(\sigma)=\frac{A_m(\underline{x},\sigma)\Gamma(m+\sigma+1)}{(m+\sigma)\Gamma(\sigma)}\,\sigma^{-2\beta-1}\,exp\left(-\frac{1}{2\sigma^2}\right)\left(\hat{T}_p\right)^{-m-\sigma+\frac{1}{2}}\Phi(z)$, $\Phi(z)=\int_{\nu}^{\infty}\sqrt{z}\,(1+z)^{-m-\alpha}\,dz$ and $\nu=\frac{\lambda_i\sigma^2}{\hat{T}_p}$.

5 H. P. D. INTERVALS

In this section, our objective is to study about highest posterior density (HPD) interval for unknown parameter θ of the considered model. Since, the posterior density $\pi(\theta|\underline{x},\sigma)$ corresponding to the parameter θ is unimodel. Thus, $100(1-\tau)\%$ HPD interval $[H_1,H_2]$ for the parameter θ must satisfy the following equations simultaneously.

$$\int_{H_1}^{H_2}\pi(\theta|\underline{x},\sigma)\,d\theta=1-\tau \tag{28}$$

and

$$\pi(H_1|\underline{x},\sigma)=\pi(H_2|\underline{x},\sigma). \tag{29}$$

Now, the expression (28) & (29) rewritten as

$$\frac{1}{\Gamma(m+\alpha)}\int_{H'}^{H''}e^{-z}z^{\alpha+m-1}\,dz=1-\tau$$

$$\Rightarrow\left[\gamma\left(\alpha+m,H''\right)-\gamma\left(\alpha+m,H'\right)\right]=(1-\tau)\Gamma(m+\alpha) \tag{30}$$

and

$$\frac{H_2}{H_1}=exp\left\{-\left(\frac{\hat{T}_p}{2\,(2\alpha+2m-1)}\right)\left(\frac{1}{H_2^2}-\frac{1}{H_1^2}\right)\right\}; \tag{31}$$

where $H'=\frac{\hat{T}_p}{2H_2^2}$ and $H''=\frac{\hat{T}_p}{2H_1^2}$.

Solve simultaneously the equations (30) and (31) to obtain the highest posterior density limits H_1 and H_2.

6 Numerical Illustration

In present section, we carry out the performance of Bayes prediction lengths of bounds, Bayes predictive estimator for r^{th} order statistic and HPD intervals for the future order statistic under progressively Type - II censored sample.

1. For given values of the prior parameter α, random values of the parameter θ is generated from the prior density (5).

2. Using generated values of θ obtained in steps (1), we generate a progressively Type - II censored sample, for the case of known location parameter, of size m for a given values of censoring scheme $R_i; i = 1,2,...,m$, from the considered Rayleigh model, according to an algorithm proposed by [4]. The censoring scheme for different values of m is presented in Table (1).

3. For the different informative sample sizes $m = 10,10,20$ and future sample size $N(= 10,20)$ Table (2) displays the Bayes predictive length of bounds for the r^{th} order statistics $y_{(r)}$.

4. We consider here without loss generality $S_i = 0 \,\forall\, i = 1,2,...,M$, which represents the ordinary order statistics. The smallest, middle, and largest future ordered lifetimes, which are practically of some special interest, are only predicted.

5. The lengths of prediction bounds of $y_{(r)}$, are calculated form $1,00,000$ generated future ordered samples each of size $N = 20$ of the Rayleigh density when location parameter is known.

6. For given θ and σ with $N = 10$, generate the future ordered samples of size $m(= 10,10,20)$ using following relation $x_i = \sigma\sqrt{2\theta^2\, log\,(1-U_i)}$. Here, U_i are independently distributed $U(0,1)$.

7. For selected values of $\sigma\,(= 0.50,1.00,5.00,10.00)$ and prior parameter $\alpha(= 0.50,1.00,\ 5.00,10.00)$; the prediction length of bounds are obtained and presented in Table (2) only for $(\alpha = \sigma = 0.50,10.00)$.

8. It is observed from Table (2) that, the length of bounds tend to be wider as the value of scale parameter increases when other parametric values are consider to be fixed. Opposite trend has been seen when prior parameter α increases. It is also noted further that when confidence level τ decreases the length of intervals tends to be closer.

   In case, when both parameters are considered to be random variable, the length of bounds are obtained as follows:

1. We generate location parameter σ from (22) for given values of the prior parameter β. Using generated values of σ, obtained the values of the parameter θ by using (21).

2. Following Step (2) to (6), the lengths of the Bayes prediction bounds for r^{th} order statistics are obtained and presented in Table (3) for selected parametric values.

3. All the behaviors are seen to be similar as compared to the known case of location parameter. The gains in magnitude in the length of the prediction bounds are robust.

4. Table (4) and (5) shows the estimate values of the $y_{(r)}$ when location parameter are known and unknown respectively. Following above steps, the estimate values are obtained. All the properties have been seen similar as discuss above.

5. Based on above steps, the HPD intervals have been obtained also and presents in Table (6). All the properties have been seen similar as discuss above for Bayes prediction length of bounds.

Table 1: Censoring Scheme for Different Values of m

Case	m	$R_i \,\forall\, i = 1,2,...,m$
1	10	1 2 1 0 0 1 2 0 0 0
2	10	1 0 0 3 0 0 1 0 0 1
3	20	1 0 2 0 0 1 0 2 0 0 0 1 0 0 0 1 0 0 1 0

Table 2: Bayes Prediction Lengths of Bounds for $y_{(r)}$ (When σ Known)

$\sigma \rightarrow$			0.50			10.00		
$\alpha \downarrow$	m	$\tau \rightarrow$	90%	95%	99%	90%	95%	99%
	10	$y_{(1)}$	0.2748	0.2814	0.3286	0.4561	0.4671	0.5454
		$y_{(5)}$	0.1201	0.1230	0.1436	0.1993	0.2042	0.2384
		$y_{(10)}$	0.1132	0.1159	0.1353	0.1879	0.1924	0.2246
0.50	10	$y_{(1)}$	0.2720	0.2814	0.3286	0.4569	0.4727	0.5520
		$y_{(5)}$	0.1212	0.1254	0.1464	0.2036	0.2107	0.2459
		$y_{(10)}$	0.1120	0.1159	0.1353	0.1881	0.1947	0.2273
	20	$y_{(1)}$	0.2701	0.2766	0.3230	0.4510	0.4619	0.5394
		$y_{(5)}$	0.1178	0.1206	0.1408	0.1967	0.2014	0.2351
		$y_{(10)}$	0.1062	0.1088	0.1270	0.1773	0.1817	0.2121
	10	$y_{(1)}$	0.1337	0.1355	0.1373	0.4519	0.4561	0.4918
		$y_{(5)}$	0.0584	0.0592	0.0600	0.1852	0.1994	0.2150
		$y_{(10)}$	0.0550	0.0557	0.0565	0.1746	0.1879	0.2025
10.00	10	$y_{(1)}$	0.1377	0.1382	0.1387	0.4245	0.4616	0.4977
		$y_{(5)}$	0.0614	0.0616	0.0618	0.1892	0.2058	0.2217
		$y_{(10)}$	0.0567	0.0569	0.0571	0.1748	0.1901	0.2049
	20	$y_{(1)}$	0.1291	0.1316	0.1342	0.4190	0.4511	0.4864
		$y_{(5)}$	0.0564	0.0575	0.0586	0.1827	0.1967	0.2120
		$y_{(10)}$	0.0508	0.0518	0.0528	0.1647	0.1774	0.1912

Table 3: Bayes Prediction Lengths of Bounds for $y_{(r)}$ (When σ Unknown)

$\beta \downarrow$	m	$\tau \rightarrow$	90%	95%	99%
	10	$y_{(1)}$	0.3019	0.3120	0.3446
		$y_{(5)}$	0.1320	0.1364	0.1506
		$y_{(10)}$	0.1244	0.1285	0.1419
0.50	10	$y_{(1)}$	0.2964	0.3067	0.3581
		$y_{(5)}$	0.1321	0.1367	0.1596
		$y_{(10)}$	0.1221	0.1263	0.1475
	20	$y_{(1)}$	0.2968	0.3067	0.3387
		$y_{(5)}$	0.1294	0.1337	0.1477
		$y_{(10)}$	0.1167	0.1206	0.1332
	10	$y_{(1)}$	0.1469	0.1502	0.1506
		$y_{(5)}$	0.0642	0.0656	0.0658
		$y_{(10)}$	0.0604	0.0618	0.0620
10.00	10	$y_{(1)}$	0.1499	0.1505	0.1510
		$y_{(5)}$	0.0668	0.0671	0.0673
		$y_{(10)}$	0.0617	0.0620	0.0622
	20	$y_{(1)}$	0.1418	0.1459	0.1472
		$y_{(5)}$	0.0620	0.0638	0.0643
		$y_{(10)}$	0.0558	0.0574	0.0579

Table 4: Estimate values of $y_{(r)}$ Under SELF (When σ Known)

$\alpha\downarrow$	m	$\sigma\rightarrow$	0.50	1.00	5.00	10.00
		$\hat{y}_{(1)}$	1.0546	1.1695	1.2969	1.4382
	10	$\hat{y}_{(5)}$	1.1234	1.2458	1.3816	1.5322
		$\hat{y}_{(10)}$	1.1655	1.2925	1.4334	1.5896
		$\hat{y}_{(1)}$	1.0602	1.1757	1.3038	1.4459
0.50	10	$\hat{y}_{(5)}$	1.1223	1.2446	1.3802	1.5306
		$\hat{y}_{(10)}$	1.1666	1.2937	1.4347	1.5910
		$\hat{y}_{(1)}$	1.0513	1.1659	1.2930	1.4339
	20	$\hat{y}_{(5)}$	1.1223	1.2446	1.3802	1.5306
		$\hat{y}_{(10)}$	1.1578	1.2840	1.4239	1.5791
		$\hat{y}_{(1)}$	0.9656	1.0708	1.1875	1.3169
	10	$\hat{y}_{(5)}$	1.0286	1.1407	1.2650	1.4029
		$\hat{y}_{(10)}$	1.0672	1.1835	1.3125	1.4555
		$\hat{y}_{(1)}$	0.9708	1.0766	1.1939	1.3240
10.00	10	$\hat{y}_{(5)}$	1.0276	1.1396	1.2638	1.4015
		$\hat{y}_{(10)}$	1.0682	1.1846	1.3137	1.4569
		$\hat{y}_{(1)}$	0.9626	1.0675	1.1838	1.3128
	20	$\hat{y}_{(5)}$	1.0276	1.1396	1.2638	1.4015
		$\hat{y}_{(10)}$	1.0601	1.1756	1.3037	1.4458

Table 5: Estimate values of $y_{(r)}$ Under SELF (When σ Unknown)

m	$\beta\rightarrow$	0.50	1.00	5.00	10.00
	$y_{(1)}$	1.1801	1.2969	1.4408	1.5980
10	$y_{(5)}$	1.2570	1.3815	1.5349	1.7024
	$y_{(10)}$	1.3041	1.4333	1.5924	1.7662
	$y_{(1)}$	1.1863	1.3038	1.4485	1.6065
10	$y_{(5)}$	1.2558	1.3802	1.5333	1.7006
	$y_{(10)}$	1.3054	1.4347	1.5939	1.7677
	$y_{(1)}$	1.1764	1.2929	1.4365	1.5932
20	$y_{(5)}$	1.2558	1.3802	1.5333	1.7006
	$y_{(10)}$	1.2955	1.4239	1.5819	1.7545

Table 6: Highest Posterior Density Intervals ($\beta = 10.00$)

m	$\alpha\downarrow$	H_1			H_2		
	0.50	1.1756	1.1600	1.1447	2.2847	2.2111	2.1398
	1.00	1.1636	1.1481	1.1328	2.1856	2.1152	2.0471
10	5.00	1.1530	1.1377	1.1225	2.1063	2.0385	1.9728
	10.00	1.1053	1.0906	1.0761	1.9771	1.9136	1.8520
	0.50	1.0703	1.0561	1.0422	2.2544	2.1818	2.1114
	1.00	1.0624	1.0482	1.0344	2.1566	2.0871	2.0200
10	5.00	1.0571	1.0431	1.0292	2.0785	2.0115	1.9466
	10.00	1.0029	0.9896	0.9765	1.9510	1.8881	1.8273
	0.50	1.0658	1.0519	1.0378	1.8724	1.7071	1.6213
	1.00	1.0464	1.0325	1.0189	1.8380	1.6782	1.6002
20	5.00	1.0313	1.0177	1.0042	1.8089	1.6583	1.5817
	10.00	0.9682	0.9553	0.9426	1.7342	1.5898	1.5163

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