Travelling Wave Solution of the Unsteady BGK Model for a Rarefied Gas Affected by a Thermal Radiation Field.

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Abstract: In the present study, a development of the paper [JNET, 2011, 36 (1), 75-98] is introduced. The non-stationary BGK (Bhatnager- Gross- Krook) model of the kinetic equations for a rarefied gas affected by nonlinear thermal radiation field is solved instead of the stationary equations. The travelling wave solution method is used to get the exact solution of the nonlinear partial differential equations. These equations were produced from applying the moment method to the unsteady Boltzmann equation. Now, a system of nonlinear partial differential equations should be solved in place of nonlinear ordinary differential equations, which represent an arduous task. The unsteady solution gives the problem a great generality and more applications. The new problem is investigated to follow the behavior of the macroscopic properties of the rarefied gas such as the temperature and concentration. The non-equilibrium thermodynamic properties of the system (gas + the heated plate) are investigated. The entropy, entropy flux, entropy production, thermodynamic forces, kinetic coefficients are obtained for the mixture. The verification of the Boltzmann H-theorem, Le Chatelier principle and the second law of thermodynamic for the system, are presented. The ratios between the different contributions of the internal energy changes based upon the total derivatives of the extensive parameters are estimated via the Gibbs formula. 3D-graphics illustrating the calculated variables are drawn to predict their behavior and the results are discussed.

Keywords: Rarefied gases; Thermal Radiation field; B.G.K. model; Exact solutions; Nonlinear partial differential equations; Unsteady Boltzmann kinetic equation; Moment Method; Irreversible thermodynamics; Gibbs formula.

1.Introduction

All matter emits thermal radiation (TR) continuously, and consequently TR is an inherent part of our environment. Radiative heat transfer is important in system analysis particularly when high temperatures are involved, cryogenic systems are also considered, when radiation is being utilized as a source flux, or when radiative transfer is the primary mode of heat rejection. Some application examples where TR transfer is of primary importance include solar collectors, boilers and furnaces, spacecraft cooling systems, and cryogenic fuel storage systems [1].

The radiative processes play a major role in the thermodynamics of the Earth system. For this purpose, researchers have used simple blackbody (BB) types of planetary models to, theoretically, estimate planetary entropy production rates. The analysis of simple radiative models of the Earth system provides insight into its thermodynamic behavior even though it is complex [2].

From a thermodynamic perspective, thermal radiation (TR) exchange, i.e., incoming sunlight and outgoing TR, is the only significant form of energy transfer between the Earth and the universe. Further, processes such as absorption and emission dominate planetary entropy production, and the non-uniform absorption of solar radiation (SR) on the Earth causes circulation of the atmosphere and oceans [2]. They have analyzed simple blackbody type radiative models to investigate the thermodynamic behavior of the Earth's system and to estimate planetary temperature and entropy production rates [2]. It is more accurate to model the Earth system as a gray-body because absorption of sunlight and emission of TR are substantially less than that of a blackbody [2].

Some authors in both linearized and non-linearized radiation heat flux formulas [3-8] investigated the gas, influenced by a thermal radiation field. Usually, they consider that the gas is dense, so that it obeys Navier-Stokes equations. However, to the best of my knowledge, the situation when a nonlinear thermal radiation force

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acting on a rarefied neutral gas has not yet been investigated within the framework of the molecular gas dynamics with the unsteady kinetic Boltzmann equation. Harmonious with this great importance of studying the effect of thermal radiation field on gases, the enhancement and the development of the previous paper [8] are introduced in this paper. I solve the non-stationary Krook kinetic equation model for a rarefied gas affected by nonlinear thermal radiation field, instead of the stationary equation.

Our aim in this paper is as follows: in section (2) to introduce a new unsteady approach for studying the influence of thermal radiation field on a rarefied neutral gas. For this purpose, I use the unsteady kinetic Boltzmann equation instead of the Navier–Stokes equations, which are satisfied only for the dense gases [9-10]. We insert the radiation field effect into the term force of the Boltzmann equation as a radiation force [8]. This idea was applied on an unsteady problem of the half space filled by a neutral gas specified by a flat rested heated plate in a frame co-moving with the gas. I use Liu-Lees model for two-stream Maxiwallian distribution functions and the moment method to predict the behavior of the macroscopic properties of the gas in an unsteady state. Specifically, the temperature and concentration that in turn are substitute into the corresponding distribution functions. Tackling this, in section (3), permits us to study the important non-equilibrium thermodynamic properties of the system (gas + heated plate). Namely, we obtain the entropy, entropy flux, entropy production, thermodynamic forces, and kinetic coefficients. We search out the verification of the Boltzmann H-theorem, Le Chatelier principle, the second law of thermodynamic and the celebrated Onsager's reciprocity relation for the system. The ratios between the different contributions of the internal energies changes based upon the total derivatives of the extensive parameters are predicted via the Gibbs's formula. Sections (4 and 5) show the discussion and conclusions of the results applied to the Helium gas for various plate temperatures.

2.The Physical Problem and Mathematical Formulation

Let us assume that the upper half of the space ($y \geq 0$), which is bounded by an infinite immobile flat plate (y=0), is filled with a monatomic neutral dilute gas with a uniform pressure Ps [11-15] and the plate is heated suddenly to produce thermal radiation field. The flow is considered unsteady, one-dimensional, and compressible. In a frame, co-moving with the fluid the behavior of the gas is studied under the assumptions that [12]:

(i) At the rest plate boundary, the velocities of the incident and reflected particles are equal; but of opposite sign. This is happened according to Maxwell formula of momentum defuse reflection. On the other hand, the exchange will be due to only the temperature difference between the particles and the heated plate, taking the form of full energy

accommodation [12].

(ii) The gas is considered gray absorbing-emitting but not a scattering medium.

(iii) A thermal radiation force is acting from the plate on the gas in vector notation [16-18] as follows:

$$\vec{F} = \frac{-4\sigma_s}{3n_s c}\vec{\nabla}T^4(y,t) \Rightarrow F_y = \frac{-16\sigma_s T^3}{3n_s c}\frac{\partial T(y,t)}{\partial y} \qquad (1)$$

For unsteady motion, the process in the system under study subject to a thermal radiation force Fy can be expressed in terms of the Boltzmann kinetic equation in the BGK model written in the form:

$$\frac{\partial f}{\partial t} + C_y\frac{\partial f}{\partial y} + \frac{F_y}{m}\frac{\partial f}{\partial C_y} = \frac{(f_0 - f)}{\tau} \qquad (2)$$

Where

$$f_0 = \frac{n(y,t)}{(2\pi RT(y,t))^{\frac{3}{2}}}\exp\left[\frac{-C^2}{(2RT(y,t))}\right] \qquad (3)$$

$$: C^2 = C_x^2 + C_y^2 + C_z^2$$

Lee's moment method [19-27] for the solution of the Boltzmann's equation is employed here. One of the most important advantages of this method is that the surface boundary conditions are easily satisfied. Maxwell converted the Maxwell-Boltzmann equation into an integral equation of transfer, or moment equation, for any quantity Q that is a function only of the molecular velocity components. The distribution function used there should be considered as a suitable weighting function that is not the exact solution of the Maxwell-Boltzmann equation in general. Lees found that the distribution function employed in Maxwell's moment equation must satisfy the following basic requirements [20,27]: (i) It must have the "two-sided" character that is an essential feature of highly rarefied gas flows.(ii) It must be capable of providing a smooth transition from free molecule flows to the continuum regime.(iii) It should lead to the simplest possible set of differential equations and boundary conditions consistent with conditions (i) and (ii). When the application of heat to a gas causes it to expand, it is thereby rendered rarer than the neighboring parts of the gas; and it tends to form an upward current of the heated gas, which is of course accompanied with a current of the more remote parts of the gas in the opposite direction. The fresh portions of gas are brought into the neighborhood of the source of heat, carrying their heat with them into other regions [28]. We assume the temperature of the upward going gas particles is T_1 while the temperature of the downward going gas particles is T_2. The corresponding concentrations are n_1 and n_2.

Making use of the Liu-Lees model of the two–stream Maxwellian distribution function near the plate suggested by Kashmarov [29] in the form:

$$f = \begin{cases} f_1 = \dfrac{n_1(y,t)}{(2\pi R T_1(y,t))^{\frac{3}{2}}} exp[\dfrac{-C^2}{(2RT_1(y,t))}], \\ \text{For } C_y > 0 \quad \uparrow \\ f_2 = \dfrac{n_2(y,t)}{(2\pi R T_2(y,t))^{\frac{3}{2}}} exp[\dfrac{-C^2}{(2RT_2(y,t))}], \\ \text{For } C_y < 0 \quad \downarrow \end{cases} \quad (4)$$

The velocity distribution function f is not directly of interest to us, in this stage, but the moments of the distribution function are of interest [9,10]. Therefore, we derive the Maxwell's Moment equations by multiplying the Boltzmann equation by a function of velocity $Q_i(C)$ and integrating over the velocity space. How many and what forms of Q_i will be used is dependent on how many unknown variables need to be determined and how many equations need to be solved [27]. Multiplying equation (2) by some functions of velocity $Q_i = Q_i(C)$, and integrating with respect to C taking into consideration the discontinuity of the distribution function caused by the cone of influence [29]. Jeans [30], and Chapman and cowling [31] showed that the resulting equation can then be written as:

$$\frac{\partial}{\partial t}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} Q_i f_2 d\underline{C} + \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} Q_i f_1 d\underline{C}\right]$$

$$+\frac{\partial}{\partial y}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} Q_i C_y f_2 d\underline{C} + \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} Q_i C_y f_1 d\underline{C}\right]$$

$$-\frac{1}{m}(F_y)\left[\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f_2 \frac{dQ_i}{dC_y} d\underline{C} + \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} f_1 \frac{dQ_i}{dC_y} d\underline{C}\right] = \quad (2)$$

$$\frac{1}{\tau}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Q_i f_0 d\underline{C} - \left(\int_{-\infty}^{\infty}\int_{-\infty}^{0}\int_{-\infty}^{\infty} f_2 + \int_{-\infty}^{\infty}\int_{0}^{\infty}\int_{-\infty}^{\infty} f_1\right)Q_i d\underline{C}\right],$$

where $d\underline{C} = dC_x dC_y dC_z$

Where F_y is the external force defined by Eq. (1). The previous equation is called the general equation of transfer or the transfer equation.

We can obtain the dimensionless forms of the variables by taking:

$$y = \bar{y}\left(\frac{5\sqrt{\pi}}{4}\frac{\mu_s}{n_s T_s}\sqrt{2RT_s}\right), t = \bar{t}\frac{\mu_s}{n_s T_s},$$

$$\vec{C} = \bar{\bar{C}}\sqrt{2RT_s}, f_i = \frac{\bar{f}_i(2\pi RT_s)^{\frac{3}{2}}}{n_s}, i = 0,1,2 \quad (6)$$

$$T_1 = \bar{T}_1 T_s \;, n_1 = \bar{n}_1 n_s, T_2 = \bar{T}_2 T_s \;, n_2 = \bar{n}_2 n_s,$$

$$\text{and} \quad dU = \overline{dU}\left(K_B T_s\right).$$

It is assumed that the temperature differences within the gas are sufficiently small such that the non-dimensional temperature may be expressed as a linear function of the temperature. This is accomplished by expanding F_y in a Taylor series about T_∞ and neglecting higher-order terms [9-10,32-33], thus $T^4(y,t) \cong 4T_\infty^3 T(y,t) - 4T_\infty^4$ This implies that:

$$F_y = \frac{-4\sigma_s}{3n_s c}\frac{\partial}{\partial y}\left[4T_\infty^3 T(y,t) - 4T_\infty^4\right]$$

$$= \frac{-16\sigma_s T_\infty^3}{3n_s c}\frac{\partial T(y,t)}{\partial y}$$

Once the expressions for f_0, f_1 and f_2 are introduced, macroscopic quantities such as density, velocity, temperature, etc… can be computed from the appropriate weighted integral of the distribution functions as follows ;

Number density [29]:

$$n(y,t) = \int f(y,t,C_x,C_y,C_z)d\underline{C}$$

$$= \frac{(n_1(y,t) + n_2(y,t))}{2} \quad (7)$$

Hydrodynamic (bulk) velocity:

$$u(y,t) = \frac{1}{n}\int C_y f(y,t,C_x,C_y,C_z)d\underline{C}$$

$$= \frac{(n_1\sqrt{T_1} - n_2\sqrt{T_2})}{(n_1 + n_2)} \quad (8)$$

Temperature:

$$T(y,t) = \frac{1}{3n}\int C^2 f(y,t,C_x,C_y,C_z)d\underline{C}$$

$$= \left(\frac{n_1 T_1 + n_2 T_2}{n_1 + n_2}\right) \quad (9)$$

The static pressure normal to the plate:

$$P_{yy}(y,t) = \int C_y{}^2 f(y,t,C_x,C_y,C_z)d\underline{C}$$
$$= \frac{1}{2}(n_1T_1 + n_2T_2) \tag{10}$$

The heat flux component:

$$Q_y(y,t) = \int C_y C^2 f(y,t,C_y)d\underline{C}$$
$$= \left(n_1T_1^{\frac{3}{2}} - n_2T_2^{\frac{3}{2}} \right) \tag{11}$$

In Eq. (4) there are four unknown functions $T_1(y,t), T_2(y,t), n_1(y,t)$ and $n_2(y,t)$ needed to be determined. Thus, we need four equations to solve our problem. We make two moment equations by taking $Q_i = C^2$ and $\frac{1}{2}C_y C^2$, and substitute by Eq. (4) into Eq. (5).

After dropping the bars, we get the following two equations, the conservation of energy:

$$\frac{\partial}{\partial t}(n_1T_1 + n_2T_2) + \frac{\partial}{\partial y}\left[\left(n_1T_1^{\frac{3}{2}} - n_2T_2^{\frac{3}{2}} \right) \right] +$$
$$N\frac{\partial}{\partial y}\left(\frac{n_1T_1 + n_2T_2}{n_1 + n_2} \right)\left(n_1T_1^{\frac{1}{2}} - n_2T_2^{\frac{1}{2}} \right) = 0 \tag{12}$$

And the heat flux in the y-direction

$$\frac{\partial}{\partial t}\left(n_1T_1^{\frac{3}{2}} - n_2T_2^{\frac{3}{2}} \right) + \frac{5}{4}\frac{\partial}{\partial y}\left[\left(n_1T_1^2 + n_2T_2^2 \right) \right]$$
$$+ \frac{3}{2}N\frac{\partial}{\partial y}\left(\frac{n_1T_1 + n_2T_2}{n_1 + n_2} \right)(n_1T_1 + n_2T_2) = \tag{13}$$
$$\left[\frac{-2}{\sqrt{\pi}K_n}\left(n_1T_1^{\frac{3}{2}} - n_2T_2^{\frac{3}{2}} \right) \right],$$

The above two equations are complemented by the equation of state [11-15],

$$P = nT = const. \tag{14}$$

And with the condition that, we shall study the problem in a coordinate system of the phase space in which the bulk velocity u is located at the origin. Thus, using Eq. (8), we get the fourth equation:

$$\left(n_1\sqrt{T_1} - n_2\sqrt{T_2} \right) = 0 \tag{15}$$

Where $\tau = \frac{5\sqrt{\pi}}{4}\frac{\mu_s}{n_sT_s}K_n$ [32], where K_n is the hydrodynamic Knudsen number defined by:

$$K_n = \frac{\text{Mean free path}}{\text{Hydrodynamic length}} = \frac{l}{\left(\dfrac{\mu_s}{n_sT_s}V_T \right)} \text{ And}$$

$$N = \frac{16\sigma_s T_\infty{}^3}{3n_s cmR} \text{ is a non-dimensional constant.}$$

We will use the traveling wave solution method [10,33-36] considering

$$\xi = ly - mt \tag{16}$$

Such that to make all the dependent variables as functions of ξ. Here l and m are transformation constants, which do not depend on the properties of the fluid but as parameters to be determined by the boundary and initial conditions [33]. From Eq. (37) we get the derivatives:

$$\frac{\partial}{\partial t} = -m\frac{\partial}{\partial \xi}, \ \frac{\partial}{\partial y} = l\frac{\partial}{\partial \xi} \text{ and } \frac{\partial^a}{\partial t^a} =$$
$$(-1)^a m^a \frac{\partial^a}{\partial \xi^a}, \ \frac{\partial^a}{\partial y^a} = l^a \frac{\partial^a}{\partial \xi^a} \tag{17}$$

Where a is a positive integer.

Substituting from Eqs. (16-17) into Eqs. (12, 13) we get:

$$-m\frac{\partial}{\partial \xi}(n_1T_1 + n_2T_2) + l\frac{\partial}{\partial \xi}\left(n_1T_1^{\frac{3}{2}} - n_2T_2^{\frac{3}{2}} \right) +$$
$$Nl\frac{\partial}{\partial \xi}\left(\frac{n_1T_1 + n_2T_2}{n_1 + n_2} \right)\left(n_1T_1^{\frac{1}{2}} - n_2T_2^{\frac{1}{2}} \right) = 0 \tag{18}$$

And

$$-m\frac{\partial}{\partial \xi}\left(n_1T_1^{\frac{3}{2}} - n_2T_2^{\frac{3}{2}} \right) + \frac{5}{4}l\frac{\partial}{\partial \xi}\left[\left(n_1T_1^2 + n_2T_2^2 \right) \right]$$
$$+ \frac{3}{2}Nl\frac{\partial}{\partial \xi}\left(\frac{n_1T_1 + n_2T_2}{n_1 + n_2} \right)(n_1T_1 + n_2T_2) = \tag{19}$$
$$\left[\frac{-2}{\sqrt{\pi}K_n}\left(n_1T_1^{\frac{3}{2}} - n_2T_2^{\frac{3}{2}} \right) \right],$$

We intend to solve Eqs. (14, 15, 18 and 19) to estimate the

four unknowns T_1, T_2, n_1 and n_2.

From Eq. (15), we have

$$n_2 \sqrt{T_2} = n_1 \sqrt{T_1}. \tag{20}$$

Substitution from Eqs. (14 and 15), with the help of Eq. (20), into Eq. (18), we obtain:

$$l \frac{\partial}{\partial \xi} \left(n_1 T_1^{\frac{3}{2}} - n_2 T_2^{\frac{3}{2}} \right) = l \frac{\partial}{\partial \xi} \left(n_2 \sqrt{T_2} (T_1 - T_2) \right) = 0 \tag{21}$$

Integrating Eq. (21), with respect to ξ, we obtain after factorization

$$\left(n_1 T_1^{\frac{3}{2}} - n_2 T_2^{\frac{3}{2}} \right) = \left(n_2 \sqrt{T_2} \left(\sqrt{T_1} + \sqrt{T_2} \right) \left(\sqrt{T_1} - \sqrt{T_2} \right) \right)$$
$$= \theta_1 \theta_2 = C_2, \tag{22}$$

Where, we put

$$\theta_1 = n_2 \sqrt{T_2} \left(\sqrt{T_1} + \sqrt{T_2} \right), \theta_2 = \left(\sqrt{T_1} - \sqrt{T_2} \right) \tag{23}$$

And C_2 is the integration constant. It's easy to show that θ_1, θ_2 are constants, this comes from the assumption of the pressure uniformity since

$$P_{yy} = \frac{n_2}{2} \sqrt{T_2} \left(\sqrt{T_1} + \sqrt{T_2} \right) = \frac{\theta_1}{2} = \text{constant, using Eq.}$$

(22), this implies that θ_2 is a constant as well [11,37] .

 For sampling the calculation, and making the better usage of Eq. (20), we assume a function $G(\xi)$ in the form [8, 12]:

$$G(\xi) = n_2 \sqrt{T_2} = n_1 \sqrt{T_1}. \tag{24}$$

From Eqs. (23) And (24) we can obtain by simple algebra that

$$T_1(\xi) = \frac{(\theta_1 + \theta_2 G)^2}{4G^2}, T_2(\xi) = \frac{(\theta_1 - \theta_2 G)^2}{4G^2},$$
$$n_1(\xi) = \frac{2G^2}{(\theta_1 + \theta_2 G)} \text{ and } n_2(\xi) = \frac{2G^2}{(\theta_1 - \theta_2 G)}. \tag{25}$$

After performing some algebraic manipulations, we can integrate Eq. (19) with respect to ξ, with the help of Eqs. (14 and 22), to obtain:

$$-m \left(n_1 T_1^{\frac{3}{2}} - n_2 T_2^{\frac{3}{2}} \right) + \frac{5}{4} l \left[\left(n_1 T_1^2 + n_2 T_2^2 \right) \right] +$$
$$\frac{3Nl}{4} \frac{(n_1 T_1 + n_2 T_2)^2}{(n_1 + n_2)} = -\frac{2}{\sqrt{\pi}} \frac{\theta_1 \theta_2 \xi}{K_n} + \theta_3 \tag{26}$$

Where θ_3 is the integration constant.

Substituting from Eqs. (25) into Eq. (26), gives:

$$\left(Kn\sqrt{\pi} \right) \begin{pmatrix} l(5+3N)\theta_1^3 - 3l(N-5)\theta_1 \theta_2^2 G^2 \\ -4G^2 \begin{pmatrix} m \begin{pmatrix} \theta_1 \begin{pmatrix} \sqrt{\frac{(\theta_1 + G\theta_2)^2}{G^2}} - \\ \sqrt{\frac{(\theta_1 - G\theta_2)^2}{G^2}} \end{pmatrix} \\ +G\theta_2 \begin{pmatrix} \sqrt{\frac{(\theta_1 + G\theta_2)^2}{G^2}} + \\ \sqrt{\frac{(\theta_1 - G\theta_2)^2}{G^2}} \end{pmatrix} \end{pmatrix} + 4\theta_3 \end{pmatrix} \end{pmatrix}$$
$$+ 32\theta_1 \theta_2 G^2 \xi = 0 \tag{27}$$

Solving it by the aid of symbolic software, we obtain the roots for $G(\xi)$. We keep into consideration the root that preserves the positive signs of both temperature and concentration [8, 12].

The values of the constants θ_1, θ_2 and θ_3 can be estimated under the initial and boundary conditions (as $(y,t) = (0,0) \Rightarrow \xi = 0$):

$$\frac{(n_1(\xi = 0) + n_2(\xi = 0))}{2} = 1 \tag{28}$$

$$\left(\frac{n_1(\xi = 0)T_1(\xi = 0) + n_2(\xi = 0)T_2(\xi = 0)}{n_1(\xi = 0) + n_2(\xi = 0)} \right) = 1 \tag{29}$$

$$\left(n_1(\xi = 0)T_1(\xi = 0)^{\frac{1}{2}} - n_2(\xi = 0)T_2(\xi = 0)^{\frac{1}{2}} \right) = 0 \tag{30}$$

The temperature of the incident particles is assumed to be T2 while the temperature of the reflected particles from the plate is the temperature T1, they are related such that [27, 29]:

$$T_2(\xi = 0) = \chi T_1(\xi = 0) : 0 < \chi \leq 1, \tag{31}$$

where χ_β is the ratio between the plate and gas temperatures.

The parameter χ can take arbitrary positive value less than unity to guaranty that the plate is hotter than the gas [12].

We can obtain by solving the algebraic system of Eqs. (28-31) that

$$n_1(\xi=0)=\left(2-\frac{2}{1+\sqrt{\chi}}\right), n_2(\xi=0)=\left(\frac{2}{1+\sqrt{\chi}}\right),$$

$$T_1(\xi=0)=\left(\frac{1}{\sqrt{\chi}}\right) \text{ and } T_2(\xi=0)=\left(\sqrt{\chi}\right). \tag{32}$$

The above four quantities represent the boundary conditions.

By substituting from (28) into (19), to obtain

$$\theta_1=2 \quad ,\theta_2=\chi^{\frac{-1}{4}}-\chi^{\frac{1}{4}} \quad , \tag{33}$$

Then from (32) and (33) into (26), we get

$$\theta_3=\frac{4m(\sqrt{\chi}-1)\chi^{\frac{1}{4}}+l(5+(3N-5)\sqrt{\chi}+5\chi)}{2\sqrt{\chi}} \tag{34}$$

By the way of introducing the obtained quantities T_1, T_2, n_1 and n_2 into the two stream Maxwellian distribution function;

$$\begin{cases} f_1=\dfrac{n_1}{T_1^{\frac{3}{2}}}exp[\dfrac{-C^2}{T_1}] \text{ ,For } C_y>0 \\[2em] f_2=\dfrac{n_2}{T_2^{\frac{3}{2}}}exp[\dfrac{-C^2}{T_2}] \text{ ,For } C_y<0 \end{cases} \tag{35}$$

We can get the sought distribution functions. These estimated distribution functions of the gas particles enable one to study their behavior in the investigated system, which is not possible by taking the way of the solution of Navier–Stokes equations. This will be the starting point to predict the irreversible thermodynamic behavior of the system in the next section [12].

3. The Non-Equilibrium Thermodynamic Properties of the System

The problems of the thermodynamics of irreversible processes continue to arouse great interest [38-43]. This is associated both with the general theoretical importance of this theory and its numerous applications in various branches of science.

Starting from the evaluation of the entropy per unit mass S, which is written as:

$$S(y,t)=-\int f \ Logf \ d\underline{C}$$

$$=\frac{\pi^{\frac{3}{2}}}{8}\left(n_1\left(3-4\ln\left(\frac{n_1}{T_1^{\frac{3}{2}}}\right)\right) +n_2\left(3-4\ln\left(\frac{n_2}{T_2^{\frac{3}{2}}}\right)\right)\right), \tag{36}$$

The y-component of the entropy flux vector has the form

$$J_y(y,t)=-\int C_y f \ Logf \ d\underline{C}=$$

$$\frac{\pi}{2}\left(n_1\sqrt{T_1}\left(1-\ln\left(\frac{n_1}{T_1^{\frac{3}{2}}}\right)\right)+ n_2\sqrt{T_2}\left(1-\ln\left(\frac{n_2}{T_2^{\frac{3}{2}}}\right)\right)\right), \tag{37}$$

While the Boltzmann's entropy production [40-43] in the unsteady state is expressed as:

$$\sigma_S=\frac{\partial S}{\partial t}+\underline{\nabla}\cdot\underline{J} \tag{38}$$

Following the general theory of irreversible thermodynamics [44-49], we could estimate the thermodynamic force corresponding to the change in concentration:

$$X_1=\frac{\delta y}{n}\frac{\partial n}{\partial y}, \tag{39}$$

The thermodynamic force corresponding to the change in temperature

$$X_2=\frac{\delta y}{T}\frac{\partial T}{\partial y}, \tag{40}$$

The thermodynamic force corresponding to the change in the radiation field energy

$$X_3=\frac{\delta y}{U_R}\frac{\partial U_R}{\partial y}, \tag{41}$$

Where $U_R = \left(\dfrac{16\sigma T_\infty^3 T_s}{3cn_s K_B T_s}\right) T$ is the dimensionless

radiation field energyinfluences the gas particles and δy is the thickness of the layer adjacent to the flat plate in units of the mean free path (the distance between two collisions of the gas particles) in the dimensionless form.

After calculating the thermodynamic forces and the entropy production we can get the kinetic coefficients Lij from the relationship between the entropy production and the thermodynamic forces which has the form [42]:

$$\sigma_S = \sum_i \sum_j L_{ij} X_i X_j =$$

$$\begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}\begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \qquad (42)$$

The restrictions on the signs of phenomenological coefficients L_{ij} which arise as a consequence of the second law of the thermodynamics that yields the quadratic form [42]:

$$\sigma_S = \sum_i \sum_j L_{ij} X_i X_j \geq 0 \qquad (43)$$

Can be deduced from the standard results in algebra. The necessary and sufficient conditions for $\sigma_s \geq 0$ are fulfilled by the determinant

$$\left| L_{ij} + L_{ji} \right| \geq 0 \qquad (44)$$

And all its principal minors are non-negative too.

Another restriction on L_{ij} was established by Onsager (1931) [42], is that, besides the restrictions on the signs, the phenomenological coefficients verify important symmetry properties. Invoking the principle of microscopic reversibility and using the theory of fluctuations, Onsager was able to demonstrate the symmetry property

$$L_{ij} = L_{ji} \qquad (45)$$

Which is called the Onsager's reciprocal relations [42, 47-50].

The Gibbs's formula for the variation of the internal energy applied to the system (gas + heated plate) is

$$dU = dU_S + dU_V + dU_R \qquad (46)$$

Where the internal energy change due to the variation of the extensive variables entropy, and volume in addition to the temperature gradient produced by the thermal radiation field are respectively

$$dU_S = TdS, dU_V = -PdV, dU_R = \omega \dfrac{\partial T^4}{\partial y} \Delta y,$$

$$\text{where } \omega = \dfrac{16\sigma T_\infty^3 T_s}{3cn_s K_B T_s}. \qquad (47)$$

The pressure and change in volume are

$$P = n\,T, dV = -\dfrac{dn}{n^2}, dn = \dfrac{\partial n}{\partial t}\delta t + \dfrac{\partial n}{\partial y}\delta y,$$

$$dS = \dfrac{\partial S}{\partial t}\delta t + \dfrac{\partial S}{\partial y}\delta y \quad \text{and } \delta y = 1, \delta t = 1.$$

4. Discussion

In a frame co-moving with the gas, we have investigated the behavior of the gas under the influence of a thermal radiation field in the unsteady state of a plane heat transfer problem in the system (gas + heated plate). The thermal radiation is introduced in the force term in the Boltzmann equation for the case of a neutral gas. In all calculations and figures, we take the following parameters values for the Helium gas:

$$\sigma_s = 5.6705 \times 10^{-8}\ Wm^{-2}K^{-4}; Kn = 1;$$

$$R = 8.3145\ JK^{-1}mol^{-1}; \rho_s = mn_s = 7.344 \times 10^{-11} kg/m^3;$$

$$c = 2.9979 \times 10^8 m/\sec; n_s = 10^{16} m^{-3};$$

Using the idea of the shooting numerical calculation method [35, 36], we evaluate the transformation constants to obtain $m = 2,\ l = 1$. The dimensionless N has the values.

$$N(300K) = 0.0446049, N(600K) =$$
$$0.356839, N(1000K) = 1.65203.$$

Although we calculate all the sought variables in three various radiation field intensities due to different plate temperatures (T = 300K, 600K, 1000K), we particularize our graphics in one case corresponding to (T =1000K), to economize the illustrations. All figures indicate that the boundary and initial conditions are hold.
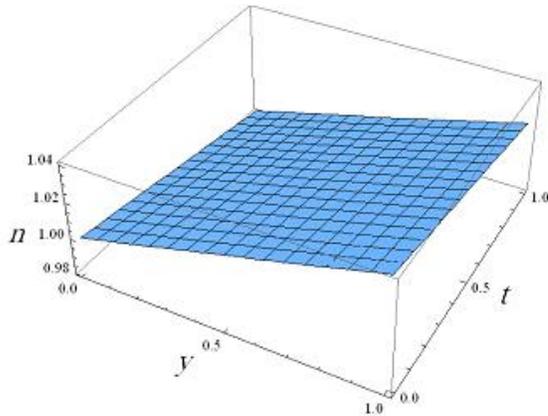
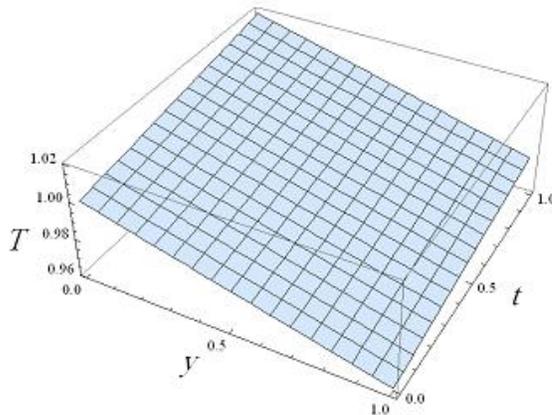**Figure. 1:** Concentration $n(1000\,\mathrm{K})$ vs. y and t at $\chi=0.9$.

Figure. 2: Temperature $T(1000\,\mathrm{K})$ vs. y and t at $\chi=0.9$.
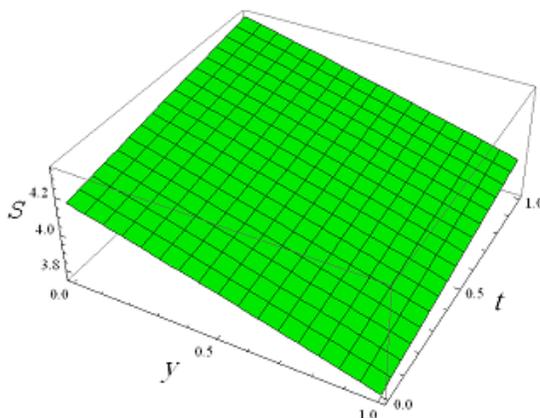
Figure. 3: Entropy $S(1000\,\mathrm{K})$ vs. y and t at $\chi=0.9$.

We will discuss the behavior of the gas particles in the non-

equilibrium state. The number density $n(y,t)$ increases with increasing the distance from the plate, while it decreases with time. On the contrary, the temperature $T(y,t)$ decreases with increasing the distance from the plate, while it increases with time. This is due to that when the application of heat to a gas causes it to expand; it is thereby rendered rarer than the neighboring parts of the gas. It tends to form an upward current of the heated gas, which is of course accompanied with a current of the more remote parts of the gas in the opposite direction. The fresh portions of gas are brought into the neighborhood of the source of heat, carrying their heat with them into other regions. In other words, heat will transfer from the hot surface into the gas, and the layer adjacent to the solid surface will be heated up. The next layer will also heat up, but to a lesser extent. In this manner, a temperature gradient will be set up across the gas. But the temperature gradient will create a density gradient in the reverse direction, that is, with the slightest gas density near the plate, see figures (1,2). Accordingly, the thermodynamic force due to the gradient of temperature X_T will have the opposite direction to the thermodynamic force due to the gradient of the density X_n, see figures (8,9).
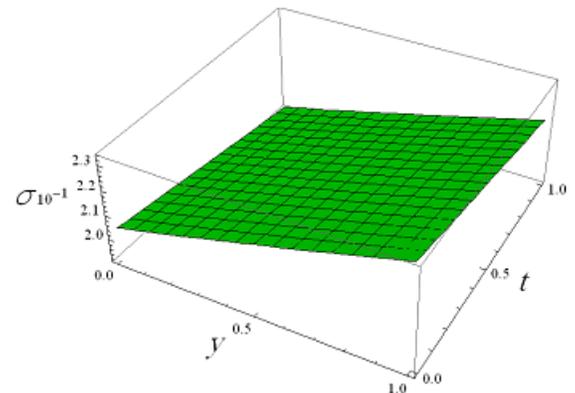

Figure. 4: Entropy production $\sigma_S(1000\mathrm{K})$ vs. y and t at $\chi=0.9$.

The entropy S increases with time, which gives a good agreement with the second law of thermodynamics, see figure (3). The entropy production behavior is fulfilled the famous Boltzmann H-Theorem, where $(\sigma \geq 0)$ for all values of y and t, see Fig. (4).
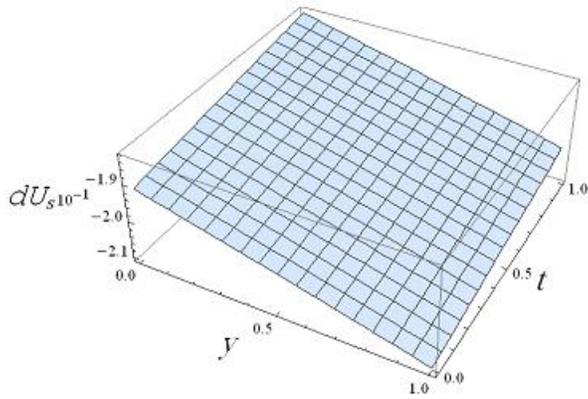
Figure. 5: The internal energy change dU_S (1000 K) vs. y and t at $\chi=0.9$.

The behavior of the different contributions of the change in internal energies can be illustrated as follows; the internal energies changes caused by the variations in the temperature dU_S and radiation energy dU_R have a negative sign. This is due to the fact that they have the same direction as the thermodynamic forces XT and XR formed by the gradient of temperature and thermal radiation, see figures (5,7,9,10).The internal energy change dU_V caused by the variation in density has a positive sign . This is because it takes the same direction as the thermodynamic force X_n due to the gradient of density, see figures (6,8).
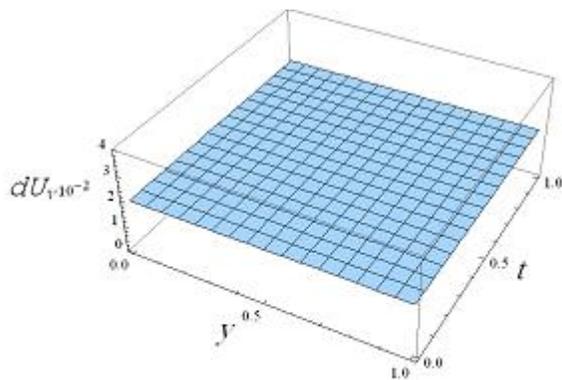
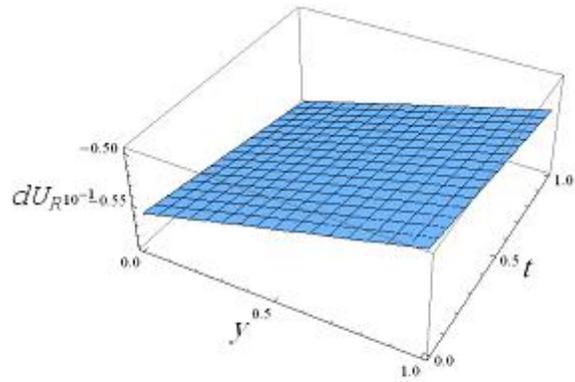**Figure. 6:** The internal energy change dU_V (1000 K) vs. y and t at $\chi=0.9$.

Figure. 7: The internal energy change dU_R (1000 K) vs. y and t at $\chi=0.9$.

The change in internal energy dU_S , dU_V and dU_R are all decrease in magnitude, with time, they go towards the equilibrium state, which gives an agreement with the Le Chatelier principle, see figures (5,6,7).. The same behavior and the same agreement with the Le Chatelier principle holds for the thermodynamic forces X_T , X_n and X_R , see figures (8,9,10).
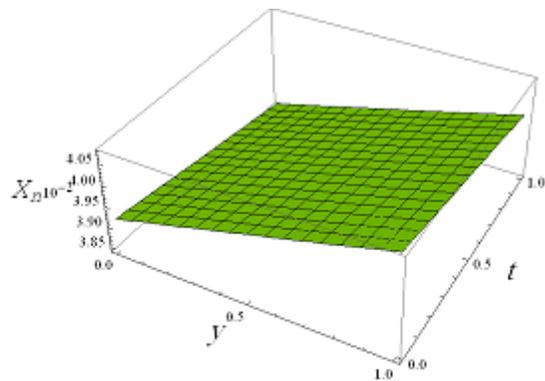

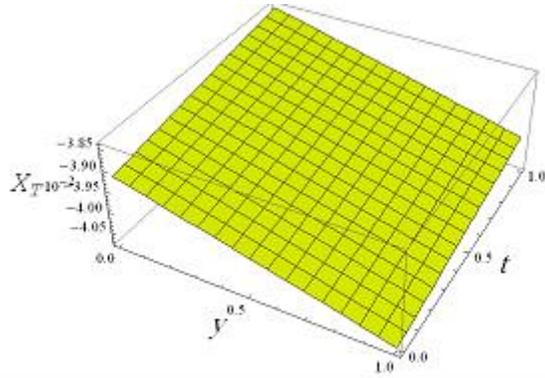**Figure. 8:** Thermodynamic force X_n(1000 K) vs. y and t at $\chi=0.9$.

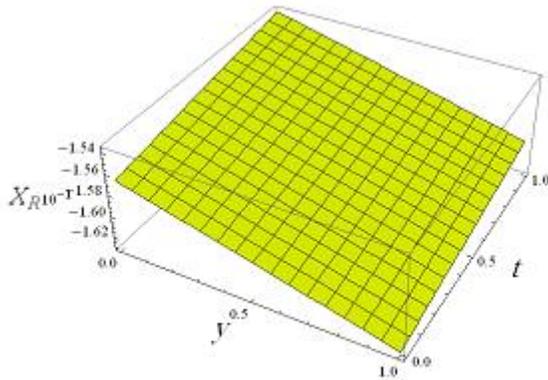**Figure. 9:** Thermodynamic force X_T (1000 K) vs. y and t at χ=0.9.

Figure. 10: Thermodynamic force X_R (1000 K) vs. y and t at χ=0.9.

The numerical ratios between the different contributions of the internal energy changes based upon the total derivatives of the extensive parameters are predicted via the Gibbs's formula illustrated in figures (8, 9, 10). Taking into consideration their different tendencies, the maximum numerical values of the three contributions at various radiation field intensity corresponding to various plate temperatures are ordered in magnitude as follows:

$dU_s(300K) : dU_V(300K) : dU_R(300K) =$

$1 : 0.85\ 10^{-1} : 0.33\ 10^{-1}$

$dU_s(600K) : dU_V(600K) : dU_R(600K) =$

$1 : 0.91\ 10^{-1} : 2.5\ 10^{-1}$

$dU_s(1000K) : dU_V(1000K) : dU_R(1000K) =$

$1 : 0.96\ 10^{-1} : 1.22.$

According to our calculations, the restrictions imposed on the kinetic coefficients L_{ij} are satisfied for all values of y and t, where $L_{11} \geq 0$, $L_{22} \geq 0$ and $L_{33} \geq 0$, see figures

(11,12,13). The celebrated Onsager's reciprocal relations are also satisfied for all values of y and t, where $\left(L_{12} \equiv L_{21}, L_{13} \equiv L_{31} \text{ and } L_{32} \equiv L_{23}\right)$, see figures (14,15,16).

5. Conclusions

In a frame co-moving with the fluid, we have investigated the behavior of the neutral monatomic gas under the influence of a nonlinear thermal radiation field in the unsteady state of a plane heat transfer problem in the system (gas + heated plate). By analyzing the results, we concluded that:

a) At high relativity temperature ($T \geq 1000K$), the radiation energy contribution in the total internal energy change in the considered system become the dominated one and cannot be ignored at all.

b) At small relativity temperature ($T < 600K$), the radiation energy contribution in the total internal energy change in the considered system become less by orders of degree than other contributions and may be ignored in calculations as a very acceptable approximate.

g) The second law of thermodynamics, the Boltzmann H-theorem, the Le-Chatelier principle, and the Onsager's reciprocal relations, all are satisfied for the system under consideration.

f) The negative sign at some of the kinetic coefficients, corresponding to cross effects, imply that in these cases there is a mass flow or a heat flux opposite to the main flow or flux due to the imposed thermodynamic force (gradient). For example, the negative sign in front of $L_{Tn} = L_{12}$ and $L_{Rn} = L_{13}$, imply that there is flow caused by the temperature gradient, from a lower to a higher temperature, known as thermal creep and thermal diffusion (or Soret effect) flows, respectively, which gives a qualitative agreement with [50].
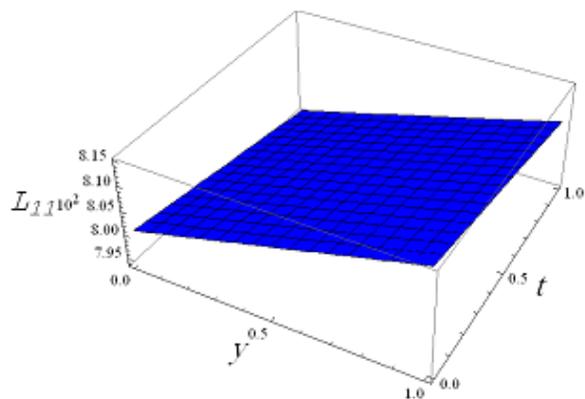
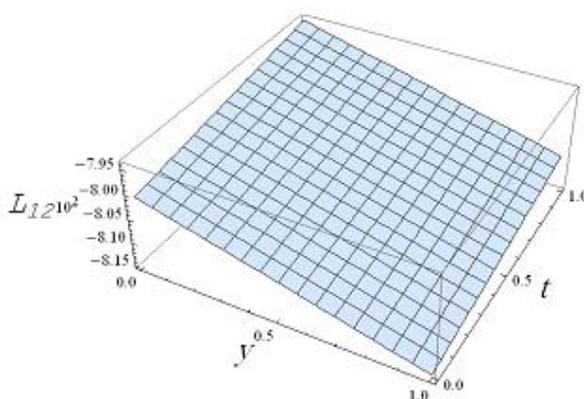**Figure. 11:** Kinetic coefficient $L_{11} \geq 0$ vs. y and t at $\chi=0.9$.

Figure. 14: Kinetic coefficient $L_{12} \equiv L_{21}$ vs. y and t at $\chi=0.9$.
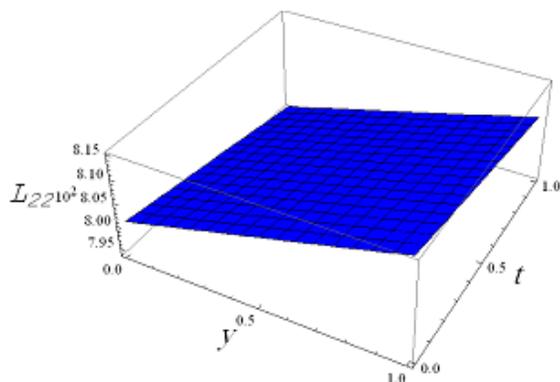

Figure. 12: Kinetic coefficient $L_{22} \geq 0$ vs. y and t at $\chi=0.9$.
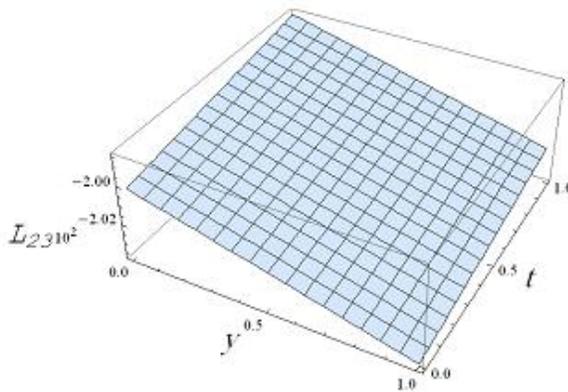

Figure. 15: Kinetic coefficient $L_{23} \equiv L_{32}$ vs. y and t at $\chi=0.9$.
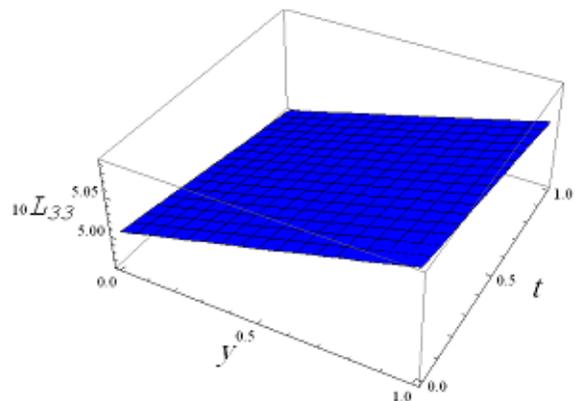

Figure. 13: Kinetic coefficient $L_{33} \geq 0$ vs. y and t at $\chi=0.9$.
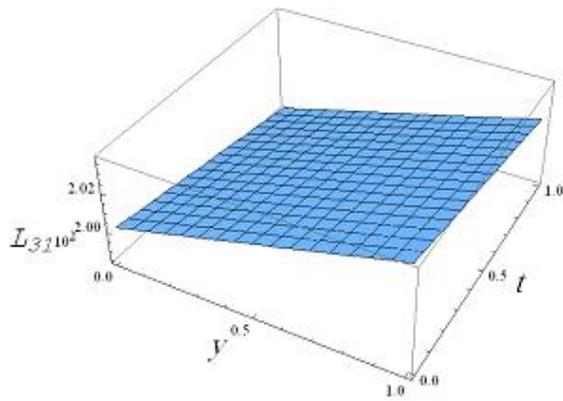

Figure. 16: Kinetic coefficient $L_{13} \equiv L_{31}$ vs. y and t at $\chi=0.9$.

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