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On a Mixed Quasilinear Nonlocal Problem With Integral Condition for a Second Order Parabolic Equation.

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Abstract: We prove the existence and uniqueness of a strong solution for a parabolic equation with integral boundary conditions. The proof uses a priori estimate and the density of the range of the operator generated by the problem considered by applying an iterative process based on the obtained results for the linear problem, we establish the existence and uniqueness of the weak solution of the nonlinear problem.

Keywords: Energy inequality, Integral boundary conditions, Non linear parabolic equation, Periodic condition.
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1 Introduction

Over the last few years, many physical phenomena were formulated by means of nonlocal mathematical models with integral boundary conditions. These integral boundary conditions appear when the data on the body can not be measured directly, but their average values are known. For instance, in some cases, describing the solution u (pressure, temperature, $\cdots$) pointwise is not possible, because only the average value of the solution can be estimate along the boundary or along a part of it. These mathematical models are encountered in many engineering models such as heat conduction [8,9,21,19], plasma physics [28], thermoelasticity [30], electrochemistry [11], chemical diffusion [12] and undergound water flow [17,23,31]. The importance of this kind of problems have been also pointed out by Samarskii [28]. The first paper, devoted to second-order partial differential equation with nonlocal integral condition goes back to Cannon [9].This type of boundary value problems with combined Dirichlet or Newmann and integral condition, or with purely integral conditions has been investigated in [3,4,5,6,7,9,10,11,21,19,20,22,29,33] for parabolic equations, for hyperbolic equations in [4,25,26,27,32], and in [13,14,15,16,18] for mixed type equations. Problems for elliptic equations with operator nonlocal conditions were considered by Mikhailov and Gushin [1], A.L.Skubachevski, G.M. Steblov [2], Peneiah [24].
In this paper we prove the existence and uniqueness of the weak solution of a class of mixed quasilinear non local problem in which we combine periodic and integral conditions for a second order quasilinear parabolic equation. We start by solving the associated linear problem. The existence and uniqueness of a strong solution is proved by means of an energy estimate and a density argument, which requires appropriate multipliers and functional spaces. On the basis of the obtained results of the linear problem, we apply an iterative process to establish the existence and uniqueness of the weak solution of the considered quasilinear problem.

2 Statement of the problem

In the rectangle Q=$(0,1)\times(0,T)$, with $T<+\infty$, we consider the equation

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)=f(x,t,u,\frac{\partial u}{\partial x}), \qquad (1)$$

with the initial condition

$$lu=u(x,0)=\varphi(x), \quad \forall x\in(0,1), \qquad (2)$$

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the periodic boundary condition

$$u(0,t)=u(1,t), \quad \forall t\in(0,T), \tag{3}$$

and the integral condition

$$\int_0^1 u(x,t)dx=0, \qquad \forall t\in(0,T). \tag{4}$$

In addition, we assume that the function $a(x,t)$ and its derivatives satisfies the conditions

$$\begin{cases} 0<a_0\le a(x,t)\le a_1 & \forall x,\ t\in Q,\\ c_2\le \dfrac{\partial a}{\partial t}(x,t)\le c_1, & \forall x,t\in Q,\\ \left|\dfrac{\partial a}{\partial x}(x,t)\right|\le b. \end{cases} \tag{5}$$

Here, we assume that the known function φ satisfies the compatibility conditions given by (3) and (4), and there exists a positive constant d such that

$$|f(x,t,u_1,v_1)-f(x,t,u_2,v_2)|\le d\,(|u_1-u_2|+|v_1-v_2|) \tag{6}$$

3 Associated linear problem

In this section we study the linear problem related to (1)-(4) and establish the existence and uniqueness of the strong solution.Thus we consider

$$\frac{\partial u}{\partial t}-\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)=f(x,t), \tag{7}$$

with the initial condition

$$lu=u(x,0)=\varphi(x), \quad x\in(0,\ 1), \tag{8}$$

the periodic boundary condition

$$u(0,t)=u(1,t), \quad \forall t\in(0,T), \tag{9}$$

and the integral condition

$$\int_0^1 u(x,t)dx=0, \qquad t\in(0,T). \tag{10}$$

The problem (7)-(10) can be considered as solving of the operator equation

$$Lu=(\pounds u,lu)=(f,\varphi)=\mathscr{F}, \tag{11}$$

where the operator L has domain of definition $D(L)$ consisting of functions $u\in L^2(\Omega)$ such that $\dfrac{\partial u}{\partial t}$, $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial^2 u}{\partial x\partial t}(x,t)\in L^2(Q)$ and satisfying the conditions (9) and (10).
The operator L is an operator defined on E into F, where

E is the Banach space of function $u\in L^2(Q)$, with the finite norm

$$\|u\|_E^2=\int_Q x^2(1-x)^2\left[\left|\frac{\partial u}{\partial t}\right|^2+\left|\frac{\partial^2 u}{\partial x^2}\right|^2\right]dxdt+\sup_t\int_0^1\left[x^2(1-x)^2\left|\frac{\partial u}{\partial x}\right|^2+|u|^2\right]dx. \tag{12}$$

F is the Hilbert space of vector valued functions obtained by completion of the space $L^2(Q)\times H^1(0,1)$, with respect to the norm

$$\|\mathscr{F}\|_F^2=\int_Q x^2(1-x)^2|f(x,t)|^2dxdt+\int_0^1\left[x^2(1-x)^2\left|\frac{d\varphi}{dx}\right|^2+|\varphi|^2\right]dx. \tag{13}$$

3.1 An energy inequality and its application

Theorem 1. *There exists a positive constant k, such that for each function $u\in D(L)$ we have*
for all $x,t\in Q$.

$$\|u\|_E\le k\|Lu\|_F. \tag{14}$$

Proof. Let

$$Mu=x^2(1-x)^2\frac{\partial u}{\partial t}+\int_x^1\frac{d\zeta}{a}\int_0^x\frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta\frac{d\eta}{a}}{\int_0^1\frac{d\zeta}{a}}\frac{\partial u}{\partial t}d\zeta+\int_0^x\frac{d\zeta}{a}\int_x^1\left[\lambda-\frac{2(2\zeta-1)(\zeta-\zeta^2)+\lambda\int_0^\zeta\frac{d\eta}{a}}{\int_0^1\frac{d\zeta}{a}}\right]\frac{\partial u}{\partial t}d\zeta,$$

We consider the quadratic form obtained by multiplying equation (7) by $exp(-ct)\overline{Mu}$, where $c>0$ and λ is a scalar parameter satisfying

$$\lambda>2a_1, \tag{15}$$

and integrating over $\Omega^s=[0,1]\times[0,s]$ with $0\le s\le T$, and taking the real part, formally

$$\Phi(u,u)=\int_{\Omega^s}\exp(-ct)f(x,t)\overline{Mu}dxdt$$
$$=\int_{\Omega^s}exp(-ct)\frac{\partial u}{\partial t}\overline{Mu}dxdt-\int_{\Omega^s}exp(-ct)\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\overline{Mu}dxdt. \tag{16}$$

Substituting Mu by its expression in the right hand-side of (16), integrating by parts with respect to x, and t using (8), (9) and the integral condition (10), we obtain

$$\int_{\Omega^s}x^2(1-x)^2exp(-ct)\left|\frac{\partial u}{\partial t}\right|^2dxdt$$
$$+\tfrac{1}{2}\int_0^s exp(-ct)\left(\sqrt{\int_0^1\left(12(x-x^2)-2+\frac{\lambda}{a}\right)\left|\int_0^x\frac{\partial u}{\partial t}d\zeta\right|^2dx}-\sqrt{\int_0^1\tfrac{\lambda}{a}\left|\int_0^x\frac{\partial u}{\partial t}d\zeta\right|^2dx}\right)^2dt$$
$$+\int_{\Omega^s}\frac{x^2(1-x)^2}{2}\left(ca-\frac{\partial a}{\partial t}\right)exp(-ct)\left|\frac{\partial u}{\partial x}\right|^2dxdt+\tfrac{\lambda c}{2}\int_{\Omega^s}\exp(-ct)|u|^2dxdt$$

$$+a_0\int_0^1 exp(-cs)\,\frac{x^2(1-x)^2}{2}\left|\frac{\partial u}{\partial x}\right|^2 dx\Big|_{t=s}+\frac{\lambda}{2}\int_0^1 exp(-cs)\,|u|^2\,dx\Big|_{t=s}\leq$$

$$\int_{\Omega^s} exp(-ct)\,f\overline{Mu}dxdt+a_1\int_0^1\frac{x^2(1-x)^2}{2}\left|\frac{\partial\varphi}{\partial x}\right|^2 dx+\frac{\lambda}{2}\int_0^1|\varphi|^2\,dx. \tag{17}$$

Substituting Mu by its expression in the first term in the right-hand side of (17) and using the Holder inequality, we deduce

$$\int_{\Omega^s} exp(-ct)\,f\overline{Mu}dxdt\leq$$

$$\left(2\left(\frac{64a_1^2}{a_0^2}+\frac{16\lambda^2a_1^2}{a_0^4}\right)+1\right)\int_{\Omega^s}x^2(1-x)^2\exp(-ct)\,|f|^2\,dxdt$$

$$+\frac{1}{2}\int_{\Omega^s}\exp(-ct)x^2(1-x)^2\left|\frac{\partial u}{\partial t}\right|^2 dxdt. \tag{18}$$

The combination of (18) and (17) yields

$$\frac{1}{2}\int_{\Omega^s}x^2(1-x)^2\,exp(-ct)\left|\frac{\partial u}{\partial t}\right|^2 dxdt$$

$$+\int_{\Omega^s}\frac{x^2(1-x)^2}{2}\left(ca-\frac{\partial a}{\partial t}\right)exp(-ct)\left|\frac{\partial u}{\partial x}\right|^2 dxdt+\frac{\lambda c}{2}\int_{\Omega^s}\exp(-ct)\,|u|^2\,dxdt$$

$$+a_0\int_0^1 exp(-cs)\,\frac{x^2(1-x)^2}{2}\left|\frac{\partial u}{\partial x}\right|^2 dx\Big|_{t=s}+\frac{\lambda}{2}\int_0^1 exp(-cs)\,|u|^2\,dx\Big|_{t=s}\leq$$

$$\left(\frac{128a_1^2}{a_0^2}+\frac{32\lambda^2a_1^2}{a_0^4}+1\right)\int_{\Omega^s}x^2(1-x)^2\exp(-ct)\,|f|^2\,dxdt$$

$$+a_1\int_0^1\frac{x^2(1-x)^2}{2}\left|\frac{\partial\varphi}{\partial x}\right|^2 dx+\frac{\lambda}{2}\int_0^1|\varphi|^2\,dx.$$

By choosing c such that

$$ca_0-c_1>0, \tag{19}$$

we obtain

$$\int_{\Omega^s}x^2(1-x)^2\left|\frac{\partial u}{\partial t}\right|^2 dxdt+\int_{\Omega^s}x^2(1-x)^2\left|\frac{\partial u}{\partial x}\right|^2 dxdt$$

$$+\int_0^1\left(x^2(1-x)^2\left|\frac{\partial u}{\partial x}\right|^2+|u|^2\right)dx\Big|_{t=s}\leq$$

$$m\left(\int_Q x^2(1-x)^2\,|f|^2\,dxdt+\int_0^1 x^2(1-x)^2\left|\frac{\partial\varphi}{\partial x}\right|^2 dx+\int_0^1|\varphi|^2\,dx\right), \tag{20}$$

where

$$m=\frac{\max\left(\left(\frac{128a_1^2}{a_0^2}+\frac{32\lambda^2a_1^2}{a_0^4}\right)+1,\frac{\lambda}{2}\right)\exp(cT)}{\min\left(\frac{1}{2},\frac{ca_0-c_1}{2},\frac{a_0}{2}\right)}.$$

From equation (7) and inequality (20), we deduce

$$\int_{\Omega^s}x^2(1-x)^2\left[\left|\frac{\partial u}{\partial t}\right|^2+\left|\frac{\partial^2u}{\partial x^2}\right|^2\right]dxdt+\int_{\Omega^s}x^2(1-x)^2\left|\frac{\partial u}{\partial x}\right|^2 dxdt$$

$$+\int_0^1\left(x^2(1-x)^2\left|\frac{\partial u}{\partial x}\right|^2+|u|^2\right)dx\Big|_{t=s}\leq$$

$$k^2\left[\int_Q x^2(1-x)^2\,|f|^2\,dxdt+\int_0^1 x^2(1-x)^2\left|\frac{\partial\varphi}{\partial x}\right|^2 dx+\int_0^1|\varphi|^2\,dx\right], \tag{21}$$

If we drop the second term in the last inequality and by taking the least upper bound of the left side with respect to s from 0 to T, we get the desired estimate (14) with $k^2=m+\frac{2+4m+4b^2m}{a_0^2}$.

It can be proved in a standard way that the operator L: $E\to F$ is closable. Let $\overline{L}$ be the closure of this operator, with the domain of definition $D(\overline{L})$.

Definition 1.*A solution of the operator equation $\overline{L}u=\mathscr{F}$ is called a strong solution of problem (7)-(10).*

The a priori estimate (14) can be extended to strong solutions, that is we have the inequality

$$\|u\|_E\leq c\left\|\overline{L}u\right\|_F,\quad \forall u\in D(\overline{L}).$$

This last inequality implies the following corollaries.

Corollary 1.*If a strong solution of (7)-(10) exists, it is unique and depends continuously on $\mathscr{F}=(f,\varphi)$.*

Corollary 2.*The range $R(\overline{L})$ of $\overline{L}$ is closed in F and $\overline{R(L)}=R(\overline{L})$.*

Corollary 2 shows that, to prove that problem (7)-(10) has a strong solution for arbitrary $\mathscr{F}$, it suffices to prove that the set $R(L)$ is dense in F.

3.2 Solvability of Problem (7)-(10).

To prove the solvability of problem (7)-(10), it sufficient to show that $R(L)$ is dense in F. The proof is based on the following lemma.

Lemma 1.*Suppose that $a(x,t)$ and its derivatives are bounded.*
Let $D_0(L)=\{u\in D(L),\ u(x,0)=0\}$. If, for $u\in D_0(L)$ and for some function $w\in L^2(\Omega)$, we have

$$\int_Q\frac{x^2(1-x)^2}{a(x,t)}\pounds u\overline{w}dxdt=0, \tag{22}$$

then $w=0$.

*Proof.*From (22) we have

$$\int_Q \frac{x^2(1-x)^2}{a(x,t)}\frac{\partial u}{\partial t}\overline{w}dxdt = \int_Q \frac{x^2(1-x)^2}{a(x,t)}\frac{\partial}{\partial x}\left(a(x,t)\frac{\partial u}{\partial x}\right)\overline{w}dxdt. \tag{23}$$

Now, for a given $w(x,t) \in L^2(Q)$, we introduce the function

$$v(x,t) = x(1-x)w + \int_0^x a(\zeta,t)\frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)w(\zeta,t)\,d\zeta,$$

then

$$\int_0^x \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)v(\zeta,t)\,d\zeta = \frac{x(1-x)}{a(x,t)}\int_0^x a(\zeta,t)\frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)w(\zeta,t)\,d\zeta,$$

which implies

$$\int_0^1 \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)v(\zeta,t)\,d\zeta = 0. \tag{24}$$

Then, (23) can be transformed as follows

$$\int_Q \frac{\partial u}{\partial t}\overline{Nv}dxdt = \int_Q A(t)u\overline{v}, \tag{25}$$

where

$$\begin{cases} A(t)u = \frac{\partial}{\partial x}\left((x-x^2)\frac{\partial u}{\partial x}\right), \\ Nv = \frac{x-x^2}{a(x,t)}v + \int_0^x \frac{\partial}{\partial \zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)v(\zeta,t)\,d\zeta = \frac{x^2(1-x)^2}{a(x,t)}w. \end{cases} \tag{26}$$

We introduce the smoothing operators $J_\varepsilon^{-1} = \left(I+\varepsilon\frac{\partial}{\partial t}\right)^{-1}$ and $\left(J_\varepsilon^{-1}\right)^* = \left(I-\varepsilon\frac{\partial}{\partial t}\right)^{-1}$, with respect to t, then these operators provide the solution of the problems:

$$\begin{cases} u_\varepsilon(t) + \varepsilon\frac{\partial u_\varepsilon}{\partial t} = u(t) & u_\varepsilon(0) = 0, \\ v_\varepsilon^*(t) - \varepsilon\frac{\partial v_\varepsilon^*}{\partial t} = v(t) & v_\varepsilon^*(T) = 0. \end{cases} \tag{27}$$

We also have the following properties: for any $g \in L^2(0,T)$, the functions $J_\varepsilon^{-1}g$, $\left(J_\varepsilon^{-1}\right)^* g \in W_2^1(0,T)$. If $g \in D(L)$, then $J_\varepsilon^{-1}g \in D(L)$ and we have

$$\begin{cases} \lim \left\|J_\varepsilon^{-1}g - g\right\|_{L^2(0,T)} = 0, & \text{for } \varepsilon \to 0, \\ \lim \left\|\left(J_\varepsilon^{-1}\right)^* g - g\right\|_{L^2(0,T)} = 0, & \text{for } \varepsilon \to 0. \end{cases} \tag{28}$$

Substituting the function u in (25) by the smoothing function u_ε and using the relation

$$A(t)u_\varepsilon = J_\varepsilon^{-1}A(t)u,$$

we obtain

$$\int_Q uN\frac{\overline{\partial v_\varepsilon^*}}{\partial t}dxdt = -\int_Q A(t)u\overline{v_\varepsilon^*}dxdt. \tag{29}$$

The left hand side of (29) is a continuous linear functional of u, hence the function v_ε^* has the derivatives $(x-x^2)\frac{\partial v_\varepsilon^*}{\partial x} \in L^2(Q)$, $\frac{\partial}{\partial x}\left((x-x^2)\frac{\partial v_\varepsilon^*}{\partial x}\right) \in L^2(Q)$, and the following conditions are satisfied:

$$\begin{cases} xv_\varepsilon^*\big|_{x=0} = (1-x)v_\varepsilon^*\big|_{x=1} = 0, \\ x\left.\frac{\partial v_\varepsilon^*}{\partial x}\right|_{x=0} = (1-x)\left.\frac{\partial v_\varepsilon^*}{\partial x}\right|_{x=1}. \end{cases} \tag{30}$$

Substituting the function

$$u = \int_0^t \exp(\theta\tau)\left[\frac{x-x^2}{a(x,t)}v_\varepsilon^*(x,\tau) - \int_0^x \left(\zeta\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,\tau)}\right) + \frac{\zeta-\zeta^2}{a(\zeta,\tau)}\right)v_\varepsilon^*(\zeta,\tau)\,d\zeta - \int_x^1 \left(\frac{\zeta-\zeta^2}{a(\zeta,\tau)} - (1-\zeta)\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,\tau)}\right)\right)v_\varepsilon^*(\zeta,\tau)\,d\zeta\right]d\tau. \tag{31}$$

in (25), where the constant θ satisfies

$$c_1 - \theta a_0 \le 0, \tag{32}$$

and using the properties of the smoothing operators, we have

$$\int_Q \frac{\partial u}{\partial t}\overline{Nv}dxdt = \int_Q A(t)u\overline{v_\varepsilon^*}dxdt - \varepsilon\int_Q A(t)u\frac{\overline{\partial v_\varepsilon^*}}{\partial t}dxdt. \tag{33}$$

Integrating by parts each term in the right-hand side of (33) with respect to x and t, using (30), we obtain

$$\int_Q A(t)u\overline{v_\varepsilon^*}dxdt - \varepsilon\int_Q A(t)u\frac{\overline{\partial v_\varepsilon^*}}{\partial t}dxdt = -\frac{1}{2}\int_0^1 a(x,T)\exp(-\theta T)\left|\frac{\partial u}{\partial x}\right|^2 dx + \frac{1}{2}\int_Q\left(\frac{\partial a}{\partial t} - \theta a\right)\exp(-\theta t)\left|\frac{\partial u}{\partial x}\right|^2 dxdt - \varepsilon\int_Q a\exp(-\theta t)\left|\frac{\partial^2 u}{\partial x\partial t}\right|^2 dxdt. \tag{34}$$

We replace u by its representation (31) in the right-hand side of (33), we obtain

$$\int_Q \frac{\partial u}{\partial t}\overline{Nv}dxdt = \int_Q \frac{x-x^2}{a(x,t)}\exp(\theta t)v_\varepsilon^*\overline{Nv}dxdt - \int_Q \exp(\theta t)\int_0^x\left(\zeta\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right) + \frac{\zeta-\zeta^2}{a(\zeta,t)}\right)v_\varepsilon^* d\zeta\,\overline{Nv}dxdt - \int_Q \exp(\theta t)\int_x^1\left(\left(\frac{\zeta-\zeta^2}{a(\zeta,\tau)} - (1-\zeta)\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,\tau)}\right)\right)v_\varepsilon^*(\zeta,\tau)\,d\zeta\right)v_\varepsilon^* d\zeta\,\overline{Nv}dxdt. \tag{35}$$

Substituting Nv by its expression in each term in the right-hand side of (35), integrating with respect to x and using the condition (24), we obtain :

$$\int_Q \frac{x-x^2}{a(x,t)}\exp(\theta t)v_\varepsilon^*\overline{Nv}dxdt = \int_Q \exp(\theta t)\left(\frac{x-x^2}{a}\right)^2|v|^2 dxdt + \int_Q \frac{x-x^2}{a(x,t)}v\exp(\theta t)\int_0^x \frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)vd\zeta dxdt + \int_Q \exp(\theta t)\frac{x-x^2}{a(x,t)}Nv\left(\overline{v_\varepsilon^* - v}\right)dxdt. \tag{36}$$

then

$$-\int_Q \exp(\theta t)\int_0^x \left(\zeta\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)+\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)v_\varepsilon^* d\zeta\,\overline{Nv}dxdt =$$
$$-\frac{1}{2}\int_0^T \exp(\theta t)\left|\int_0^1 \frac{\zeta-\zeta^2}{a(\zeta,t)}vd\zeta\right|^2 dt-\frac{1}{2}\int_0^T \exp(\theta t)\left|\int_0^1 dx\int_0^x \frac{\zeta-\zeta^2}{a(\zeta,t)}vd\zeta\right|^2 dt$$
$$+\int_0^T\int_0^1 \exp(\theta t)x\left|\int_0^x \frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)vd\zeta\right|^2 dxdt$$
$$-\int_Q \exp(\theta t)\frac{x-x^2}{a(x,t)}v\int_0^x \zeta\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)\overline{v}d\zeta dxdt$$
$$+\int_Q \exp(\theta t)\int_0^x \frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)vd\zeta\int_0^x \frac{\zeta-\zeta^2}{a(\zeta,t)}\overline{v}d\zeta dxdt$$
$$-\int_Q \exp(\theta t)\int_0^x \left(\zeta\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)+\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)(v_\varepsilon^*-v)\,d\zeta\,\overline{Nv}dxdt. \quad (37)$$

and

$$-\int_Q \exp(\theta t)\int_x^1 \left(\frac{\zeta-\zeta^2}{a(\zeta,t)}-(1-\zeta)\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)\right)v_\varepsilon^*\overline{Nv}dxdt =$$
$$-\frac{1}{2}\int_0^T \exp(\theta t)\left|\int_0^1 dx\int_0^x \frac{\zeta-\zeta^2}{a(\zeta,t)}vd\zeta\right|^2 dt-\frac{1}{2}\int_0^T \exp(\theta t)\left|\int_0^1 dx\int_0^x \frac{\zeta-\zeta^2}{a(\zeta,t)}vd\zeta\right|^2 dt$$
$$+\int_0^T\int_0^1 \exp(\theta t)(1-x)\left|\int_0^x \frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)vd\zeta\right|^2 dxdt$$
$$+\int_Q \exp(\theta t)\frac{x-x^2}{a(x,t)}v\int_x^1 (1-\zeta)\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)\overline{v}d\zeta dxdt$$
$$+\int_Q \exp(\theta t)\int_0^x \frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)vd\zeta\int_x^1 \frac{\zeta-\zeta^2}{a(\zeta,t)}\overline{v}d\zeta dxdt$$
$$-\int_Q \exp(\theta t)\int_x^1 \left((1-\zeta)\frac{\partial}{\partial\zeta}\left(\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)-\frac{\zeta-\zeta^2}{a(\zeta,t)}\right)(v_\varepsilon^*-v)\,d\zeta\,\overline{Nv}dxdt. \quad (38)$$

from (32), (34)-(38), for sufficiently small ε we have

$$0\le\int_Q \exp(\theta t)|Nv|^2 dxdt-\int_0^T \exp(\theta t)dt\left|\int_0^1 Nvdx\right|^2\le 0,$$

then

$$\int_Q \exp(\theta t)|Nv|^2 dxdt-\int_0^T \exp(\theta t)dt\left|\int_0^1 Nvdx\right|^2 =$$
$$\frac{1}{2}\int_0^T\int_0^1\int_0^1 \exp(\theta t)|(Nv)(x,t)-(Nv)(y,t)|^2 dxdydt=0$$

We conclude that

$$Nv(x,t)=Nv(y,t)\ \forall x,y\in[0,1],\quad t\in[0,T],$$

then $Nv=0$ a.e.; hence from (26), we deduce that $w=0$ a.e., which ends the proof of the lemma.

Theorem 2.*The range $R(\overline{L})$ of the operator $\overline{L}$ coincides with F.*

*Proof.*Since F is a Hilbert space, we have $R(\overline{L})=F$ if and only if the relation

$$\int_Q x^2(1-x)^2 f\overline{g}dxdt+\int_0^1 x^2(1-x)^2\frac{dlu}{dx}\frac{\overline{d\varphi_1}}{dx}dx+\int_0^1 lu\overline{\varphi_1}dx=0, \quad (39)$$

for arbitrary $u\in D(L)$ and $(g,\varphi_1)\in F$, implies that $g=0$ and $\varphi_1=0$.
Putting $u\in D_0(L)$ in (39), we conclude from the Lemma 1 that $g=\frac{w}{a}=0$, then $g=0$. a.e.
Taking $u\in D(L)$ in (39) yields

$$\int_0^1 x^2(1-x)^2\frac{dlu}{dx}\frac{\overline{d\varphi_1}}{dx}dx+\int_0^1 lu\overline{\varphi_1}dx=0,$$

since the range of the trace operator l is everywhere dense in Hilbert space with the norm

$$\int_0^1 x^2(1-x)^2\left|\frac{dlu}{dx}\right|^2 dx+\int_0^1 |lu|^2 dx,$$

hence, $\varphi_1=0$.

4 Study of the nonlinear problem

In this section, we prove the existence, uniqueness and continuous dependance of the weak solution on the data of the problem (1)-(4).
It is clear that if the solution of problem (1)-(4) exists, it can be expressed in the form $u=w+U$, where U is a solution of the homogeneous problem

$$\pounds U=\frac{\partial U}{\partial t}-\frac{\partial}{\partial x}\left(a\frac{\partial U}{\partial x}\right)=0, \quad (40)$$
$$lU=U_0=U(x,0)=\varphi(x), \quad (41)$$
$$U(0,t)=U(1,t), \quad (42)$$
$$\int_0^1 U(x,t)dx=0. \quad (43)$$

and w is a solution of the problem

$$\pounds w=\frac{\partial w}{\partial t}-\frac{\partial}{\partial x}\left(a\frac{\partial w}{\partial x}\right)=F\left(x,t,w,\frac{\partial w}{\partial x}\right), \quad (44)$$
$$lw=w(x,0)=0, \quad (45)$$
$$w(0,t)=w(1,t), \quad (46)$$
$$\int_0^1 w(x,t)dx=0. \quad (47)$$

where $F\left(x,t,w,\frac{\partial w}{\partial x}\right)=f\left(x,t,w+U,\frac{\partial(w+U)}{\partial x}\right)$ and satisfied the condition

$$|F(x,t,u_1,v_1)-F(x,t,u_2,v_2)|\le d(|u_1-u_2|+|v_1-v_2|)\quad \text{for all } x,t\in Q. \quad (48)$$

According to Theorem 1 and Lemma 2, the problem (40)-(43) has a unique solution that depend continuously on $U_0\in V^{1,0}(0,1)$ where $V^{1,0}(0,1)$ is a Hilbert space with the scalar product

$$(u,v)_{V^{1,0}(0,1)}=\int_0^1 x^2(1-x)^2\frac{\partial u}{\partial x}\frac{\overline{\partial v}}{\partial x}dx+\int_0^1 u\overline{v}dx,$$

and with associated norm

$$\|u\|_{V^{1,0}(0,1)}=\int_0^1 x^2(1-x)^2\left|\frac{\partial u}{\partial x}\right|^2 dx+\int_0^1 |u|^2 dx.$$

We shall prove that the problem (44)-(47) has a weak solution by using an iterative process and passing to the limit.
Assume that v and $w \in C^1(Q)$, and the following conditions are satisfies

$$\begin{cases} v(x,T)=0, \int_0^1 v(x,t)\,dx=0, \\ w(x,0)=0, w(0,t)=w(1,t). \end{cases} \tag{49}$$

Taking the scalar product in $L^2(Q)$ of equation (44) with the integrodifferential operator

$$Mv=\int_x^1 \frac{d\zeta}{a}\int_0^x \frac{\zeta-\zeta^2}{\int_0^1 \frac{d\zeta}{a}} v\,d\zeta-\int_0^x \frac{d\zeta}{a}\int_x^1 \frac{\zeta-\zeta^2}{\int_0^1 \frac{d\zeta}{a}} v\,d\zeta,$$

and by taking the real part, we obtain

$$H(w,v)=\int_Q F\left(x,t,w,\frac{\partial w}{\partial x}\right)\overline{Mv}dxdt$$

$$=\int_Q \frac{\partial w}{\partial t}\overline{Mv}dxdt-\int_Q \frac{\partial}{\partial x}\left(a\frac{\partial w}{\partial x}\right)\overline{Mv}dxdt. \tag{50}$$

Substituting the expression of Mv in the first integral of the right hind-side of (50), integrating with respect to t, using the condition (49), we get

$$\int_Q \frac{\partial w}{\partial t}\overline{Mv}=-\int_Q w\left(\int_x^1 \frac{d\zeta}{a}\int_0^x \frac{\zeta-\zeta^2}{\int_0^1 \frac{d\zeta}{a}}\overline{\frac{\partial v}{\partial t}}d\zeta-\int_0^x \frac{d\zeta}{a}\int_x^1 \frac{\zeta-\zeta^2}{\int_0^1 \frac{d\zeta}{a}}\overline{\frac{\partial v}{\partial t}}d\zeta\right)dxdt$$
$$+\int_Q w\frac{\int_x^1 \frac{1}{a^2}\frac{\partial a}{\partial t}d\zeta\int_0^1 \frac{dx}{a}-\int_0^1 \frac{1}{a^2}\frac{\partial a}{\partial t}d\zeta\int_x^1 \frac{d\zeta}{a}}{\left(\int_0^1 \frac{d\zeta}{a}\right)^2}\int_0^x (\zeta-\zeta^2)\overline{\nabla}d\zeta dxdt$$
$$-\int_Q w\frac{\int_0^x \frac{1}{a^2}\frac{\partial a}{\partial t}d\zeta\int_0^1 \frac{dx}{a}-\int_0^1 \frac{1}{a^2}\frac{\partial a}{\partial t}d\zeta\int_0^x \frac{d\zeta}{a}}{\left(\int_0^1 \frac{d\zeta}{a}\right)^2}\int_x^1 (\zeta-\zeta^2)\overline{\nabla}d\zeta dxdt. \tag{51}$$

Substituting the expression of Mv in the second integral of the right hind-side of (50), integrating with respect to x, using the condition (49), we get

$$-\int_Q \frac{\partial}{\partial x}\left(a\frac{\partial w}{\partial x}\right)\overline{Mv}dxdt=\int_Q (x-x^2)\,a\frac{\partial w}{\partial x}\overline{\nabla}dxdt. \tag{52}$$

Inserting expressions (51), (52) into (50) yields

$$H(w,v)=$$
$$\int_Q \frac{x-x^2}{\int_0^1 \frac{d\zeta}{a}}\overline{\nabla}\left[\int_x^1\int_\zeta^1 \frac{d\eta}{a}F\left(\zeta,t,w,\frac{\partial w}{\partial \zeta}\right)d\zeta-\int_0^x\int_0^\zeta \frac{d\eta}{a}F\left(\zeta,t,w,\frac{\partial w}{\partial \zeta}\right)d\zeta\right]dxdt, \tag{53}$$

where

$$H(w,v)=\int_Q (x-x^2)\,a\frac{\partial w}{\partial x}\overline{\nabla}dxdt$$
$$-\int_Q w\left(\int_x^1 \frac{d\zeta}{a}\int_0^x \frac{\zeta-\zeta^2}{\int_0^1 \frac{d\zeta}{a}}\overline{\frac{\partial v}{\partial t}}d\zeta-\int_0^x \frac{d\zeta}{a}\int_x^1 \frac{\zeta-\zeta^2}{\int_0^1 \frac{d\zeta}{a}}\overline{\frac{\partial v}{\partial t}}d\zeta\right)dxdt$$
$$+\int_Q w\frac{\int_x^1 \frac{1}{a^2}\frac{\partial a}{\partial t}d\zeta\int_0^1 \frac{dx}{a}-\int_0^1 \frac{1}{a^2}\frac{\partial a}{\partial t}d\zeta\int_x^1 \frac{d\zeta}{a}}{\left(\int_0^1 \frac{d\zeta}{a}\right)^2}\int_0^x (\zeta-\zeta^2)\overline{\nabla}d\zeta dxdt$$
$$-\int_Q w\frac{\int_0^x \frac{1}{a^2}\frac{\partial a}{\partial t}d\zeta\int_0^1 \frac{dx}{a}-\int_0^1 \frac{1}{a^2}\frac{\partial a}{\partial t}d\zeta\int_0^x \frac{d\zeta}{a}}{\left(\int_0^1 \frac{d\zeta}{a}\right)^2}\int_x^1 (\zeta-\zeta^2)\overline{\nabla}d\zeta dxdt. \tag{54}$$

Definition 2. *By a weak solution of problem (44)-(47) we mean a function* $w \in L^2\left(0,T:V^{1,0}(0,1)\right)$ *satisfying the identity (53) and the integral condition (47).*

We will construct an iteration sequence in the following way.
Starting with $w_0=0$, the sequence $(w_n)_{n\in\mathbb{N}}$ is defined as follows: given w_{n-1}, then for $n\geq 1$, we solve the problem

$$\pounds w_n=\frac{\partial w_n}{\partial t}-\frac{\partial}{\partial x}\left(a\frac{\partial w_n}{\partial x}\right)=F\left(x,t,w_{n-1},\frac{\partial w_{n-1}}{\partial x}\right), \tag{55}$$

$$lw_n=w_n(x,0)=0, \tag{56}$$

$$w_n(0,t)=w_n(1,t), \tag{57}$$

$$\int_0^1 w_n(x,t)\,dx=0. \tag{58}$$

From Theorem 1 and Theorem 2, we deduce that for fixed n, each problem (55) -(58) has a unique solution $w_n(x,t)$. If we set $V_n(x,t)=w_{n+1}(x,t)-w_n(x,t)$, we obtain the new problem

$$\pounds V_n=\frac{\partial V_n}{\partial t}-\frac{\partial}{\partial x}\left(a\frac{\partial V_n}{\partial x}\right)=\sigma_{n-1}, \tag{59}$$

$$lV_n=V_n(x,0)=0, \tag{60}$$

$$V_n(0,t)=V_n(1,t), \tag{61}$$

$$\int_0^1 V_n(x,t)\,dx=0. \tag{62}$$

where

$$\sigma_{n-1}=F\left(x,t,w_n,\frac{\partial w_n}{\partial x}\right)-F\left(x,t,w_{n-1},\frac{\partial w_{n-1}}{\partial x}\right). \tag{63}$$

Lemma 2. *Assume that the condition (48) holds, for the linearized problem (59)-(62), there exists a positive constant k, such that*

$$\|V_n\|_{L^2\left(0,T:V^{1,0}(0,1)\right)}\leq k\|V_{n-1}\|_{L^2\left(0,T:V^{1,0}(0,1)\right)}. \tag{64}$$

Proof. We denote by

$$MV_n=x^2(1-x)^2\frac{\partial V_n}{\partial t}+\int_x^1 \frac{d\xi}{a}\int_0^x \frac{2(2\xi-1)(\xi-\xi^2)+\lambda\int_0^\xi \frac{d\eta}{a}}{\int_0^1 \frac{dx}{a}}\frac{\partial V_n}{\partial t}d\xi$$
$$+\int_0^x \frac{d\xi}{a}\int_x^1 \left(\lambda-\frac{2(2\xi-1)(\xi-\xi^2)+\lambda\int_0^\xi \frac{d\eta}{a}}{\int_0^1 \frac{dx}{a}}\right)\frac{\partial V_n}{\partial t}d\xi.$$

Using the same arguments as in the proof of Theorem 1 we get

$$\|V_n\|^2_{L^2\left(0,T:V^{1,0}(0,1)\right)}\leq k^2\|V_{n-1}\|^2_{L^2\left(0,T:V^{1,0}(0,1)\right)}, \tag{65}$$

where

$$k^2=2d^2\frac{\left(\frac{128a_1^2}{a_0^2}+\frac{32\lambda^2a_1^2}{a_0^4}\right)+1}{\min\left(\frac{ca_0-c_1}{2},\frac{\lambda c}{2}\right)}exp(cT).$$

Since $V_n(x,t) = w_{n+1}(x,t) - w_n(x,t)$, then the sequence $w_n(x,t)$ can be written as follows

$$w_n(x,t) = \sum_{k=1}^{k=n-1} V_k + w_0(x,t),$$

the sequence $w_n(x,t)$ converge to an element $w \in L^2\left(0,T:V^{1,0}(0,1)\right)$ if

$$d^2 < \frac{\min\left(\frac{ca_0-c_1}{2}, \frac{\lambda c}{2}\right)}{2\left(\frac{128a_1^2}{a_0^2} + \frac{32\lambda^2 a_1^2}{a_0^4}\right)+1} exp(-cT.)$$

Now to prove that this limit function w is a solution of the problem under consideration (59)-(62), we should show that w satisfies (47) and (53).
For problem (55)-(58), we have

$$H(w_n - w, v) + H(w, v) =$$
$$\int_Q \frac{x-x^2}{\int_0^1 \frac{d\zeta}{a}} \overline{v}\left[\int_x^1 d\zeta \int_\zeta^1 \frac{d\eta}{a}\left(F\left(\eta,t,w_{n-1},\frac{\partial w_{n-1}}{\partial \eta}\right) - F\left(\eta,t,w,\frac{\partial w}{\partial \eta}\right)d\eta\right)dxdt\right.$$
$$\left. + \int_0^x d\zeta \int_0^\zeta \frac{d\eta}{a}\left(F\left(\eta,t,w_{n-1},\frac{\partial w_{n-1}}{\partial \eta}\right) - F\left(\eta,t,w,\frac{\partial w}{\partial \eta}\right)d\eta\right)dxdt\right]$$
$$+ \int_Q \frac{x-x^2}{\int_0^1 \frac{d\zeta}{a}} \overline{v}\left(\int_x^1 d\zeta \int_\zeta^1 \frac{d\eta}{a} F\left(\eta,t,w,\frac{\partial w}{\partial \eta}\right)d\eta + \int_0^x d\zeta \int_0^\zeta \frac{d\eta}{a} F\left(\eta,t,w,\frac{\partial w}{\partial \eta}\right)d\eta\right)dxdt. \tag{66}$$

Integrating with respect to t and x, using the conditions (49) and $\varepsilon-$inequalities, we obtain

$$|H(w_n - w, v)| \le C\|w_n - w\|_{L^2(0,T:V^{1,0}(0,1))}\left(\int_Q \left(x^2(1-x)^2\left|\frac{\partial v}{\partial t}\right|^2 + |v|^2\right)dxdt\right)^{\frac{1}{2}}, \tag{67}$$

where

$$C = \max\left(1, \frac{4a_1^2}{a_0^2}\max(|c_1|,|c_2|), a_1\right).$$

Integrating with respect to x, the first two terms in (66), using (49), (67) and by passing to the limit as $n \to \infty$, we obtain

$$H(w,v) =$$
$$\int_Q \frac{x-x^2}{\int_0^1 \frac{d\zeta}{a}} \overline{v}\left(\int_x^1 d\zeta \int_\zeta^1 \frac{d\eta}{a} F\left(\eta,t,w,\frac{\partial w}{\partial \eta}\right)d\eta + \int_0^x d\zeta \int_0^\zeta \frac{d\eta}{a} F\left(\eta,t,w,\frac{\partial w}{\partial \eta}\right)d\eta\right)dxdt$$

now we show that (47) holds. Since $\lim_{n\to+\infty}\|w_n - w\|_{L^2(0,T:V^{1,0}(0,1))} = 0$, then

$$\lim_{n\to+\infty}\left|\int_0^1 (w_n - w)\,dx\right| \le \lim_{n\to+\infty}\int_0^1 |w_n - w|^2\,dx \to 0. \tag{68}$$

from (68) we conclude that $\int_0^1 w dx = 0$.

Thus, we have proved the following

Theorem 3.*If condition (48) is satisfied, then the solution of problem (44)-(47) is unique.*

*Proof.*Suppose that $w_1, w_2 \in L^2\left(0,T:V^{1,0}(0,1)\right)$ are two solution of (44)-(47), then the function $v = w_1 - w_2$ is in $L^2\left(0,T:V^{1,0}(0,1)\right)$ and satisfies

$$\frac{\partial v}{\partial t} - \frac{\partial}{\partial x}\left(a\frac{\partial v}{\partial x}\right) = G(x,t), \tag{69}$$
$$v(x,0) = 0, \tag{70}$$
$$v(0,t) = v(1,t), \tag{71}$$
$$\int_0^1 v dx = 0. \tag{72}$$

Where $G(x,t) = F\left(x,t,w_1,\frac{\partial w_1}{\partial x}\right) - F\left(x,t,w_2,\frac{\partial w_2}{\partial x}\right)$.
Taking the inner product in $L^2(Q)$ of equation (69) with the integro-differential operator

$$Mv = x^2(1-x)^2\frac{\partial v}{\partial t} + \int_x^1 \frac{d\xi}{a}\int_0^x \frac{2(2\xi-1)(\xi-\xi^2) + \lambda\int_0^\xi \frac{d\eta}{a}}{\int_0^1 \frac{dx}{a}}\frac{\partial v}{\partial t}d\xi$$
$$+ \int_0^x \frac{d\xi}{a}\int_x^1 \left(\lambda - \frac{2(2\xi-1)(\xi-\xi^2) + \lambda\int_0^\xi \frac{d\eta}{a}}{\int_0^1 \frac{dx}{a}}\right)\frac{\partial v}{\partial t}d\xi.$$

where $\lambda > 2a_1$ and following the same procedure done in the proof of Lemma 1, we get

$$\|v\|^2_{L^2(0,T:V^{1,0}(0,1))} \le k^2\|v\|^2_{L^2(0,T:V^{1,0}(0,1))}.$$

where

$$k^2 = \frac{2\left(\frac{128a_1^2}{a_0^2} + \frac{16\lambda^2 a_1^2}{a_0^4}\right)+1}{\min\left(\frac{ca_0-c_1}{2}, \frac{\lambda c}{2}\right)} d^2 exp(cT).$$

Since $k^2 < 1$, then $v = 0$, which implies that $w_1 = w_2 \in L^2\left(0,T:V^{1,0}(0,1)\right)$.

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