Applied Mathematics & Information Sciences Letters
An International Journal

http://dx.doi.org/10.12785/amisl/030204

The Generalized Random Variable Appears the Trace of Fractional Calculus in Statistics

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Received: 5 Aug. 2014, Revised: 14 Mar. 2015, Accepted: 17 Mar. 2015
Published online: 1 May 2015

Abstract: In this paper, introducing a generalized random variable, we obtained a new relationship between fractional calculus and statistics. With this definition, the domain of shape parameter space for some of distributions, such as Gamma, Beta and Weibull, were expanded from $(0,\infty)$ to $(-1,\infty)$. It is shown that the expectation of such random variable , $\Phi_\alpha(x)$, coincides with the fractional integral of probability density function (PDF), at the origin, for $\alpha > 0$, and also the fractional derivative of PDF, at the origin, for $-1 < \alpha < 0$. We also, presented PDFs of such distributions as a product of fractional derivation of Dirac delta function of shape parameter order, $\delta^{(\alpha)}(.)$. Finally, we showed that the Liouville fractional differintegral operator on the moment generating function (MGF) of positive random variable, at the origin, gives fractional moments.

Keywords: Fractional calculus, Dirac delta function, Fractional moments, Weibull distribution, Gamma distribution, Beta distribution

1 Introduction

The study of generalized functions is now widely used in applied mathematics and engineering sciences. Generalized functions are defined as a linear functional on a space X of conveniently chosen test functions.
For every locally integrable function $f \in \mathscr{L}^1_{loc}(\mathbb{R})$, there exists a distribution $F_f : X \to \mathbb{C}$ defined by:

$$F_f(\varphi) = \langle f,\varphi\rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)dx \tag{1}$$

where $\varphi \in X$ is test function from a suitable space X of test functions. A distribution that corresponds to functions via equation (1) are called regular distributions. Examples for regular distributions are the convolution kernels $K^\alpha_\pm \in \mathscr{L}^1_{loc}(\mathbb{R})$ defined as:

$$K^\alpha_\pm = H(\pm x)\frac{\pm x^{\alpha-1}}{\Gamma(\alpha)} \tag{2}$$

for $\alpha > 0$ and $H(x)$ is a Heaviside unit step function. Distributions that are not regular are sometimes called singular. An example for a singular distribution is the Dirac delta function which is defined as:

$$\delta : X \to \mathbb{C}$$
$$\int \delta(x)\varphi(x)dx = \varphi(0)$$

for every test function $\varphi \in X$. The test function space X is usually chosen as a subspace of $C^\infty(\mathbb{R})$, the space of infinitely differentiable functions [6].
In the present work, we suppose that X is a positive random variable and α is the shape parameter of distribution. The new random variable is represented as the function $\Phi_\alpha(x)$ and defined by $\Phi_\alpha(x) = \frac{X_+^{\alpha-1}}{\Gamma(\alpha)}$. The function $\Phi_\alpha(x)$ can be extended to all complex values of α as a pseudo function and is a distribution whose support is $[0,\infty)$ except for the case $\alpha = 0,-1,\ldots$. Since distributions or generalized functions in mathematical analysis are not really functions in the classical sense, we also call our proposed random variable a generalized random variable. The expectation of this generalized random variable coincides with Riemann-Liouville left fractional integral of the PDF, at the origin, for $\alpha > 0$ and Marchaud fractional derivative of the PDF, at the origin, for $-1 < \alpha < 0$. The generalized random variable appears in some distributions that belong to the

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exponential family like Weibull, Gamma, Beta, Binomial and poisson distributions. By this generalized random variable, we more expand the domain of shape parameter space. For example, the domain of shape parameter which has been already expanded for Gamma and Weibull distributions of $\alpha > 0$ to $\alpha > -1$ and for Beta distribution from $(0,\infty)\times(0,\infty)$ to $(-1,\infty)\times(0,\infty)$. In case of negative values of the shape parameter space, the relationship between fractional derivatives and statistics theory can be obtained, this means we can write the PDF of distributions like Gamma, Weibull, Beta as a product of fractional derivatives of Dirac delta function. The characteristics of this generalized random variable are as following:

(a) By rewriting Binomial and poisson distributions in terms of the generalized random variable, they are respectively transformed into Gamma and Beta distributions with a discrete parameter space, that is

$$[P(x;\lambda);\Phi_\lambda(x)] = [\Gamma(\lambda;x,1);\Phi_x(\lambda)],\ \ x=0,1,...$$

and

$$[Bin(x;p);\Phi_p(x)] = [B(p;x,n-x+2);\Phi_x(p)]$$

where $x=1,2,...,n$ and $[.,\Phi]$ is the rewriting form of generalized random variable.

(b) The expectation of this generalized random variable, $\Phi_\alpha(x)$, coincides with Riemann-Liouville left fractional integral of the PDF at the origin for $\alpha > 0$ and Marchaud fractional derivative of the PDF at the origin for $-1<\alpha<0$, that is, we have

$$E[\Phi_\alpha(X)] = \begin{cases} (I_-^\alpha f)(0), & \alpha > 0 \\ (D_-^\alpha f)(0), & -1<\alpha<0 \end{cases} \tag{3}$$

where

$$(I_-^\alpha f)(x) = \frac{1}{\Gamma(\alpha)}\int_0^\infty t^{\alpha-1} f(x+t)dt \tag{4}$$

is the Riemann-Liouville left fractional integral, while

$$(\mathbf{D}_-^\alpha f)(x) = \frac{1}{\Gamma(-\alpha)}\int_0^\infty t^{-\alpha-1}\{f(x+t)-f(x)\}dt \tag{5}$$

is the Marchaud fractional derivative.

(c) The integer and fractional derivatives of this generalized random variable are the generalized random variable, too.

(d) Another property is

$$\Phi_\alpha(t-a) *\Phi_\beta(t) = \Phi_{\alpha+\beta}(t-a), \tag{6}$$

where $\alpha > -1$ and $\beta > -1$ such that $\alpha+\beta > -1$ and we used the star notation for the convolution operation.
The proof is easy for $\alpha > 0$ and $\beta > 0$, as instance, see [2]. Other values of α and β can be proved using analytic continuation.

(e) If $X_1,X_2,...,X_n$ be $i.i.d.$ $W(x;\alpha,\theta)$ then

$$\Phi_{\alpha+1}(x) \sim \Gamma(1,\Gamma(\alpha+1)\theta),$$

and

$$\frac{\Phi_{\alpha+1}(x)}{\Sigma_1^n \Phi_{\alpha+1}(x_i)} \sim B(1,n-1).$$

Also, we demonstrate that the αth moment, $\alpha\in\mathbb{R}$, of the positive random variable X can be obtained directly from the Liouville fractional derivation and integrations of the MGF at the origin.

2 Preliminaries

In this section, we introduce notations, definitions and preliminary facts which are used throughout this paper. We need some basic definitions and properties of the fractional calculus theory and the generalized functions theory which are used further in this paper. As mentioned in references [1] and [3], the definitions 1 and 2 of fractional calculus are as following:

Definition 1. For a function f defined on an interval $[a,b]$, the Riemann-Liouville (R-L) integrals $I_{a+}^\alpha f$ and $I_{b-}^\alpha f$ of order $\alpha\in\mathbb{C}$, $(\mathscr{R}(\alpha)>0)$ are defined, respectively, by

$$(I_{a+}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\xi)^{\alpha-1} f(\xi)d\xi \tag{7}$$

and

$$(I_{b-}^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (\xi-t)^{\alpha-1} f(\xi)d\xi. \tag{8}$$

Also, the left and right R-L fractional derivations $D_{a+}^\alpha f$ and $D_{b-}^\alpha f$ of order $\alpha\in\mathbb{C}$, $(\mathscr{R}(\alpha)>0)$ are defined, respectively, by

$$(D_{a+}^\alpha f)(t) = (\frac{d}{dt})^n (I_{a+}^{n-\alpha} f)(t), \tag{9}$$

and

$$(D_{b-}^\alpha f)(t) = (\frac{-d}{dt})^n (I_{a+}^{n-\alpha} f)(t). \tag{10}$$

If in above definition, respectively, $a=-\infty$ and $b=\infty$, then we get Liouville fractional differintegral.

Definition 2. The Liouville fractional differintegral $D_\pm^\alpha$ is defined by

$$D_{+}^{\alpha}\{f(x)\} = \begin{cases} frac1\Gamma(-\alpha)\int_{-\infty}^{x}(x-t)^{-\alpha-1}f(t)dt, \\ \frac{d^n}{dx^n}\{D_{+}^{\alpha-n}\{f(x)\} \end{cases} \tag{11}$$

where the first expression satisfies for $\mathscr{R}(\alpha) < 0$ and the second expression satisfies for $\mathscr{R}(\alpha) > 0; n = [\mathscr{R}(\alpha)] + 1$. Also

$$D_{-}^{\alpha}\{f(x)\} = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_{x}^{\infty}(x-t)^{-\alpha-1}f(t)dt, \\ \frac{(-1)^n d^n}{dx^n}\{D_{-}^{\alpha-n}\{f(x)\} \end{cases} \tag{12}$$

with $\mathscr{R}(\alpha) < 0$ for the first expression and $\mathscr{R}(\alpha) > 0$ such that $n = [\mathscr{R}(\alpha)] + 1$, for the second expression satisfies. In particular, when $\alpha = n \in \mathbb{N}_0$, then

$$D_{+}^{0}\{f(x)\} = f(x), D_{+}^{n}\{f(x)\} = f^{(n)}(x), \tag{13}$$

and

$$D_{-}^{0}\{f(x)\} = f(x), D_{-}^{n}\{f(x)\} = (-1)^n f^{(n)}(x). \tag{14}$$

Definition 3. Let f be a generalized function $f \in C_0^{\infty}(\mathbb{R})'$ with $suppf \subset \mathbb{R}^+$. Then its fractional integral is the distribution $I_{0+}^{\alpha} f$ defined as:

$$\langle I_{0+}^{\alpha} f, \varphi\rangle = \langle I^{\alpha} f, \varphi\rangle = \langle K_{+}^{\alpha} * f, \varphi\rangle \tag{15}$$

for $\mathscr{R}(\alpha) > 0$ [6]. Also, the fractional derivative of order α with lower limit 0 is the distribution $D^{\alpha}\{f(z)\}$defined as:

$$\langle D_{0+}^{\alpha} f, \varphi\rangle = \langle D^{\alpha} f, \varphi\rangle = \langle K_{+}^{-\alpha} f, \varphi\rangle \tag{16}$$

where $\alpha \in \mathbb{C}$ and

$$K_{+}^{\alpha}(x) = \begin{cases} H(x)\frac{x^{\alpha-1}}{\Gamma(\alpha)}, & \mathscr{R}(\alpha) > 0 \\ \frac{d^n}{dx^n}[H(x)\frac{x^{\alpha+n-1}}{\Gamma(\alpha+n)}], & \mathscr{R}(\alpha)+n > 0; n \in \mathbb{N} \end{cases} \tag{17}$$

is the kernel distribution. For $\alpha = 0$ one can finds $K_{+}^{0}(x) = (\frac{d}{dx})H(x) = \delta(x)$ and $D_{0+}^{0} = I$ as the identity operator. For the $\alpha = -n;\ n \in \mathbb{N}$, one can finds

$$K_{+}^{-n}(x) = \delta^{(n)}(x) \tag{18}$$

where $\delta^{(n)}$ is the nth derivative of the δ distribution. The kernel distribution in equation (17) is given by:

$$K_{+}^{\alpha}(x) = \frac{d}{dx}[H(x)\frac{x^{-\alpha}}{\Gamma(1-\alpha)}] = \frac{d}{dx}K_{+}^{1-\alpha}(x) \tag{19}$$

for $0 < \alpha < 1$. [6]

Now if $f \in C_0^{\infty}(\mathbb{R})'$ with $suppf \subset \mathbb{R}^+$, then

$$D_{0+}^{\alpha} f = D_{0+}^{\alpha}(If) = (K_{+}^{-\alpha} \cdot K_{+}^{0}) \cdot f = \delta^{(\alpha)} \cdot f \tag{20}$$

for all $\alpha \in \mathbb{C}$. Also, the differentiation rule

$$D_{0+}^{\alpha} K_{+}^{\beta} = K_{+}^{\beta-\alpha} \tag{21}$$

holds for all $\beta, \alpha \in \mathbb{C}$. It contains

$$DK_{+}^{\beta} = K_{+}^{\beta-1}$$

for all $\beta \in \mathbb{C}$ as a special case [6].

3 The generalized Weibull random variable

The Wiebull distribution was originally used for modeling fatigue data and is at present used extensively in engineering problems for modeling, the distribution of the lifetime of an object which consists of several parts and that fails if any component part fails [7].
It is important to note that the fractional calculus has also been applied in describing several PDFs in mathematical statistics in terms of fractional integral and derivative of exponential functions. For instance, the Wiebull distribution can be written in terms of the fractional derivative or integral of $e^{-\theta x^{\alpha}}$, as

$$\theta^{1-\alpha}\Gamma(\alpha+1)\Phi_{\alpha}(x)(D_{-;x^{\alpha}}^{\alpha} e^{-\theta t^{\alpha}})(x), \tag{22}$$

or

$$\theta^{1+\alpha}\Gamma(\alpha+1)\Phi_{\alpha}(x)(I_{-;x^{\alpha}}^{\alpha} e^{-\theta t^{\alpha}})(x), \tag{23}$$

where the subscript x^{α} indicates fractional derivative or integral with respect to the variable x^{α}. In this way representation of the PDF might introduce novel statistical interpretations in the study of statistical problems involving such these PDFs (at least in some cases).
We will define the Weibull distribution as a two-parameter family of distribution functions, in which the parameter $\theta > 0$ is the scale parameter and $-1 < \alpha$ is the shape parameter. So, this distribution can be defined as

$$\Gamma^2(\alpha+1)\Phi_{\alpha+1}(\theta)D\Phi_{\alpha+1}(x)e^{-\Gamma^2(\alpha+1)\Phi_{\alpha+1}(\theta)\Phi_{\alpha+1}(x)}.$$

On the other hand, the scale parameter θ, can also be considered as a random variable. This means that, depending on the variable which is being differentiated, we have the distribution of that random variable. Therefore, the PDF can be written as follows:

$$-D_x e^{-\Gamma^2(\alpha+1)\Phi_{\alpha+1}(\theta)\Phi_{\alpha+1}(x)}. \tag{24}$$

And also, if X be random variable of Wiebull distribution as following:

$$\Gamma^2(\alpha+1)\Phi_{\alpha+1}(\theta)D\Phi_{\alpha+1}(x)e^{-\Gamma^2(\alpha+1)\Phi_{\alpha+1}(\theta)\Phi_{\alpha+1}(x)}, \tag{25}$$

by using equation (21) with $\beta=0$ and equation (17), we can rewrite the PDF in terms of the fractional derivation of Dirac delta function, as following definition.

Definition 3.1. Suppose that X be a random variable of Weibull distribution as a two-parameter family of distribution functions, in which the parameter θ is the scale parameter and α is the shape parameter. Then, *PDF* of this distribution can be defined as

$$f(x)=\begin{cases}\Gamma^2(\alpha+1)\Phi_{\alpha+1}(\theta)K_+^{\alpha}(x)e^{-\Gamma^2(\alpha+1)\Phi_{\alpha+1}(\theta)\Phi_{\alpha+1}(x)}\\ \Gamma^2(1-\alpha)\Phi_{1-\alpha}(\theta)\delta^{(\alpha)}(x)e^{-\Gamma^2(1-\alpha)\Phi_{1-\alpha}(\theta)\Phi_{1-\alpha}(x)}\end{cases} \tag{26}$$

where the frist expression satisfies for $0<\alpha$ and the second expression satisfies for $-1<\alpha\le 0$ (For comfortable computationally, we are showing $-1<\alpha\le 0$ with $-\alpha$.). It can write, for $0<\alpha<1$,

$$f_X(x)=\Gamma^2(\alpha)\Phi_{\alpha}(\theta)\delta^{(\alpha-1)}(x)e^{-\Gamma^2(\alpha)\Phi_{\alpha}(\theta)\Phi_{\alpha}(x)}.$$

Now we are showing that $\int_0^{\infty}f_X(x)dx=1$, for the case $-1<\alpha\le 0$, too. We have

$$\int_0^{\infty}f_X(x)dx=\Gamma(1-\alpha)\theta^{-\alpha}\int_0^{\infty}\delta^{(\alpha)}(x)\times e^{-\Gamma(1-\alpha)\theta^{-\alpha}\frac{x^{-\alpha}}{\Gamma(1-\alpha)}}dx,$$

by using equation (19)

$$=\Gamma(1-\alpha)\theta^{-\alpha}\int_0^{\infty}\delta^{(\alpha)}(x).e^{-\Gamma(1-\alpha)\theta^{-\alpha}\delta^{(\alpha-1)}(x)}dx,$$

under the substitution $u=\delta^{(\alpha-1)}(x)$,

$$=\Gamma(1-\alpha)\theta^{-\alpha}\int_0^{\infty}e^{-\Gamma(1-\alpha)\theta^{-\alpha}u}du,$$
$$=\Gamma(1-\alpha)\theta^{-\alpha}.\Gamma^{-1}(1-\alpha)\theta^{\alpha},$$

which proves our result.
Also the failure rate (or hazard function) for distribution is given by

$$h(x)=\begin{cases}\Gamma^2(\alpha+1)\Phi_{\alpha+1}(\theta)K_+^{\alpha}(x), & \alpha>0\\ \Gamma^2(1-\alpha)\Phi_{1-\alpha}(\theta)\delta^{(\alpha)}(x), & -1<\alpha\le 0.\end{cases} \tag{27}$$

So, we succeed to represent the PDF of Weibull distribution with the extended shape parameter space, $-1<\alpha$, as a product of fractional derivation of Dirac delta function of shape parameter order. Also the scale parameter can be considered as a random variable.

4 The generalized Gamma random variable

In this section, we will define the Gamma distribution as a two-parameter family of distribution functions, in which the parameter $\beta>0$ is the scale parameter, and $-1<\alpha$ is the shape parameter. This distribution is defined as

$$\Gamma(\alpha+1)\Phi_{\alpha+1}(\beta)D\Phi_{\alpha+1}(x)e^{-\Phi_2(\beta)\Phi_2(x)},$$

The scale parameter β, can also be considered as a random variable. This means that, depending on which variable is under differentiation, we have the distribution of that random variable; that is, the PDF can be rewritten as:

$$\Gamma(\alpha+1)\Phi_{\alpha+1}(\beta)D_x\Phi_{\alpha+1}(x)exp-\Phi_2(\beta)\Phi_2(x), \tag{28}$$

or as

$$\Gamma(\alpha+1)D_{\beta}\Phi_{\alpha+1}(\beta)\Phi_{\alpha+1}(x)e^{-\Phi_2(\beta)\Phi_2(x)}. \tag{29}$$

Similarly as a general case of Weibull distribution, if X be a random variable, by using equation (21) with $\beta=0$ and equation (17), we can write its PDF as following definition:

Definition 4.1. Suppose that X be a random variable of Gamma distribution as a two-parameter family of distribution functions, in which the parameter β is the scale parameter and α is the shape parameter. Then, *PDF* of this distribution can be defined as

$$f_X(x)=\begin{cases}\Gamma(\alpha+1)\Phi_{\alpha+1}(\beta)K_+^{\alpha}(x)e^{-\Phi_2(\beta)K_+^2(x)}\\ \Gamma(1-\alpha)\Phi_{1-\alpha}(\beta)\delta^{(\alpha)}(x)e^{-\Phi_2(\beta)K_+^2(x)}\end{cases} \tag{30}$$

where the first expression satisfies for $0<\alpha$ and the second expression satisfies for $-1<\alpha\le 0$.
It can write, for $0<\alpha<1$,

$$f_X(x)=\Gamma(\alpha)\Phi_{\alpha}(\beta)\delta^{(\alpha-1)}(x)e^{-\Phi_2(\beta)K_+^2(x)}.$$

The MGF of the Gamma generalized random variable is as following:

$$M_X(-t)=\beta^{-\alpha}(t+\beta)^{\alpha}, \qquad -1<\alpha\le 0. \tag{31}$$

Since $\mathbf{L}\{\delta^{(\alpha)}(x)\}=s^{\alpha}$, which implied that $\int_0^{\infty}f_X(x)dx=1$ for $-1<\alpha\le 0$ and the MGF be as above, such that we have

$$\int_0^{\infty}f_X(x)dx=\beta^{-\alpha}\int_0^{\infty}\delta^{(\alpha)}(x).e^{-\beta x}dx \tag{32}$$
$$=\beta^{-\alpha}.\beta^{\alpha}, \tag{33}$$

which proves our result. And also

$$E[e^{-tX}] = \beta^{-\alpha}\int_0^\infty \delta^{(\alpha)}(x).e^{-\beta x}.e^{-tx}dx \quad (34)$$
$$= \beta^{-\alpha}\int_0^\infty \delta^{(\alpha)}(x).e^{-(\beta+t)x}dx \quad (35)$$
$$= \beta^{-\alpha}(\beta+t)^{\alpha}. \quad (36)$$

Since the Fourier transform $\mathbf{F}\{\delta^{(\alpha)}(x)\} = (iw)^{\alpha}$, which implied that the characteristic function (CF) of this distribution equals to $\beta^{-\alpha}(it+\beta)^{\alpha}$, for $-1<\alpha\leq 0$.
Therefore, this form of the PDF of Gamma distribution indicates that the extended shape parameter appears the relationship between fractional calculus and Statistics. Also, it allows the representation of the PDF as a product of fractional derivation of Dirac delta function of shape parameter order. Also, the scale parameter can also be considered as a random variable.

5 The generalized Beta random variable

The Beta distribution is a two-parameter (α and β) family of density functions $f_X(x) = \dfrac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}$ for $0\leq x\leq 1$ [7]. It is often used to represent the proportions and percentages. It has the following probability density function:

$$\Gamma(\alpha+\beta)D_x\Phi_{\alpha+1}(x)D_x\Phi_{\beta+1}(y), \quad y=1-x \quad (37)$$

The PDF by using equation (21) with $\beta=0$ and equation (17), can be written as following definition.

Definition 5.1. suppose that X be a random variable of Beta distribution as a two-parameter family of distribution functions, in which the parameter β and α are the shape parameters. Then *PDF* of this distribution can be defined as

$$f(x) = \begin{cases} \Gamma(\alpha+\beta)K_+^{\alpha}(x)K_+^{\beta}(y), \\ \Gamma(\beta-\alpha)\delta^{(\alpha)}(x)K_+^{\beta}(y), \\ \Gamma(\alpha-\beta)\delta^{(\beta)}(y)K_+^{\alpha}(x), \end{cases} \quad (38)$$

where the first expression satisfies for $0<\alpha$ and $0<\beta$. The second expression satisfies for $-1<\alpha\leq 0,\ 0<\beta$ and $0<\alpha+\beta$.

The third expression satisfies for $-1<\beta\leq 0,\ 0<\alpha$ and $0<\alpha+\beta$.

To show that $\int_0^\infty f_X(x)dx = 1$ for $-1<\alpha\leq 0$, by considering $-1<\alpha\leq 0,\ 0<\beta$ and $0<\alpha+\beta$, we get

$$\int_0^\infty f_X(x)dx = \Gamma(\beta-\alpha)\int_0^1 \delta^{(\alpha)}(x)\frac{(1-x)^{\beta-1}}{\Gamma(\beta)}dx,$$

by using equation (6) with $t=1$ and $a=0$; we have

$$= \Gamma(\beta-\alpha)\int_0^1 \frac{x^{-\alpha-1}}{\Gamma(-\alpha)}.\frac{(1-x)^{\beta-1}}{\Gamma(\beta)}dx,$$
$$= \Gamma(\beta-\alpha).\Gamma^{-1}(\beta-\alpha),$$

which proves our result. This result has obtained with similar way for $-1<\beta\leq 0,\ 0<\alpha$ and $0<\alpha+\beta$ the case.

Therefore, as previous cases, we represented the PDF of Beta distribution as the product of fractional derivation of Dirac delta function. Also, we extended the parameter space from $(0,\infty)\times(0,\infty)$ to $(-1,\infty)\times(0,\infty)$.

6 the Liouville fractional differintegral operator on the MGF of positive random variable, at the origin, is giving fractional moments

Recently, fractional moments of the type $E[X^{nq}]$, where $n\in\mathbb{N}$ and $0<q\leq 1$, have been introduced [4], such quantities have important characteristics: (i) they are exact natural generalization of integer moments as like as fractional differential operators generalize the classical differential calculus; (ii) the interesting point is the relationship between fractional moments and the fractional special functions. Also, in [5] it was shown that complex fractional moments, which are complex moments of order nqth of a certain distribution, are equivalent to Caputa fractional derivation of generalized characteristic function in origin, such that when $q=1$ the case was reduced to the complex moments. The fractional moments mentioned above are moments of positive real order, but in here, they are moments of real order.
Now we demonstrate that the fractional moments of a positive random variable, the αth moments of X, can be obtained directly from Liouville fractional integrals and derivations of the MGF at the origin, as in following theorem.

Theorem 6.1. Suppose that X be a positive and continuous random variable and its MGF be infinite. then for $0\leq\alpha$, the αth moments of X can be obtained as following:

$$I_+^{\alpha}\{M_X(t)\}\mid_{t=0} = I_-^{\alpha}\{M_X(-t)\}\mid_{t=0} = E[X^{-\alpha}], \quad \alpha>0 \quad (39)$$

and

$$D_+^{\alpha}\{M_X(t)\}\mid_{t=0} = D_-^{\alpha}\{M_X(-t)\}\mid_{t=0} = E[X^{\alpha}], \quad 0\leq\alpha \quad (40)$$

in particular when $\alpha=n\in\mathbb{N}$,

$$D_+^{n}\{M_X(t)\}\mid_{t=0} = M_X^{(n)}(t)\mid_{t=0} = E[X^n], \quad (41)$$

and

$$D_{-}^{n}\{M_X(-t)\}\mid_{t=0}=(-1)^n M_X^{(n)}(-t)\mid_{t=0}=E[X^n]. \quad (42)$$

Proof. We have:

$$\begin{aligned}I_{-}^{\alpha}\{M_X(-t)\}&=I_{-}^{\alpha}\{\int_0^\infty e^{-xt}f_X(x)dx\}\\&=\frac{1}{\Gamma(\alpha)}\int_t^\infty(u-t)^{\alpha-1}\int_0^\infty e^{-xu}f_X(x)dxdu\\&=\frac{1}{\Gamma(\alpha)}\int_t^\infty\int_0^\infty(u-t)^{\alpha-1}e^{-xu}f_X(x)dxdu,\end{aligned}$$

by using the the Fubini theorem, we get

$$=\frac{1}{\Gamma(\alpha)}\int_0^\infty\int_t^\infty(u-t)^{\alpha-1}e^{-xu}f_X(x)dudx,$$

under the substitution $u-t=\frac{y}{x}$, we have

$$\begin{aligned}&=\frac{1}{\Gamma(\alpha)}\int_0^\infty\int_0^\infty y^{\alpha-1}x^{-\alpha}e^{-y}e^{-xt}f_X(x)dydx\\&=\int_0^\infty x^{-\alpha}e^{-xt}f_X(x)dx\\&=E[X^{-\alpha}e^{-Xt}],\end{aligned}$$

then

$$I_{-}^{\alpha}\{M_X(-t)\}\mid_{t=0}=E[X^{-\alpha}e^{-Xt}]\mid_{t=0}=E[X^{-\alpha}],$$

for $\alpha>0, X>0$. With similar way we get

$$\begin{aligned}I_{+}^{\alpha}\{M_X(t)\}&=I_{+}^{\alpha}\{\int_0^\infty e^{xt}f_X(x)dx\}\\&=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^t(t-u)^{\alpha-1}\int_0^\infty e^{xu}f_X(x)dxdu\\&=\frac{1}{\Gamma(\alpha)}\int_{-\infty}^t\int_0^\infty(t-u)^{\alpha-1}e^{xu}f_X(x)dxdu,\end{aligned}$$

again, by using the Fubini theorem, we have

$$=\frac{1}{\Gamma(\alpha)}\int_0^\infty\int_{-\infty}^t(t-u)^{\alpha-1}e^{xu}f_X(x)dudx,$$

and under the substitution $t-u=\frac{y}{x}$

$$\begin{aligned}&=\frac{1}{\Gamma(\alpha)}\int_0^\infty\int_0^\infty y^{\alpha-1}x^{-\alpha}e^{-y}e^{xt}f_X(x)dydx\\&=\int_0^\infty x^{-\alpha}e^{xt}f_X(x)dx\\&=E[X^{-\alpha}e^{Xt}],\end{aligned}$$

then

$$I_{+}^{\alpha}\{M_X(t)\}\mid_{t=0}=E[X^{-\alpha}e^{Xt}]\mid_{t=0}=E[X^{-\alpha}],$$

for $\alpha>0, X>0$. Also, we have

$$\begin{aligned}D_{-}^{\alpha}\{M_X(-t)\}&=(-1)^m(\frac{d}{dt})^m I_{-}^{m-\alpha}\{M_X(-t)\}\\&=(-1)^m(\frac{d}{dt})^m I_{-}^{m-\alpha}\{\int_0^\infty e^{-xt}f_X(x)dx\}\\&=(-1)^m(\frac{d}{dt})^m\frac{1}{\Gamma(m-\alpha)}\\&\quad\times\int_t^\infty(u-t)^{m-\alpha-1}\int_0^\infty e^{-xu}f_X(x)dxdu\\&=(-1)^m(\frac{d}{dt})^m\frac{1}{\Gamma(m-\alpha)}\\&\quad\times\int_t^\infty\int_0^\infty(u-t)^{m-\alpha-1}e^{-xu}f_X(x)dxdu,\end{aligned}$$

and by using the Fubini theorem, we have

$$\begin{aligned}&=(-1)^m(\frac{d}{dt})^m\frac{1}{\Gamma(m-\alpha)}\\&\quad\times\int_0^\infty\int_t^\infty(u-t)^{m-\alpha-1}e^{-xu}f_X(x)dudx,\end{aligned}$$

under the substitution $u-t=\frac{y}{x}$

$$\begin{aligned}&=(-1)^m(\frac{d}{dt})^m\frac{1}{\Gamma(m-\alpha)}\\&\quad\times\int_0^\infty\int_0^\infty y^{m-\alpha-1}x^{\alpha-m}e^{-y}e^{-xt}f_X(x)dydx\\&=\int_0^\infty x^{\alpha}e^{-xt}f_X(x)dx\\&=E[X^{\alpha}e^{-Xt}],\end{aligned}$$

then

$$D_{-}^{\alpha}\{M_X(-t)\}\mid_{t=0}=E[X^{\alpha}e^{-Xt}]\mid_{t=0}=E[X^{\alpha}],$$

for $X>0$ and $m-1\leq\alpha<m$. Also similarly

$$\begin{aligned}D_{+}^{\alpha}\{M_X(t)\}&=(\frac{d}{dt})^m I_{+}^{m-\alpha}\{M_X(t)\}\\&=(\frac{d}{dt})^m I_{+}^{m-\alpha}\{\int_0^\infty e^{xt}f_X(x)dx\}\\&=(\frac{d}{dt})^m\frac{1}{\Gamma(m-\alpha)}\\&\quad\times\int_t^\infty(t-u)^{m-\alpha-1}\int_0^\infty e^{xu}f_X(x)dxdu\\&=(\frac{d}{dt})^m\frac{1}{\Gamma(m-\alpha)}\\&\quad\times\int_t^\infty\int_0^\infty(t-u)^{m-\alpha-1}e^{xu}f_X(x)dxdu,\end{aligned}$$

by using the Fubini theorem, we have

$$= (\frac{d}{dt})^m \frac{1}{\Gamma(m-\alpha)} \times \int_0^\infty \int_t^\infty (t-u)^{m-\alpha-1} e^{xu} f_X(x) du dx,$$

under the substitution $t-u = \frac{y}{x}$

$$= (\frac{d}{dt})^m \frac{1}{\Gamma(m-\alpha)} \times \int_0^\infty \int_0^\infty y^{m-\alpha-1} x^{\alpha-m} e^{-y} e^{xt} f_X(x) dy dx$$
$$= \int_0^\infty x^\alpha e^{xt} f_X(x) dx$$
$$= E[X^\alpha e^{Xt}],$$

then

$$D_+^\alpha \{M_X(t)\} |_{t=0} = E[X^\alpha e^{Xt}] |_{t=0} = E[X^\alpha],$$

for $X > 0$ and $m-1 \leq \alpha < m$.□

7 Perspective

In this paper, we obtained a new relationship between fractional calculus and statistics by introducing a generalized random variable. With this definition, the domain of shape parameter space of distributions such as Gamma, Weibull and Beta were expanded from $(0,\infty)$ to $(-1,\infty)$. Also, we showed that the Liouville fractional differintegral operator on the MGF of a positive random variable, at the origin, gives fractional moments.

Acknowledgement

We would like to thank the referees for a careful reading of our paper and lot of valuable suggestions on the first draft of the manuscript.

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