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Subclasses of Starlike Functions with a Fixed Point Involving q-hypergeometric Function

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Abstract: Recently, Kanas and Ronning introduced the classes of functions starlike and convex, which are normalized with $f(w) = f'(w) - 1 = 0$, w is a fixed point in the open disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$. The aim of this paper is to continue this investigation and introduce a new class $\mathscr{S}\mathscr{T}_l^m(\alpha, \beta, w)$, of functions which are analytic in Δ. We obtain various results including coefficients estimates, distortion and covering theorems, radii of close-to-convexity, starlikeness and convexity for functions belonging to this class.

Keywords: Analytic function, Starlike function, Convex function, Uniformly convex function, Convolution product, $q-$hypergeometric function.

1 Introduction and Motivations

Let $w(|w| = d)$ be a fixed point in the unit disc $\Delta := \{z \in \mathbb{C} : |z| < 1\}$. Denote by $\mathscr{H}(\Delta)$ the class of functions which are regular and $\mathscr{A}(w) = \{f \in H(\Delta) : f(w) = f'(w) - 1 = 0\}$. Also denote by $\mathscr{S}_w = \{f \in \mathscr{A}(w) : f$ is univalent in $\Delta\}$, the subclass of $\mathscr{A}(w)$ consisting of the functions of the form

$$f(z) = (z-w) + \sum_{n=2}^{\infty} a_n (z-w)^n, \tag{1}$$

which are analytic in Δ. Denote by $\mathscr{T}_w$ the subclass of $\mathscr{S}_w$ consisting of the functions of the form

$$f(z) = (z-w) - \sum_{n=2}^{\infty} a_n (z-w)^n \ \ (a_n \geq 0). \tag{2}$$

Note that $\mathscr{S}_0 = \mathscr{S}$ and $\mathscr{T}_0 = \mathscr{T}$ be subclasses of $\mathscr{A} = \mathscr{A}(0)$ consisting of univalent functions in Δ. By $\mathscr{S}_w^*(\beta)$ and $\mathscr{K}_w(\beta)$, respectively, we mean the classes of analytic functions that satisfy the analytic conditions $\Re\left(\frac{(z-w)f'(z)}{f(z)}\right) > \beta$, $\Re\left(1 + \frac{(z-w)f''(z)}{f'(z)}\right) > \beta$ and $(z-w) \in \Delta$ for $0 \leqq \beta < 1$ introduced and studied by Kanas and Ronning [10]. The class $\mathscr{S}_w^*(0)$ is defined by geometric property that the image of any circular arc centered at w is starlike with respect to $f(w)$ and the corresponding class $\mathscr{K}_w(0)$ is defined by the property that the image of any circular arc centered at w is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [8] and [9] for uniformly starlike and convex functions, except that in this case the point w is fixed. In particular, $\mathscr{K} = \mathscr{K}_0(0)$ and $\mathscr{S}_0^* = \mathscr{S}^*(0)$ respectively, are the well-known standard classes of convex and starlike functions (see [11,15]).

For complex parameters $a_1, \ldots, a_l$ and $b_1, \ldots, b_m$ $(b_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$ the *q-hypergeometric function* ${}_l\Psi_m(z)$ is defined by

$$\begin{aligned} &{}_l\Psi_m(a_1, \ldots a_l; b_1, \ldots, b_m; q, z) \\ &:= \sum_{n=0}^{\infty} \frac{(a_1,q)_n \ldots (a_l,q)_n}{(q,q)_n (b_1,q)_n \ldots (b_m,q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+m-l} z^n \end{aligned} \tag{3}$$

with $\binom{n}{2} = \frac{n(n-1)}{2}$ where $q \neq 0$ when $l > m+1 (l, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \Delta$.

The $q-$ shifted factorial is defined for $a, q \in \mathbb{C}$ as a product of n factors by

$$(a;q)_n = \begin{cases} 1 & n = 0 \\ (1-a)(1-aq)\ldots(1-aq^{n-1}) & n \in \mathbb{N} \end{cases}$$

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and in terms of basic analogue of the gamma function

$$(q^a;q)_n = \frac{\Gamma_q(a+n)(1-q)^n}{\Gamma_q(a)}, n > 0. \quad (4)$$

Now for $z \in \Delta, 0 < |q| < 1$ and $l = m+1$, the basic hypergeometric function defined in (3) takes the form

$$ {}_l\Psi_m(a_1;\ldots a_l;b_1,\ldots,b_m;q,z) = \sum_{n=0}^{\infty} \frac{(a_1,q)_n \ldots (a_l,q)_n}{(q,q)_n(b_1,q)_n \ldots (b_m,q)_n} z^n,$$

which converges absolutely in the open unit disk Δ. It is of interest to note that $\lim_{q\to 1^-} \frac{(q^a;q)_n}{(1-q)^n} = (a)_n$ $= a(a+1)...(a+n-1)$, the familiar Pochhammer symbol and

$$ {}_l\Psi_m(a_1,\ldots a_l;b_1,\ldots,b_m;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \ldots (a_l)_n}{(b_1)_n \ldots (b_m)_n} \frac{z^n}{n!}. \quad (5)$$

For $a_i = q^{\alpha_i}, b_j = q^{\beta_j}, \alpha_i, \beta_j \in \mathbb{C}$ and $\beta_j \neq 0,-1,-2,...,(i=1,...,l, j=1,..,m)$ and $q \to 1$, we obtain the well-known Dziok-Srivastava linear operator [6,7] (for $l = m+1$). For $l=1, m=0, a_1 = q$, many (well known and new) integral and differential operators can be obtained by specializing the parameters, for example the operators introduced in [4,5,11,12,14].

Motivated by the recent work of Mohammed and Darus [13], we define a linear operator

$$\mathscr{I}_m^l[a_l,q]f(z) : \mathscr{A}(w) \longrightarrow \mathscr{A}(w)$$

given by

$$\mathscr{I}_m^l[a_l,q]f(z) = \mathscr{I}(a_l,b_m;q;z-w) * f(z) = (z-w)\ {}_l\Psi_m(a_1;\ldots a_l;b_1,\ldots,b_m;q,(z-w)) * f(z), \quad (6)$$

$$\mathscr{I}_m^l f(z) = \mathscr{I}_m^l[a_l,q]f(z) = (z-w) + \sum_{n=2}^{\infty} \Upsilon_n^{l,m}[a_1,q]a_n(z-w)^n \quad (7)$$

where

$$\Upsilon(n) = \Upsilon_n^{l,m}[a_1,q] = \frac{(a_1;q)_{n-1} \ldots (a_l;q)_{n-1}}{(q;q)_{n-1}(b_1;q)_{n-1} \ldots (b_m,q)_{n-1}}. \quad (8)$$

Making use of the operator $\mathscr{I}_m^l f(z)$ and motivated by the results discussed in [1,2,3,16] we introduce a new subclass $\mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$ of analytic functions with negative coefficients.

For $0 \leqq \beta < 1,\ \alpha \geqq 0$, and for a fixed point w, let the class $\mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$, consists of functions $f \in \mathscr{T}_w$, satisfying the condition

$$\Re\left(\frac{(z-w)(\mathscr{I}_m^l f(z))'}{\mathscr{I}_m^l f(z)} - \beta\right) > \alpha\left|\frac{(z-w)(\mathscr{I}_m^l f(z))'}{\mathscr{I}_m^l f(z)} - 1\right|, (z-w) \in \Delta. \quad (9)$$

In this present paper, we obtain a characterization, coefficients estimates, distortion theorem and covering theorem, extreme points and radii of close-to-convexity, starlikeness and convexity for functions belonging to the class $\mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$.

2 Characterization and Coefficient estimates

Theorem 1. *Let $f \in \mathscr{T}_w$. Then $f \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$, $0 \leqq \beta < 1$ and $\alpha \geqq 0$, if and only if*

$$\sum_{n=2}^{\infty} [n(\alpha+1)-(\alpha+\beta)]\,(r+d)^{n-1}\Upsilon(n)|a_n| \leqq 1-\beta. \quad (10)$$

where $0 \leqq |z-w| \leqq |z|+|w| < r+d < 1$,

Proof. We employ the technique adopted by [1,16]. We have $f \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$, if and only if the condition (9) is satisfied, which is equivalent to

$$\Re\left(\frac{(1-\alpha e^{i\theta})(z-w)(\mathscr{I}_m^l f(z))' + \alpha e^{i\theta}\mathscr{I}_m^l f(z)}{\mathscr{I}_m^l f(z)}\right) > \beta, \qquad -\pi \leqq \theta < \pi. \quad (11)$$

Now, letting $G(z) = (1-\alpha e^{i\theta})(z-w)(\mathscr{I}_m^l f(z))' + \alpha e^{i\theta}\mathscr{I}_m^l f(z)$ and $F(z) = \mathscr{I}_m^l f(z)$, equation (11) is equivalent to

$$|G(z)+(1-\beta)F(z)| > |G(z)-(1+\beta)F(z)|,\ 0 \leqq \beta < 1.$$

By simple computation we have

$$
\begin{aligned}
&|G(z)+(1-\beta)F(z)| \\
&\geqq |(2-\beta)(z-w)| - \left|\sum_{n=2}^{\infty}(n+1-\beta)\Upsilon(n)a_n(z-w)^n\right| \\
&\qquad - \left|\alpha e^{i\theta}\sum_{n=2}^{\infty}(n-1)\Upsilon(n)a_n(z-w)^n\right| \\
&\geqq (2-\beta)|z-w| - \sum_{n=2}^{\infty}(n+1-\beta)\Upsilon(n)a_n|z-w|^n \\
&\qquad -\alpha\sum_{n=2}^{\infty}(n-1)\Upsilon(n)a_n|z-w|^n \\
&\geqq (2-\beta)|z-w| \\
&\qquad -\sum_{n=2}^{\infty}\Big[n(\alpha+1)-(\alpha+\beta)+1\Big]\Upsilon(n)a_n|z-w|^n
\end{aligned}
$$

and similarly,

$$
\begin{aligned}
&|G(z)-(1+\beta)F(z)| \\
&\leqq \beta|z-w| + \sum_{n=2}^{\infty}\Big[n(\alpha+1)-(\alpha+\beta)-1\Big]\Upsilon(n)a_n|z-w|^n.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&|G(z)+(1-\beta)F(z)| - |G(z)-(1+\beta)F(z)| \\
&\geqq 2(1-\beta)|z-w| \\
&\qquad -2\sum_{n=2}^{\infty}\Big[n(\alpha+1)-(\alpha+\beta)\Big]\Upsilon(n)a_n|z-w|^n \\
&\geqq (1-\beta)- \\
&\quad \sum_{n=2}^{\infty}\Big[n(\alpha+1)-(\alpha+\beta)\Big]\Upsilon(n)a_n|z-w|^{n-1} \geqq 0.
\end{aligned}
$$

By putting $|z-w| \leqq |z|+|w| < r+d$ with $0<r<1$ and $|w|=d$ in above inequality, we obtain (10).

On the other hand, for all $-\pi \leqq \theta < \pi$, we must have

$$
\Re\left(\frac{(z-w)F'(z)}{F(z)}(1+\alpha e^{i\theta}) - \alpha e^{i\theta}\right) > \beta.
$$

Now, choosing the values of $(z-w)$ on the positive real axis, where $0 \leqq |z-w| \leqq |z|+|w| < r+d < 1$, and using $\Re\{-e^{i\theta}\} \geqq -|e^{i\theta}| = -1$, the above inequality can be written as

$$
\Re\left(\frac{(1-\beta)-\sum_{n=2}^{\infty}[n(\alpha+1)-(\alpha+\beta)]\Upsilon(n)a_n(r+d)^{n-1}}{1-\sum_{n=2}^{\infty}\Upsilon(n)a_n(r+d)^{n-1}}\right) \geqq 0.
$$

hence we get the desired result.

Corollary 1. *If $f \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$, then*

$$
a_n \leqq \frac{1-\beta}{\big[n(\alpha+1)-(\alpha+\beta)\big](r+d)^{n-1}\Upsilon(n)},\ n \geqq 2, \quad (12)
$$

where $0 \leqq \beta < 1$ and $\alpha \geqq 0$. Equality in (12) holds for the function

$$
f(z) = (z-w) - \frac{1-\beta}{\big[n(\alpha+1)-(\alpha+\beta)\big](r+d)^{n-1}\Upsilon(n)}(z-w)^n. \quad (13)
$$

In the following section we state Distortion and Growth theorems without proof.

3 Distortion and Covering Theorems

Theorem 2. *Let $\Upsilon(n)$ be defined as in (8). Then, for $f \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$, with $|z-w| \leqq |z|+|w| < r+d < 1$ in Δ, we have*

$$
\begin{aligned}
(r+d)-B(\alpha,\beta,\mu)(r+d)^2 &\leqq |f(z)| \\
&\leqq (r+d)+B(\alpha,\beta,\mu)(r+d)^2, \quad (14)
\end{aligned}
$$

where,

$$
B(\alpha,\beta,\mu) := \frac{1-\beta}{\Big[2(\alpha+1)-(\alpha+\beta)\Big](r+d)\Upsilon(2)}.
$$

Theorem 3. *If $f \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$, then for $|z-w| \leqq |z|+|w| < r+d < 1$*

$$
1-B(\alpha,\beta,\mu)(r+d) \leqq |f'(z)| \leqq 1+B(\alpha,\beta,\mu)(r+d), \quad (15)
$$

where $B(\alpha,\beta,\mu)$ as in Theorem 2.

Note that in Theorem 2 and Theorem 3 equality holds for the function

$$
f(z) = (z-w) - \frac{1-\beta}{\Big[2(\alpha+1)-(\alpha+\beta)\Big](r+d)\Upsilon(2)}(z-w)^2.
$$

Theorem 4. *If $f \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$, then $f \in \mathscr{S}^*(\delta)$, where*

$$
\delta = 1 - \frac{1-\beta}{\big[2(\alpha+1)-(\alpha+\beta)\big](r+d)\Upsilon(2)-(1-\beta)}.
$$

This result is sharp with the extremal function being

$$
f(z) = (z-w) - \frac{1-\beta}{\big[2(\alpha+1)-(\alpha+\beta)\big](r+d)\Upsilon(2)}(z-w)^2.
$$

Proof. It is sufficient to show that (10) implies $\sum_{n=2}^{\infty}(n-\delta)a_n \leqq 1-\delta$ [15], that is,

$$
\frac{n-\delta}{1-\delta} \leqq \frac{\big[n(\alpha+1)-(\alpha+\beta)\big](r+d)^{n-1}\Upsilon(n)}{1-\beta},\ n \geqq 2. \quad (16)
$$

Since, for $n \geqq 2$, (16) is equivalent to

$$\delta \leqq 1 - \frac{(n-1)(1-\beta)}{\left[n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)-(1-\beta)} = \Phi(n)$$

and $\Phi(n) \leqq \Phi(2)$, (16) holds true for any $0 \leqq \beta < 1$ and $\alpha \geqq 0$. This completes the proof of the Theorem 4.

4 Extreme points of the class $\mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$

Theorem 5. *Let* $f_1(z) = (z-w)$ *and*

$$f_n(z) = (z-w) - \frac{1-\beta}{\left[n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)}(z-w)^n,$$

$n \geqq 2$ *and* $\Upsilon(n)$ *be as defined in* (8). *Then* $f \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$ *if and only if it can be represented in the form*

$$f(z) = \sum_{n=1}^{\infty} \omega_n f_n(z), \qquad \lambda_n \geqq 0, \qquad \sum_{n=1}^{\infty} \lambda_n = 1. \quad (17)$$

Proof. Suppose $f(z)$ can be written as in (17). Then

$$f(z) = (z-w) - \sum_{n=2}^{\infty} \lambda_n \left[\frac{1-\beta}{\left[n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)}\right](z-w)^n.$$

Now,

$$\sum_{n=2}^{\infty} \lambda_n \frac{\left[n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)(1-\beta)}{(1-\beta)\left[n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)} = \sum_{n=2}^{\infty} \lambda_n = 1-\lambda_1 \leqq 1.$$

Thus $f \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$. Conversely, let us have $f \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$. Then by using (12), we may write

$$\lambda_n = \frac{\left[n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)}{1-\beta} a_n, \quad n \geqq 2,$$

and $\lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n$. Then $f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z)$, with $f_n(z)$ is as in the Theorem.

Theorem 6. *The class* $\mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$ *is a convex set.*

Proof. Let the function

$$f_j(z) = \sum_{n=2}^{\infty} a_{n,j}(z-w)^n, \qquad a_{n,j} \geqq 0, \qquad j = 1,2,$$

be in the class $\mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$. It sufficient to show that the function $g(z)$ defined by

$$g(z) = \lambda f_1(z) + (1-\lambda) f_2(z), \qquad 0 \leqq \lambda \leqq 1,$$

is in the class $\mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$. Since

$$g(z) = (z-w) - \sum_{n=2}^{\infty} [\lambda a_{n,1} + (1-\lambda)a_{n,2}](z-w)^n,$$

an easy computation with the aid of Theorem 1 gives,

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left[n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)[\lambda a_{n,1}+(1-\lambda)a_{n,2}] \\
&\quad + (1-\lambda)\sum_{n=2}^{\infty}[n(\alpha+1)-(\alpha+\beta)](r+d)^{n-1}\Upsilon(n) \\
&\leqq \lambda(1-\beta) + (1-\lambda)(1-\beta) \\
&\leqq 1-\beta,
\end{aligned}$$

which implies that $g \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$. Hence $\mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$ is convex.

5 Modified Hadamard products

For functions of the form

$$(f_1 * f_2)(z) = (z-w) - \sum_{n=2}^{\infty} a_{n,1} a_{n,2}(z-w)^n. \quad (18)$$

we define the modified Hadamard product as

$$(f_1 * f_2)(z) = (z-w) - \sum_{n=2}^{\infty} a_{n,1} a_{n,2}(z-w)^n. \quad (19)$$

Theorem 7. *If* $f_j(z) \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$, $j = 1, 2$, *then*

$$(f_1 * f_2)(z) \in \mathscr{S}\mathscr{T}_w^m(\alpha,\xi,w),$$

where

$$\xi = \frac{(2-\beta)\left(2(\alpha+1)-(\alpha+\beta)\right)(r+d)\Upsilon(2) - 2(1-\beta)^2}{(2-\beta)\left(2(\alpha+1)-(\alpha+\beta)\right)(r+d)\Upsilon(2) - (1-\beta)^2},$$

with $\Upsilon(n)$ *be defined as in* (8).

Proof. Since $f_j(z) \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$, $j = 1, 2$, we have

$$\sum_{n=2}^{\infty} \left[n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n) a_{n,j} \leqq 1-\beta. \quad (20)$$

The Cauchy-Schwartz inequality leads to

$$\sum_{n=2}^{\infty} \frac{\left[n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)a_{n,j}}{1-\beta}\sqrt{a_{n,1}a_{n,2}} \leqq 1. \quad (21)$$

Note that we need to find the largest ξ such that

$$\sum_{n=2}^{\infty} \frac{\left[n(\alpha+1)-(\alpha+\xi)\right](r+d)^{n-1}\Upsilon(n)a_{n,j}}{1-\xi} a_{n,1}a_{n,2} \leqq 1. \quad (22)$$

Therefore, in view of (21) and (22), whenever
$\frac{n-\xi}{1-\xi}\sqrt{a_{n,1}a_{n,2}} \leqq \frac{n-\beta}{1-\beta}, \ n \geqq 2$
holds, then (22) is satisfied. We have, from (21),

$$\sqrt{a_{n,1}a_{n,2}} \leqq \frac{1-\beta}{\left[\, n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)}, \quad n \geqq 2. \quad (23)$$

Thus, if

$$\left(\frac{n-\xi}{1-\xi}\right)\left[\frac{1-\beta}{\left[\, n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)}\right] \leqq \frac{n-\beta}{1-\beta}, \quad n \geqq 2,$$

or, if

$$\xi \leqq \frac{(n-\beta)\left[\, n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)-n(1-\beta)^2}{(n-\beta)\left[\, n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)-(1-\beta)^2}, \quad n \geqq 2,$$

then (21) is satisfied. Note that the right hand side of the above expression is an increasing function on n. Hence, setting $n=2$ in the above inequality gives the required result. Finally, by taking the function $f(z) = (z-w) - \frac{1-\beta}{(2-\beta)\left[2(\alpha+1)-(\alpha+\beta)\right](r+d)\Upsilon(2)}(z-w)^2$, we see that the result is sharp.

6 Radii of close-to-convexity, starlikeness and convexity

Theorem 8. *Let the function $f \in \mathscr{T}_w$ be in the class $\mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$. Then $f(z)$ is close-to-convex of order ρ, $0 \leqq \rho < 1$ in $|z-w| =< R_1$, where*

$$R_1 = \inf_n \left[\frac{(1-\rho)\left[\, n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)}{n(1-\beta)}\right]^{\frac{1}{n-1}},$$

$n \geqq 2$, with $\Upsilon(n)$ be defined as in (8). This result is sharp for the function $f(z)$ given by (13).

Proof. It is sufficient to show that $|f'(z)-1| \leqq 1-\rho$, $0 \leqq \rho < 1$, for $|z-w| < r_1(\alpha,\beta,l,\rho)$, or equivalently

$$\sum_{n=2}^{\infty}\left(\frac{n}{1-\rho}\right)a_n|z-w|^{n-1} \leqq 1. \quad (24)$$

By Theorem 1, (24) will be true if

$$\left(\frac{n}{1-\rho}\right)|z-w|^{n-1} \leqq \frac{[n(\alpha+1)-(\alpha+\beta)\,](r+d)^{n-1}\Upsilon(n)}{1-\beta}$$

or, if

$$|z-w| = R_1 \leqq \left[\frac{(1-\rho)\left[\, n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)}{n(1-\beta)}\right]^{\frac{1}{n-1}}, \ n \geqq 2. \quad (25)$$

The theorem follows easily from (25).

Theorem 9. *Let the function $f(z) \in \mathscr{T}_w$ be in the class $\mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$. Then $f(z)$ is starlike of order ρ, $0 \leqq \rho < 1$ in $|z-w| = R_2$ where*

$$R_2 = \inf_n \left[\frac{(1-\rho)\left[\, n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)}{(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}},$$

$n \geqq 2$, with $\Upsilon(n)$ be defined as in (8). This result is sharp for the function $f(z)$ given by (13).

Proof. It is sufficient to show that

$$\left|\frac{(z-w)f'(z)}{f(z)}-1\right| \leqq 1-\rho, \quad (26)$$

or equivalently

$$\sum_{n=2}^{\infty}\left(\frac{n-\rho}{1-\rho}\right)a_n|z-w|^{n-1} \leqq 1,$$

for $0 \leqq \rho < 1$ and $|z-w| = R_2$. Proceeding as in Theorem 8, with the use of Theorem 1, we get the required result. Hence, by Theorem 1, (26) will be true if

$$\left(\frac{n-\rho}{1-\rho}\right)|z-w|^{n-1} \leqq \frac{[n(\alpha+1)-(\alpha+\beta)](r+d)^{n-1}\Upsilon(n)}{1-\beta}$$

or, if

$$|z-w| = R_2 \leqq \left[\frac{[n(\alpha+1)-(\alpha+\beta)](r+d)^{n-1}\Upsilon(n)}{(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}}, \ n \geqq 2. \quad (27)$$

The theorem follows easily from (27).

Theorem 10. *Let the function $f(z) \in \mathscr{T}_w$ be in the class $\mathscr{S}\mathscr{T}_m^l(\alpha,\beta,w)$. Then $f(z)$ is convex of order ρ, $0 \leqq \rho < 1$ in $|z-w| = R_3$, where*

$$R_3 = \inf_n \left[\frac{(1-\rho)\left[\, n(\alpha+1)-(\alpha+\beta)\right](r+d)^{n-1}\Upsilon(n)}{n(n-\rho)(1-\beta)}\right]^{\frac{1}{n-1}},$$

$n \geqq 2$, with $\Upsilon(n)$ be defined as in (8). This result is sharp for the function $f(z)$ given by (13).

Proof. It is sufficient to show that $\left|\frac{(z-w)f''(z)}{f'(z)}\right| \leqq 1-\rho$ or equivalently

$$\sum_{n=2}^{\infty}\left(\frac{n(n-\rho)}{1-\rho}\right)a_n|z-w|^{n-1} \leqq 1,$$

for $0 \leqq \rho < 1$ and $|z| < r_3(\alpha,\beta,l,\rho)$. Proceeding as in Theorem 8, we get the required result.

7 Integral transform of the class $\mathscr{S}\mathscr{T}_m^l(\alpha, \beta, w)$

For $f \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$ we define the integral transform

$$V_\mu(f)(z) = \int_0^1 \mu(t)\frac{f(tz)}{t}dt,$$

where μ is real valued, non-negative weight function normalized so that $\int_0^1 \mu(t)dt = 1$. Since special cases of $\mu(t)$ are particularly interesting such as $\mu(t) = (1+c)t^c$, $c > -1$, for which V_μ is known as the Bernardi operator and

$$\mu(t) = \frac{(c+1)^\delta}{\mu(\delta)}t^c\left(log\frac{1}{t}\right)^{\delta-1}, \; c > -1, \; \delta \geq 0$$

which gives the Komatu operator.

First we show that the class $\mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$ is closed under $V_\mu(f)$.

Theorem 11. *Let* $f \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$. *Then* $V_\mu(f) \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$.

Proof. By definition, we have

$$V_\mu(f) = \frac{(c+1)^\delta}{\mu(\delta)}* \int_0^1 (-1)^{\delta-1}t^c(logt)^{\delta-1}\left((z-w)-\sum_{n=2}^{\infty}a_n(z-w)^n t^{n-1}\right)dt$$
$$= \frac{(-1)^{\delta-1}(c+1)^\delta}{\mu(\delta)}* \lim_{r\to 0^+}\left[\int_r^1 t^c(logt)^{\delta-1}\left((z-w)-\sum_{n=2}^{\infty}a_n(z-w)^n t^{n-1}\right)dt\right],$$

and a simple calculation gives

$$V_\mu(f)(z) = (z-w) - \sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^\delta a_n(z-w)^n.$$

We need to prove that

$$\sum_{n=2}^{\infty}\frac{[n(\alpha+1)-(\alpha+\beta)](r+d)^{n-1}\,\Upsilon(n)}{1-\beta}\left(\frac{c+1}{c+n}\right)^\delta a_n < 1. \tag{28}$$

On the other hand by Theorem 1, $f \in \mathscr{S}\mathscr{T}_m^l(\alpha, \beta, w)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[n(\alpha+1)-(\alpha+\beta)](r+d)^{n-1}\,\Upsilon(n)a_n}{1-\beta} < 1.$$

Hence $\frac{c+1}{c+n} < 1$. Therefore (28) holds and the proof is complete.

Next we provide a starlike condition for functions in $\mathscr{S}\mathscr{T}_m^l(\alpha, \beta, w)$ and $V_\mu(f)$ on lines similar to Theorem 8.

Theorem 12. *Let* $f \in \mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$. *Then*
(i) $V_\mu(f)$ *is starlike of order* $0 \leq \gamma < 1$ *in* $|z-w| < R_1$ *where*

$$R_1 = \inf_n\left[\left(\frac{c+n}{c+1}\right)^\delta\frac{(1-\gamma)[n(\alpha+1)-(\alpha+\beta)](r+d)^{n-1}\,\Upsilon(n)}{(1-\beta)(n-\gamma)}\right]^{\frac{1}{n-1}}$$

(ii) $V_\mu(f)$ *is convex of order* $0 \leq \gamma < 1$ *in* $|z-w| < R_2$ *where*

$$R_2 = \inf_n\left[\left(\frac{c+n}{c+1}\right)^\delta\frac{(1-\gamma)[n(\alpha+1)-(\alpha+\beta)](r+d)^{n-1}\,\Upsilon(n)}{(1-\beta)n(n-\gamma)}\right]^{\frac{1}{n-1}}.$$

This result is sharp for the function

$$f(z) = (z-w) - \frac{1-\beta}{[n(\alpha+1)-(\alpha+\beta)](r+d)^{n-1}\Upsilon(n)}(z-w)^n, \quad n \geqq 2. \tag{29}$$

Concluding Remarks: For suitable choices α,β,l and m the family $\mathscr{S}\mathscr{T}_l^m(\alpha,\beta,w)$, eventually lead us further to new class of functions defined either by extension or by generalization.

References

[1] E. Aqlan, J.M. Jahangiri and S.R. Kulkarni, Classes of $k-$uniformly convex and starlike functions, Tamkang J. Math., 35 (2004), no.3, 1–7.
[2] M.K.Aouf and G.Murugusundaramoorthy, A subclass of uniformly convex functions defined by the Dziok-Srivastava operator, Aust.J.Math. Anal. Appln. (AJMAA), 5 (2008), no.1, Article 3, 1-17.
[3] M. K. Aouf, A. Shamndy, A. O. Mostafa and S. Madian, A subclass of m-w-starlike functions , Acta Univ. Apulensis (2010), no. 21, 135–142.
[4] S. D. Bernardi, Convex and starlike univalent functions, *Trans. Amer. Math. Soc.*, 135 (1969), 429–446.
[5] B. C. Carlson and S. B. Shaffer, Starlike and prestarlike hypergeometric functions, SIAM J. Math. Anal., 15 (1984), 737–745.
[6] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.* 103 (1999), 1–13.
[7] J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Intergral Transforms Spec. Funct.* 14 (2003), 7–18.
[8] A.W. Goodman, On uniformly starlike functions, J. Math. Anal. Appl. 155 (1991), 364-370.
[9] A.W. Goodman, On uniformly convex functions, Ann. Polon. Math. 56 (1991), no. 1, 87-92.
[10] S. Kanas and F. Ronning, Uniformaly starlike and convex function and other related classes of univalent functions, Ann. Univ. Marie Curie-Sklodowaska, Sect. A 53 (1991), 95-105.

[11] R. J. Libera, Some classes of regular univalent functions, *Proc. Amer. Math. Soc.*, 16 (1965), 755–758.

[12] A. E. Livingston, On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc., 17 (1966), 352–357.

[13] A. Mohammed and M. Darus, A generalized operator involving the q-hypergeometric function, Mathematici Vesnik, Available online 10.06.2012. 12 pages.

[14] S. Ruscheweyh, New criteria for Univalent Functions, Proc. Amer. Math. Soc., 49 (1975), 109 -115.

[15] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51(1975) 109–116.

[16] H. Silverman,G.Murugusundaramoorthy and K.Vijaya,On a certain class of analytic functions associated with a convolution structure , Kyungpook.J. Math., 49(2009), 95-106.

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