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Estimation of Unknown Function of a Class of Retarded Iterated Integral Inequalities

*Ricai Luo and Wu-sheng Wang**

School of Mathematics and Statistics, Hechi University,Yizhou, Guangxi 546300, P. R. China

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Abstract: In this paper, we establish a class of retarded iterated integral inequalities, which includes a nonconstant term outside the integrals. By adopting novel analysis techniques, the upper bound of the embedded unknown function is estimated explicitly. The derived result can be applied in the study of solutions of ordinary differential equations and integral equations.

Keywords: Integral inequality; Estimation; Integral inequality technique.

1 Introduction

It is well known that differential equations and integral equations are important tools to discuss the rule of natural phenomena. In the study of the existence, uniqueness, boundedness, stability, oscillation and other qualitative properties of solutions of differential equations and integral equations, one often deals with certain integral inequalities. One of the best known and widely used inequalities in the study of nonlinear differential equations is Gronwall-Bellman inequality [1,2], which can be stated as follows: If u and f are non-negative continuous functions on an interval $[a,b]$ satisfying

$$u(t) \le c + \int_a^t f(s)u(s)ds, \quad t \in [a,b],$$

for some constant $c \ge 0$, then

$$u(t) \le c\exp\left(\int_a^t f(s)ds\right), \quad t \in [a,b]. \tag{1}$$

In 1956, Bihari [3] studied a new nonlinear integral inequality

$$u(t) \le a + \int_0^t f(s)w(u(s))ds, \quad t > 0, \tag{2}$$

where $a > 0$ is a constant. Replacing the upper limit t of the integral with a function $\alpha(t)$ in (2), Lipovan [6] improved Bihari's results by investigating the following so-called retarded Gronwall-like inequalities

$$u(t) \le a + \int_{\alpha(t_0)}^{\alpha(t)} f(s)w(u(s))ds, \quad t_0 \le t < t_1,$$

and

$$u(t) \le a + \int_{t_0}^t f(s)w(u(s))ds + \int_{\alpha(t_0)}^{\alpha(t)} g(s)w(u(s))ds, \quad t_0 \le t < t_1.$$

Pachpatte [5] investigated the retarded inequality

$$u(t) \le k + \int_a^t g(s)u(s)ds + \int_a^{\alpha(t)} h(s)u(s)ds, \tag{3}$$

where k is a constant. Replacing k by a nondecreasing continuous function f (t) in (1), Rashid [12] studied the following retarded inequality

$$u(t) \le f(t) + \int_a^t g(s)u(s)ds + \int_a^{\alpha(t)} h(s)u(s)ds, \tag{4}$$

Their results were further generalized by Agarwal, Deng and Zhang [8] to the inequality

$$v(t) \le a(t) + \sum_{i=1}^n \int_{b_i(t_0)}^{b_i(t)} g_i(t,s)w_i(u(s))ds, \quad t_0 \le t < t_1, \tag{5}$$

In 2011, Abdeldaim et al. [10] studied a new iterated integral inequality of Gronwall-Bellman-Pachpatte type

$$u(t) \le u_0 + \int_0^t f(s)u(s)\left[u(s) + \int_0^s h(\tau)\left[u(\tau) + \int_0^\tau g(\xi)u(\xi)d\xi\right]d\tau\right]ds. \tag{6}$$

In 2014, El-Owaidy, Abdeldaim, and El-Deeb[13] investigated some new retarded nonlinear integral

* Corresponding author e-mail: wang4896@126.com

inequalities

$$u(t) \le f(t) + \int_a^t g(s)u^p(s)ds + \int_a^{\alpha(t)} h(s)u^p(s)ds, \quad (7)$$

$$u^p(t) \le f^p(t) + \int_a^{\alpha_1(t)} g(s)u(s)ds + \int_a^{\alpha_2(t)} h(s)u(s)ds, \quad (8)$$

$$u(t) \le f(t) + \int_a^{\alpha_1(t)} g(s)w_1(u(s))ds + \int_a^{\alpha_2(t)} h(s)w_2(u(s))ds, \quad (9)$$

$$u(t) \le f(t) + \int_a^{\alpha_1(t)} g(s)u(s)w_1(\ln u(s))ds + \int_a^{\alpha_2(t)} h(s)u(s)w_2(\ln u(s))ds, \quad (10)$$

$$u(t) \le f(t) + \int_a^{\alpha(t)} g(s)u(s)ds + \int_a^{\alpha(t)} g(s)u(s)[u(s) + \int_a^{\alpha(t)} h(\lambda)u(\lambda)d\lambda]ds. \quad (11)$$

During the past few years, some investigators have established a lot of useful and interesting integral inequalities in order to achieve various goals; see [3-15] and the references cited therein.

In this paper, on the basis of [10,13], we discuss a new retarded nonlinear Volterra-Fredholm type integral inequality

$$u(t) \le f(t) + \int_a^{\alpha(t)} g(s)w_1(\ln u(s))ds + \int_a^{\alpha(t)} g(s)w_1(\ln u(s))\Big[u(s) + \int_a^s h(\tau)u(\tau)w_2(\ln u(\tau))d\tau\Big]ds. \quad (12)$$

2 Result

Throughout this paper, let $\mathbf{R}_+ = [0,+\infty), I = [a,+\infty). C^1(M,S)$ denotes the class of continuously differentiable functions defined on set M with range in the set S, $C(M,S)$ denotes the class of continuously functions defined on set M with range in the set S, $\alpha'(t)$ denotes the derivative function of a function $\alpha(t)$.

For the sake of convenience, we define three functions

$$W_1(u) = \int_{\ln(1+f(a))}^u \frac{dr}{w_1(r)}, u > \ln(1+f(a)), u \in \mathbf{R}_+, \quad (13)$$

$$W_2(u) = \int_0^u \frac{w_1(W_1^{-1}(r))dr}{w_2(W_1^{-1}(r))}, u > W_1(\ln(1+f(a))), u \in \mathbf{R}_+, \quad (14)$$

$$W_3(u) = \int_{\ln(1+f(a))}^u \frac{dr}{w_2(r)}, u > \ln(1+f(a)), u \in \mathbf{R}_+. \quad (15)$$

Theorem 1 Suppose that $g,h \in C(I,\mathbf{R}_+), \alpha \in C^1(I,I)$ is nondecreasing with $\alpha(a) = a$ and $\alpha(t) \le t$ on I. Let $f \in C^1(\mathbf{R}_+,\mathbf{R}_+)$ be nondecreasing functions with $f(u) > 0$ for $u > 0$, and $w_1, w_2 \in C(\mathbf{R}_+,\mathbf{R}_+)$ be nondecreasing functions with $uw_1(\ln u) > 1, w_2(u) > 1, w_2(u)/w_1(u) > 1$ for $u > 0$. Suppose that $W_1(+\infty) = +\infty, W_2(+\infty) = +\infty$. If $u(t)$ satisfies (12), then

$$u(t) \le \exp\Big\{W_1^{-1}\Big[W_2^{-1}\Big(\int_a^t f(s)ds + \int_a^{\alpha(t)} (g(s)+h(s))ds\Big)\Big]\Big\}, t \in I, \quad (16)$$

where W_1, W_2 are defined by (13) and (14), respectively.

Proof. Define a function $z(t)$ by the right hand side of the inequality (12), i.e.

$$z(t) = f(t) + \int_a^{\alpha(t)} g(s)w_1(\ln u(s))ds + \int_a^{\alpha(t)} g(s)w_1(\ln u(s))\Big[u(s) + \int_a^s h(\tau)u(\tau)w_2(\ln u(\tau))d\tau\Big]ds. \quad (17)$$

which is a positive and nondecreasing function on I. From (12) and (17) we have

$$u(t) \le z(t), u(\alpha(t)) \le z(\alpha(t)) \le z(t), t \in I, \quad (18)$$

$$z(a) = f(a). \quad (19)$$

Differentiating $z(t)$ with respect to t, using (18) we have

$$\begin{aligned} z'(t) &= f'(t) + \alpha'(t)g(\alpha(t))w_1(\ln u(\alpha(t))) + \alpha'(t)g(\alpha(t))w_1(\ln u(\alpha(t)))\Big[u(\alpha(t)) + \int_a^{\alpha(t)} h(\tau)u(\tau)w_2(\ln u(\tau))d\tau\Big] \\ &= f'(t) + \alpha'(t)g(\alpha(t))w_1(\ln u(\alpha(t)))\Big[1 + u(\alpha(t)) + \int_a^{\alpha(t)} h(\tau)u(\tau)w_2(\ln u(\tau))d\tau\Big] \\ &\le f'(t) + \alpha'(t)g(\alpha(t))w_1(\ln z(\alpha(t)))\Big[1 + z(\alpha(t)) + \int_a^{\alpha(t)} h(\tau)z(\tau)w_2(\ln z(\tau))d\tau\Big] \\ &\le f'(t) + \alpha'(t)g(\alpha(t))w_1(\ln z(t))\Big[1 + z(t) + \int_a^{\alpha(t)} h(\tau)z(\tau)w_2(\ln z(\tau))d\tau\Big] \\ &\le f'(t) + \alpha'(t)g(\alpha(t))w_1(\ln z(t))r_1(t), \end{aligned} \quad (20)$$

where

$$r_1(t) = 1 + z(t) + \int_a^{\alpha(t)} h(\tau)z(\tau)w_2(\ln z(\tau))d\tau, \quad (21)$$

which is a positive and nondecreasing function on I. From (20) and (21) we have

$$z(t) \le r_1(t), z(\alpha(t)) \le r_1(\alpha(t)) \le r_1(t), t \in I, \quad (22)$$

$$r_1(a) = 1 + f(a). \quad (23)$$

Differentiating $r_1(t)$ with respect to t and using (20) and (22), we have

$$\begin{aligned} r_1'(t) &= z'(t)+\alpha'(t)h(\alpha(t))z(\alpha(t))w_2(\ln z(\alpha(t))) \\ &\le f'(t)+\alpha'(t)g(\alpha(t))w_1(\ln z(t))r_1(t) \\ &\quad +\alpha'(t)h(\alpha(t))z(\alpha(t))w_2(\ln z(\alpha(t))) \\ &\le f'(t)+\alpha'(t)g(\alpha(t))w_1(\ln r_1(t))r_1(t) \\ &\quad +\alpha'(t)h(\alpha(t))r_1(t)w_2(\ln r_1(t)). \end{aligned} \tag{24}$$

Since $w_1(\ln r_1(t))r_1(t)$ is a positive function. From (24) we have

$$\begin{aligned} \frac{r_1'(t)}{w_1(\ln r_1(t))r_1(t)} &\le \frac{f'(t)}{w_1(\ln r_1(t))r_1(t)}+\alpha'(t)g(\alpha(t)) \\ &\quad +\alpha'(t)h(\alpha(t))\frac{w_2(\ln r_1(t))}{w_1(\ln r_1(t))} \\ &\le f'(t)+\alpha'(t)g(\alpha(t)) \\ &\quad +\alpha'(t)h(\alpha(t))\frac{w_2(\ln r_1(t))}{w_1(\ln r_1(t))}. \end{aligned} \tag{25}$$

Integrating the inequality (25) from a to t, and making the change of variable we have

$$\begin{aligned} W_1(\ln r_1(t)) &\le W_1(\ln(1+f(a))+f(t)-f(a)+\int_a^{\alpha(t)} g(s)ds \\ &\quad +\int_a^t \alpha'(s)h(\alpha(s))\frac{w_2(\ln r_1(s))}{w_1(\ln r_1(s))}ds \\ &\le f(t)-f(a)+\int_a^{\alpha(t)} g(s)ds \\ &\quad +\int_a^t \alpha'(s)h(\alpha(s))\frac{w_2(\ln r_1(s))}{w_1(\ln r_1(s))}ds. \end{aligned} \tag{26}$$

Define a function $r_2(t)$ by

$$\begin{aligned} r_2(t) &= f(t)-f(a)+\int_a^{\alpha(t)} g(s)ds \\ &\quad +\int_a^t \alpha'(s)h(\alpha(s))\frac{w_2(\ln r_1(s))}{w_1(\ln r_1(s))}ds. \end{aligned} \tag{27}$$

which is a positive and nondecreasing function on I. From (27) we have

$$r_1(t)) \le \exp(W_1^{-1}(r_2(t))), \tag{28}$$

$$r_2(a) = 0. \tag{29}$$

Differentiating $r_2(t)$ with respect to t and using (28), we have

$$\begin{aligned} r_2'(t) &= f'(t)+\alpha'(t)g(\alpha(t))+\alpha'(t)h(\alpha(t))\frac{w_2(\ln r_1(t))}{w_1(\ln r_1(t))} \\ &\le f'(t)+\alpha'(t)g(\alpha(t)) \\ &\quad +\alpha'(t)h(\alpha(t))\frac{w_2(W_1^{-1}(r_2(t)))}{w_1(W_1^{-1}(r_2(t)))} \end{aligned} \tag{30}$$

Since $w_2(u)/w_1(u)>1$ for any $u>0$, from (30) we have

$$\begin{aligned} \frac{w_1(W_1^{-1}(r_2(t)))dr_2}{w_2(W_1^{-1}(r_2(t)))} &= \Big[f'(t)+\alpha'(t)g(\alpha(t)) \\ &\quad +\alpha'(t)h(\alpha(t))\Big]dt \end{aligned} \tag{31}$$

Integrating the inequality (31) from a to t and making the change of variable we have

$$\begin{aligned} W_2(r_2(t)) &= W_2(r_2(a))+\int_a^t f(s)ds+\int_a^{\alpha(t)}(g(s)+h(s))ds \\ &= \int_a^t f(s)ds+\int_a^{\alpha(t)}(g(s)+h(s))ds. \end{aligned} \tag{32}$$

From (18), (22), (28) and (32), we obtain

$$\begin{aligned} u(t) &\le z(t)\le r_1(t)\le \exp(W_1^{-1}(r_2(t))) \\ &= \exp\Big\{W_1^{-1}\Big[W_2^{-1}\Big(\int_a^t f(s)ds \\ &\quad +\int_a^{\alpha(t)}(g(s)+h(s))ds\Big)\Big]\Big\}. \end{aligned} \tag{33}$$

We get the required estimation (16). The proof is complete.

Theorem 2 Suppose that $h(t)\in C(I,\mathbf{R}_+), \alpha\in C^1(I,I)$ is nondecreasing with $\alpha(a)=a$ and $\alpha(t)\le t$ on I. Let $f\in C^1(\mathbf{R}_+,\mathbf{R}_+)$ be nondecreasing functions with $f(u)>0$ for $u>0$, and $w_1,w_2\in C(\mathbf{R}_+,\mathbf{R}_+)$ be nondecreasing functions with $uw_1(\ln u)>1, w_2(u)>1$ for $u>0$. Suppose that $W_1(+\infty)=+\infty, W_3(+\infty)=+\infty$.

If $u(t)$ satisfies (12) and $w_1(u)>w_2(u)$, then

$$\begin{aligned} u(t) &\le \exp\Big\{W_1^{-1}\Big[f(t)-f(a) \\ &\quad +\int_a^{\alpha(t)}[g(s)+h(s)]ds\Big]\Big\}, t\in I. \end{aligned} \tag{34}$$

If $u(t)$ satisfies (12) and $w_1(u)<w_2(u)$, then

$$\begin{aligned} u(t) &\le \exp\Big\{W_3^{-1}\Big[f(t)-f(a) \\ &\quad +\int_a^{\alpha(t)}[g(s)+h(s)]ds\Big]\Big\}, t\in I. \end{aligned} \tag{35}$$

Proof. Similarly to proof of Theorem 1. Performing the same procedure as in (17), (18), (19), (20), (21), (22) and (36), we have

$$\begin{aligned} r_1'(t) &\le f'(t)+\alpha'(t)g(\alpha(t))w_1(\ln r_1(t))r_1(t) \\ &\quad +\alpha'(t)h(\alpha(t))r_1(t)w_2(\ln r_1(t)). \end{aligned} \tag{36}$$

If $w_1(u)>w_2(u)$, then from (36) we have

$$\begin{aligned} r_1'(t) &\le f'(t)+[\alpha'(t)g(\alpha(t)) \\ &\quad +\alpha'(t)h(\alpha(t))]r_1(t)w_1(\ln r_1(t)). \end{aligned} \tag{37}$$

Since $w_1(\ln r_1(t))r_1(t)>1$ is a positive function. From (37) we have

$$\begin{aligned} \frac{r_1'(t)}{w_1(\ln r_1(t))r_1(t)} &\le \frac{f'(t)}{w_1(\ln r_1(t))r_1(t)} \\ &\quad +[\alpha'(t)g(\alpha(t))+\alpha'(t)h(\alpha(t))] \\ &\le f'(t)+[\alpha'(t)g(\alpha(t)) \\ &\quad +\alpha'(t)h(\alpha(t))]. \end{aligned} \tag{38}$$

Integrating the inequality (38) from a to t and making the change of variable we have

$$W_1(\ln r_1(t)) \le f(t)-f(a)+\int_a^{\alpha(t)}[g(s)+h(s)]ds. \tag{39}$$

From (18), (22) and (39), we obtain

$$\begin{aligned} u(t) &\le z(t) \le r_1(t) \\ &= \exp\Big\{W_1^{-1}\Big[f(t) - f(a) \\ &\quad + \int_a^{\alpha(t)} [g(s)+h(s)]ds\Big]\Big\}. \end{aligned} \tag{40}$$

We get the required estimation (34).

If $w_1(u) < w_2(u)$. Performing the same procedure as in (37)-(40). From (36) we can get the required estimation (35). The proof is complete.

3 Application

In this section, similar to the applications in [14], we apply our result in Theorem 1 to study of solutions of retarded integral equation

$$\begin{aligned} x(t) &= y(t) + \int_a^{\alpha(t)} A(s,x(s))ds \\ &\quad + \int_a^{\alpha(t)} A(s,x(s))B(s, \int_a^s C(\tau,x(\tau))d\tau)ds, \forall t \in I. \end{aligned} \tag{41}$$

Assume that

$$|y(t)| \le f(t), \tag{42}$$

$$|A(t,x(t))| \le g(t)w_1(\ln|x(t)|), \tag{43}$$

$$|C(t,x(t))| \le h(t)|x(t)|w_2(\ln|x(t)|), \tag{44}$$

$$|B(t, \int_a^t C(\tau,x(\tau))d\tau)| \le |x(s)| + \int_a^t |C(\tau,x(\tau))|d\tau, \tag{45}$$

where f,g,h,w_1,w_2,α are as defined in Theorem 1. From (42)-(45) and (41), we have

$$\begin{aligned} |x(t)| &\le f(t) + \int_a^{\alpha(t)} g(s)w_1(\ln|x(s)|)ds \\ &\quad + \int_a^{\alpha(t)} g(s)w_1(\ln|x(s)|)\Big[|x(s)| \\ &\quad + \int_a^s h(\tau)|x(\tau)|w_2(\ln|x(\tau)|)d\tau\Big]ds, \forall t \in I. \end{aligned} \tag{46}$$

By Theorem 1 we get an explicit bound on an unknown function $|(x(t)|$ in the retarded integral equation (41) such that

$$\begin{aligned} |x(t)| &\le \exp\Big\{W_1^{-1}\Big[W_2^{-1}\Big(\int_a^t f(s)ds \\ &\quad + \int_a^{\alpha(t)} (g(s)+h(s))ds\Big)\Big]\Big\}, t \in I, \end{aligned} \tag{47}$$

where W_1, W_2 are as defined in Theorem 1.

4 Conclusion

This paper establish a class of retarded iterated integral inequalities.

$$\begin{aligned} u(t) \le f(t) &+ \int_a^{\alpha(t)} g(s)\omega_1(\ln u(s))ds \\ &+ \int_a^{\alpha(t)} g(s)\omega_1(\ln u(s))[u(s) \\ &+ \int_a^s h(\tau)u(\tau)\omega_2(\ln u(\tau))d\tau]ds. \end{aligned}$$

Which includes a nonconstant term $f(t)$ outside the integrals. By adopting novel analysis techniques, the upper bound of the embedded unknown function.

$$u(t) \le expW_1^{-1}[W_2^{-1}(W_2(W_1(\ln(1+f(a)))) + \int_a^t f(s)ds + \int_a^{\alpha(t)} (g(s)+h(s))ds)], \quad t \in I,$$

Is estimated explicitly, where . The derived result can be applied in the study of solutions of ordinary differential equations and integral equations.

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Ricai Luo is a associate professor of mathematics in the School of Mathematics and Statistics at Hechi University,he received the M.S. degree in Mathematics at Guizhou University. His research interests concentrate on the integral and integro-differential inequalities of one or more than one variables and their applications, Differential equations, Artificial Neural Networks.

Wu-sheng Wang was born in 1960. He is a professor of mathematics in the School of Mathematics and Statistics at Hechi University, Guangxi, China. He received M.S., degree in 1997, from Department of Mathematics, Shanxi Normal University. He received Ph.D. in Applied Mathematics at School of Mathematics, Sichuan University in 2007. He worked as a Head of School of Mathematics and Statistics, Hechi University. He published more than 30 paper in various international journals. His research interest concentrate on the integral and integro-differential inequalities of one or more than one variables and their applications, Differential equations, Difference equations.