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A Transmission Problem for Euler-Bernoulli beam with Kelvin-Voigt Damping

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In this work we consider a transmission problem for the longitudinal displacement of a Euler-Bernoulli beam, where one small part of the beam is made of a viscoelastic material with Kelvin-Voigt constitutive relation. We use semigroup theory to prove existence and uniqueness of solutions. We apply a general results due to L. Gearhart [5] and J. Pruss [10] in the study of asymptotic behavior of solutions and prove that the semigroup associated to the system is exponentially stable. A numerical scheme is presented.

Keywords: Transmission problem, Exponencial stability, Euler-Bernoulli beam, Kelvin-Voigt damping, Semigroup, Numerical scheme.

1 Introduction

Consider a clamped elastic beam of length L. Let the interval $[0, L]$ be the reference configuration of a beam and $x \in [0, L]$ to denote its material points. We denote $u(x, t)$ the longitudinal displacement of the beam. Suppose that the stress σ is of rate type, i. e.,

$$\sigma = \alpha u_x + \gamma u_{xt} \quad \text{with} \quad \gamma > 0.$$

Then, the equation governing such a motion is given by

$$u_{tt} - \alpha u_{xx} - \gamma u_{txx} = 0 \quad \text{in} \quad (0, L) \times (0, \infty). \tag{1.1}$$

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Assuming that the beam is held fixed at both ends, $x = 0$ and $x = L$ we have the following boundary conditions

$$u(0,t) = 0, \quad u(L,t) = 0.$$

Now we observe that in the equation (1.1) the viscosity is distributed uniformly in the whole beam. However, it is desirable in practice to consider a situation where viscosity is active only in a piece of the beam. In this case, it is important to know if the dissipation is transmitted and if it is strong enough to stabilize the whole system.

In order to work out this question we consider the following model where one small part of the beam is made of a viscoelastic material with Kelvin-Voigt constitutive relation,

$$u_{tt} - \alpha u_{xx} - \gamma u_{txx} = 0 \text{ in } (0, L_0) \times (0, \infty), \tag{1.2}$$

$$v_{tt} - \beta v_{xx} = 0 \text{ in } (L_0, L) \times (0, \infty), \tag{1.3}$$

with boundary conditions

$$u(0,t) = v(L,t) = 0, \ t > 0, \tag{1.4}$$

initial data

$$u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in (0, L_0), \tag{1.5}$$

$$v(x,0) = v_0(x), \ v_t(x,0) = v_1(x), \ x \in (L_0, L), \tag{1.6}$$

transmission conditions

$$u(L_0,t) = v(L_0,t), \ t > 0, \tag{1.7}$$

$$\alpha u_x(L_0,t) = \beta v_x(L_0,t), \ t > 0, \tag{1.8}$$

and compatibility condition

$$u_{tx}(L_0,t) = 0, \ t > 0, \tag{1.9}$$

This kind of problem is known as a transmission problem.

A relevant question raised about the transmission problems and problems with locally distributed damping, is the asymptotic behavior of the solutions. Does the solution goes to zero uniformly? If this is the case, what is the rate of decay?

In [7] longitudinal and transversal vibrations of a clamped elastic beam where studied as problems with locally distributed damping. It was shown that when viscoelastic damping is distributed only on a subinterval in the interior of the domain, the exponential stability holds for the transversal but not for for the longitudinal motion.

At this point it is crucial to notice the difference between the formulation of transmission problems and problems with locally distributed damping. While in the former, the transmission conditions play decisive role establishing the way the parts of the body mingle with each other, in the latter it is expressed only by discontinuities in the coefficients of the equation. For further information on general transmission problems we refer to [4].

A exponential stability of transmission problem for waves with frictional damping was treated in [2], for the Timoshenko system it was treated in [12], the General Decay of solution for the transmission problem of viscoelastic waves with memory was treated in [13] and in [8] uniform stability is proved for the wave equation with smooth viscoelastic damping applied just around the boundary.

In this work we address the questions above to the system (1.1)-(1.8). We prove that the solution this system decay exponentially, i.e., the estimate,

$$E(t) \leq CE(0)e^{-wt}, \;\; C>0, \, w>0, \;\; \forall \; t>0 \tag{1.10}$$

holds for the total energy $E(t)$ of the system. This is equivalent [15] to establish the exponential stability for the semigroup $S(t)$ generated by the system, i. e.,

$$||S(t)|| \leq Ce^{-wt}, \;\; C>0, \, w>0, \;\; \forall \; t>0.$$

The central idea is to explore the dissipative character of the infinitesimal generator of the semigroup and make use of a Theorem due to Gearhart [5] and Pruss [10]. It is in this point that transmission conditions makes the difference between the formulations of problems with locally distributed damping and transmission problems. Therefore, our result does not contradicts that in [7].

These type of questions for viscoelastic waves with memory are studied in [14], for frictional and viscoelastic damping for a semi-linear wave equation are studied in [3] and for Timoshenko's beams with viscoelastic damping and memory are studied in [12].

The paper is organized as follows, in the section 2 we introduce the notation and the functional spaces, in the section 3 we prove the existence and uniqueness of solutions, in the section 4 we prove exponential stability of the semigroup generated by the system and finally, in the section 5 a numerical scheme is presented.

2 Notation and Functional Spaces

For the Sobolev spaces we use the standard notation as in [1]. We define $I_1 = (0, L_0)$ and $I_2 = (L_0, L)$. We also define

$$V_1 = \{u \in H^1(I_1) : u(0) = 0\},$$

$$V_2 = \{v \in H^1(I_2) : v(L) = 0\},$$

and

$$\mathcal{H} = \left\{ \begin{pmatrix} u^1 \\ v^1 \\ u^2 \\ v^2 \end{pmatrix} : u^i \in V_i,\ v^i \in L^2(I_i) \right\}.$$

In $\mathcal{H}$ we consider the following inner product

$$\langle U^1, U^2 \rangle = \int_0^{L_0} \alpha u_x^1 u_x^2 + v^1 v^2\, dx + \int_{L_0}^{L} \beta z_x^1 z_x^2 + w^1 w^2 dx, \tag{2.1}$$

where

$$U = \begin{pmatrix} u^i \\ v^i \\ z^i \\ w^i \end{pmatrix} \in \mathcal{H} \quad i = 1, 2.$$

Following [9] we define the linear operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ with

$$D(A) = \left\{ \begin{pmatrix} u^1 \\ v^1 \\ u^2 \\ v^2 \end{pmatrix} \in \mathcal{H} : u^i \in V_i \cap H^2(I_i),\ v^i \in V_i,\ \alpha u_x^1 + \gamma v_x^1 \in H^1(I_1) \right\}$$

and

$$A = \begin{pmatrix} 0 & I & 0 & 0 \\ \alpha \frac{\partial^2}{\partial x^2} & \gamma \frac{\partial^2}{\partial x^2} & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & \beta \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}.$$

The problem (1.2)-(1.3) can be written as

$$U_t - AU = 0 \tag{2.2}$$
$$U(0) = U_0 \tag{2.3}$$

where

$$U(t) = \begin{pmatrix} u \\ u_t \\ v \\ v_t \end{pmatrix}, \quad U(0) = \begin{pmatrix} u_0 \\ u_1 \\ v_0 \\ v_1 \end{pmatrix}, \quad AU = \begin{pmatrix} u_t \\ \alpha u_{xx} + \gamma u_{txx} \\ v_t \\ \beta v_{xx} \end{pmatrix}.$$

3 Existence of Solutions

From the construction of the functional spaces and from the theory of Sobolev Spaces, it follows that $D(A)$ is dense in $\mathcal{H}$.

Now we will to show that the full system is dissipative. First we define the total energy of the system by

$$E(t) = \frac{1}{2}\int_0^{L_0} |u_t|^2 + |u_x|^2 dx + \frac{1}{2}\int_{L_0}^{L} |v_t|^2 + |v_x|^2 dx,$$

and denote the energy in each part as

$$E_1(t) = \frac{1}{2}\int_0^{L_0} |u_t|^2 + |u_x|^2 dx,$$

$$E_2(t) = \frac{1}{2}\int_{L_0}^{L} |v_t|^2 + |v_x|^2 dx.$$

Multiplying equation (1.2) by u_t and performing integration by part on $(0, L_0)$ we obtain,

$$\frac{d}{dt}E_1(t) = -\gamma \int_0^{L_0} |u_{tx}|^2 dx + \alpha u_x(L_0)u_t(L_0) - \alpha u_x(0)u_t(0) + \beta u_{tx}(L_0)u_t(L_0). \quad (3.1)$$

Multiplying equation (1.3) by v_t and performing integration by part on (L_0, L) we obtain,

$$\frac{d}{dt}E_2(t) = \beta v_x(L)v_t(L) - \beta v_x(L_0)v_t(L_0). \quad (3.2)$$

Now adding (3.1)-(3.2), using boundary condition, transmissions conditions and compatibility condition, we get the dissipative character as we intend, that is,

$$\frac{d}{dt}E(t) = -\gamma \int_0^{L_0} |u_{tx}|^2 dx.$$

In this direction, the operator A has an important property which will be used in the sequel.

Lemma 3.1. *The operator $A : \mathcal{H} \to \mathcal{H}$ is dissipative.*

Proof. Using the definition of the inner product (2.1), we have

$$\langle AU, U\rangle = \int_0^{L_0} \alpha u_{tx} u_x + (\alpha u_{xx} + \gamma u_{txx}) u_t \, dx + \int_{L_0}^{L} \beta v_{tx} v_x + \beta v_{xx} v_t dx.$$

Performing integration by parts and using the boundary conditions , the transmission conditions and the compatibility condition, we get

$$\langle AU, U\rangle = -\gamma \int_0^{L_0} |u_{tx}|^2 \, dx \leq 0 \tag{3.3}$$

which proves the dissipativeness of A. □

Theorem 3.1. *The operator A generates a C_0-Semigroup of contractions $S(t) = e^{At}$.*

Proof. For any

$$F = \begin{pmatrix} f^1 \\ f^2 \\ f^3 \\ f^4 \end{pmatrix} \in H,$$

consider the equation $AU = F$, i.e.,

$$u_t = f^1 \in H^1(I_1) \tag{3.4}$$

$$\alpha u_{xx} + \gamma u_{txx} = f^2 \in L^2(I_1) \tag{3.5}$$

$$v_t = f^3 \in H^1(I_2) \tag{3.6}$$

$$\beta v_{xx} = f^4 \in L^2(I_2). \tag{3.7}$$

Using (3.4), (3.5) and observing the regularity of the stress, as settled in the definition of $D(A)$, we get

$$\alpha u_{xx} = f^2 - \gamma f^1_{xx} \in L^2(I_1).$$

Thus, by the standard results of the theory of elliptic equations we conclude that $u \in H^2(I_1)$. Then we obtain a unique solution

$$U = \begin{pmatrix} u \\ u_t \\ v \\ v_t \end{pmatrix} \in \mathcal{H},$$

such that $U \in D(A)$, $||U|| \leq k||F||$ for $k > 0$ and satisfies (3.4)-(3.7). Thus $0 \in \rho(A)$ the resolvent set of A, and then, A is invertible and A^{-1} is bounded linear operator. By the contraction mapping theorem, the operator $\lambda I - A = A(\lambda A^{-1} - I)$ is invertible for $0 < \lambda < ||A^{-1}||^{-1}$. Therefore, it follows from the Lummer-Phillips Theorem that A is the infinitesimal generator of a C_0-semigroup of contractions $S(t) = e^{At}$. □

From the theory of semigroup it follows that $U(t) = e^{At}U_0$ is the unique solution of (1.1)-(1.8) in the class

$$U \in C^0([0,\infty) : D(A)) \cap C^1([0,\infty) : \mathcal{H}).$$

4 Exponential Stability

In order to prove the exponential decay we are going to use the following result.

Theorem 4.1. *Let $S(t) = e^{At}$ be a C_0-Semigroup of contractions in a Hilbert space. Then $S(t)$ is exponentially stable if and only if,*

$$i\,\mathbb{R} = \{i\,\beta : \beta \in \mathbb{R}\} \subset \rho(A)$$

and

$$||(\lambda\, I - A)^{-1}|| \le C, \quad \forall\ \lambda \in i\,\mathbb{R}.$$

Proof. This result is due to L. Gearhart and its proof can be found in [5] or in Huang [6] and Prüss [10]. □

Now, by using the stability criterium due to Gearhart we prove the main result of this paper.

Theorem 4.2. *The C_0-Semigroup of contractions $S(t) = e^{At}$ generated by A is exponentially stable.*

Proof. Since $0 \in \rho(A)$ then, for every β with $|\beta| < ||A^{-1}||^{-1}$ the operator

$$i\,\beta - A = A(i\,\beta A^{-1} - I)$$

is invertible and $||(i\,\beta - A)^{-1}||$ is a continuous function of $\beta \in (-||A^{-1}||^{-1}, ||A^{-1}||^{-1})$.

At this point we are going to use an argument of contradiction. First we suppose that the condition $\{i\,\beta : \beta \in \mathbb{R}\} \subset \rho(A)$ is not satisfied. Then, there exists $w \in \mathbb{R}$ with $||A^{-1}||^{-1} \le w < \infty$ such that $\{i\,\beta : |\beta| < |w|\} \subset \rho(A)$ and the $Sup\left\{||(i\,\beta - A)^{-1}|| : |\beta| < |w|\right\} = \infty$.

Hence, there exists $(\beta_n) \in \mathbb{R}$ with $\beta_n \to w$, $|\beta_n| < |w|$ and a sequence of complex vector functions $U_n \in D(A)$ such that $||U_n|| = 1$ in $\mathcal{H}$ and

$$||(i\,\beta_n - A)U_n|| \to 0.$$

Taking the inner product of $(i\,\beta_n - A)U_n$ with U_n we obtain

$$i\,\beta_n||U_n||^2 - \langle AU_n, U_n\rangle \to 0.$$

Now, using (3.3) arrive at

$$i\,\beta_n||U_n||^2 + \gamma\int_0^{L_0} |v^1_{n,x}|^2\,dx \to 0. \tag{4.1}$$

Taking the real part we get

$$\gamma\int_0^{L_0} |v^1_{n,x}|^2\,dx \to 0. \tag{4.2}$$

Using (4.2) in (4.1) we can say that

$$i\,\beta_n||U_n||^2 \to 0. \tag{4.3}$$

Observing that $\beta_n \to w$, $|\beta_n| < |w|$ we conclude that $||U_n|| \to 0$ which contradicts $||U_n|| = 1$.

In order to finish the proof it remains to prove that

$$||(\lambda\, I - A)^{-1}|| \le C,\ \forall\ \lambda \in i\,\mathbb{R}.$$

Suppose that it is not true. Then there exist a sequence of vector function (V_n) such that

$$||(i\,\beta_n - A)^{-1}V_n|| > n||V_n||. \tag{4.4}$$

As $(V_n) \in H$ and $i\,\beta_n \in \rho(A)$, there exists a unique sequence $U_n \in D(A)$ such that

$$i\,\beta_n U_n - AU_n = V_n \quad \text{with} \quad ||U_n|| = 1.$$

Introducing $g_n = (i\,\beta_n - A)U_n$ and using (4.4) we obtain

$$||g_n|| \le \frac{1}{n} \ \text{ and hence } \ \ g_n \to 0.$$

By taking the inner product of g_n with U_n and using (3.3) we get

$$i\,\beta_n||U_n||^2 + \gamma\int_0^{L_0} |v^1_{n,x}|^2\,dx = \langle g_n, U_n\rangle.$$

Now, taking the real part and observing that (U_n) is bounded and that $g_n \to 0$ we deduce

$$\gamma\int_0^{L_0} |v^1_{n,x}|^2\,dx \to 0.$$

Proceeding as in the previous case we prove $||U_n|| \to 0$ which is a contradiction. This completes de proof. □

5 Numerical Results

In this section we implement a numerical scheme using the method of finite difference without any additional technic of numerical dissipation.

In order to verify our computational code we construct an analytic solution which produces source terms in the equation (1.2) and (1.3).

These solutions are defined by ,

$$u(x,t) = \left(\frac{1}{1+t}\right)\left[a_0^5 + \sum_{i=0}^{4} a_i a_0^{(4-i)}\right] \text{ in } (0,L_0)\times(0,\infty) \tag{5.1}$$

$$v(x,t) = u(x,t) \text{ in } (L_0,L)\times(0,\infty) \tag{5.2}$$

where $a_i = \cos\left[\frac{(i+1)(x-\pi)}{2}\right]$, $i = 0, ..., 4$. Observe that $u(x,t) \to 0,\ t \to \infty$.

We considered a spacial domain of length $L = 2\pi$ with a grid with 128 nodes uniformly distributed. The convergence of the solution, or permanent regime, is attained after 290.000 iterations in time, with error of 10^{-7} to u and 10^{-4} to v, with $\frac{\triangle t}{\triangle x} = 0,22$.

We made four tests. In the first one the viscosity was distributed uniformly in the whole string L while in the others viscosity was localized in the piece of length L_0. If $L_0 = L$ we have exponential stabilization for the solutions of equation (1.2) well illustrated in the figure 5.1. In the figures 5.2-5.4 the lengths for L_0 were: $\left\{\pi, \frac{\pi}{2}, \frac{\pi}{4}\right\}$, respectively. The parameters used in all the tests were $\alpha = 1, \gamma = 0,2$ and $\beta = 10$.

We could have used other sets of parameters since our code accepts any. In the sequence of the tests we observed that, as L_0 decreases, the solutions oscillate with larger amplitude, and, bigger values for γ could be considered. We finally observe that exponential decay occurs no matter what is the size of the interval $(0, L_0)$.

This indicates that the presence of any quantity of viscous material in the mixture will cause exponential stability for the problem considered here.

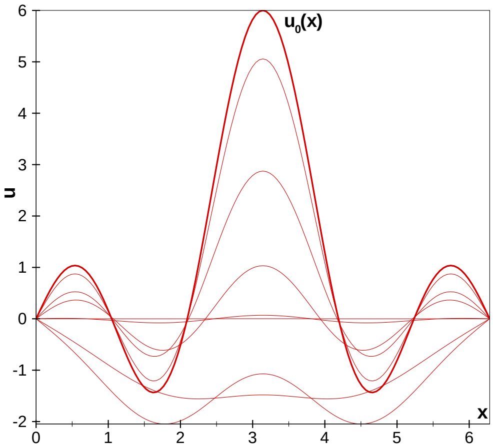

Figure 5.1: Test 1, with $L_0 = L$.

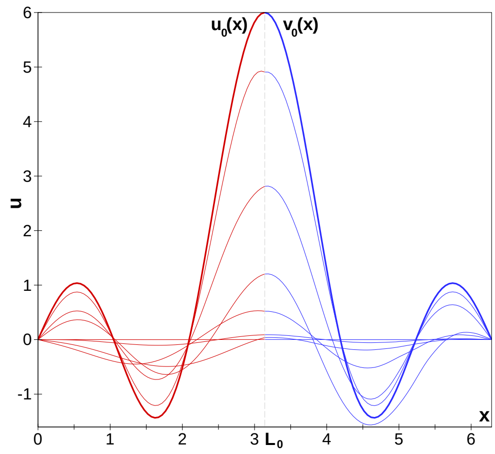

Figure 5.2: Test 2, with $L_0 = \frac{L}{2}$.

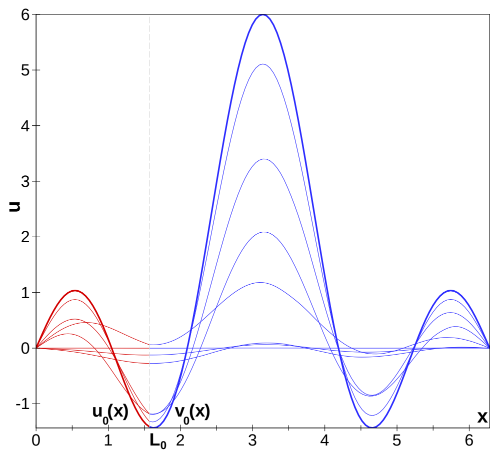

Figure 5.3: Test 3, with $L_0 = \frac{L}{4}$.

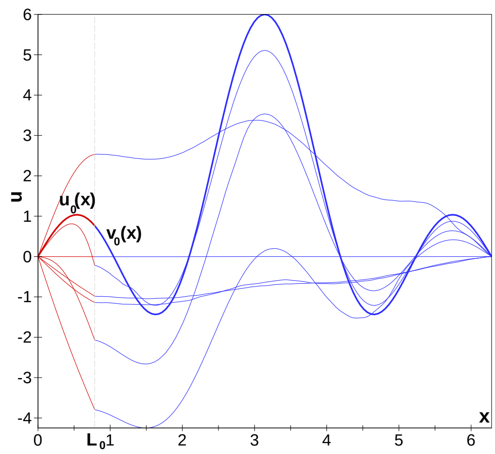

Figure 5.4: Test 4, with $L_0 = \frac{L}{8}$.

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