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# Bifurcation Analysis in Simple SIS Epidemic Model Involving Immigrations with Treatment

*Ahmed Ali Muhseen*[*]

Assistant Lecturer in Ministry of Education, First Rusafa, Baghdad, Iraq.

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**Abstract:** In this paper, local and Hopf bifurcation for an SIS epidemic model with treatment is investigated. Through theoretical analysis, we show the disease free equilibrium point has a transcritical bifurcation and the endemic equilibrium point has a saddle-node bifurcation. Also this model has a Hopf bifurcation near all equilibrium points. Applying the normal form theory and the center manifold argument, we derive the explicit formulas determining the properties of the bifurcation periodic solutions. In addition, we also study numerical simulations are also included.

**Keywords:** SIS epidemic model, Immigrations, Treatment, local and hopf bifurcation

## 1 Introduction

In recent decades, periodic oscillations (Bifurcation) received great attention due to they have been observed in the incidence of many infectious diseases, including measles, mumps, rubella, chickenpox, and influenza. In some locations, some diseases periodically occur every year, such as chickenpox, mumps, and poliomyelitis [1]. As the main reason to investigate that occurrence of this type of bifurcations it that they play a relevant role for disease control and eradication. In fact, there has been more and more interest in the analysis and prediction of consequences of public health strategies designed to control infectious diseases. The asymptotic behavior of epidemic models has been studied by many researchers. But a very different bifurcation behavior has been found in recent papers, see for example, Fred Brauer [2], J. Dushoff et al. [3], Hui and Zhu [4], P. van den Driessche and James Watmough [5], L. Liu, X.G. Li, K.J. Zhuang [6], T.L. Zhang, J.L. Liu, Z.D. Teng [7,8], B.D. Hassard, N.D. Kazarinoffand, Y.H. Wan [9], Carlo B., Massimiliano F., Luca G. [10] and Abid A., Khalid H. [11] . In this paper, from Showroq' studding she studied just local and global stability [12]. In addition to the above studied, we discuss several types of bifurcations: Saddle-node, transcritical, pitchfork, and Hopf bifurcation near all the equilibrium points.

## 2 Mathematical model

In this section, an epidemic population with a limited resource for treatment involving immigrations is proposed for study. The population is divided into two classes the susceptible individuals S(t) at time t and the infected individuals I(t) at time t. It is assume that a constant flow say A, of new members arrives into the population in unit time with the fraction p $(0 \leq p \leq 1)$ of A arriving infected. Accordingly the dynamics of SIS epidemic model with a limited resource for treatment and constant rate of immigrations, which is represented by the following system of nonlinear ordinary equations:

$$\begin{aligned} \dot{S} &= \Lambda + (1-p)A - (\mu + \beta I)S + \gamma I + T(I) \\ \dot{I} &= pA + \beta SI - (\mu + \alpha + \gamma)I - T(I) \end{aligned} \tag{1}$$

Note that, all the parameters $\Lambda, A, \mu, \gamma, \alpha$ and $\beta$ of system (1) are assumed to be positive and can be described as follows: $\Lambda$ is the recruitment rate of the susceptible population, A is the constant flow rate of immigrants, $\mu$ is the natural death rate in each classes, $\gamma$ is the nature recovery rate from infected individuals to susceptible individuals, $\alpha$ is the disease related death rate, $\beta$ is the infected coefficient, finally $T(I)$ is the treatment function which given by [13]:

$$T(I) = \begin{cases} I \;\; if \;\; 0 < I \leq I_\circ, \\ k \;\; if \;\; I > I_\circ. \end{cases} \tag{2}$$

[*] Corresponding author e-mail: aamuhseen@gmail.com

Here, $K = rI_\circ$ this means that the treatment rate is proportional to the number of infected individuals when the capacity of treatment is not reached, and other wise takes the maximal capacity. Obviously, due to the biological meaning of the components S(t) and I(t) we focus on the model in the domain:$\Re^2_+ = \left\{(S,I) \in \Re^2_+ : S \geq 0, I \geq 0\right\}$ which is positively invariant for system (1). Note that, study system (1) when $(0 < I \leq I_\circ)$ of treatment function only then can be rewritten in the following form:

$$\begin{aligned} \dot{S} &= \Lambda + (1-p)A - (\mu + \beta I)S + (\gamma + r)I \\ \dot{I} &= pA + \beta SI - (\mu + \alpha + \gamma + r)I \end{aligned} \tag{3}$$

Now, all the solutions of above system which initiate in $\Re^2_+$ are uniformly bounded and this system has two possible equilibrium points, namely $E_1 = (S_1, 0)$ is called a disease free equilibrium point and the second point is called endemic equilibrium point and denoted by $E_2 = (S_2, I_2)$ that given in [12] where:

$$S_1 = \frac{\Lambda + A}{\mu} \tag{4}$$

$$S_2 = \frac{\Lambda + A(1-p) + (\gamma + r)I_2}{\mu + \beta I_2} \tag{5}$$

$$I_2 = \frac{-a_2}{2a_1} - \frac{\sqrt{a_2^2 - 4a_1a_3}}{2a_1} \tag{6}$$

here:$a_1 = -\beta(\mu + \alpha)$ ; $a_2 = \beta(\Lambda + A) - \mu(\mu + \alpha + \gamma + r)$ ; $a_3 = p\mu A$

## 3 The local bifurcation analysis

In this section, the effect of varying the parameter on the dynamical behavior of the system (3) around each equilibrium points is studied. Recall that the existence of non-hyperbolic equilibrium point of system (3) is the necessary but not sufficient condition for bifurcation to occur. Therefore, in the following theorems are applications to the Sotomayor's theorem [14] for local bifurcation is adapted.

Theorem (1): The Jacobian matrix of system (3) at $(E_1)$ with parameter:

$$\beta^* = \frac{\mu + \alpha + \gamma + r}{S_1} \tag{7}$$

has:

1) No saddle-node bifurcation

2) No pitchfork bifurcation

3) Transcritical bifurcation

Proof: It is easy to verify that the Jacobian matrix of system (3) at $(E_1, \beta^*)$ can be written as:

$$J = Df(E_1, \beta^*)$$

$$= \begin{pmatrix} -\mu & -\beta^* S_1 + \gamma + r \\ 0 & 0 \end{pmatrix}$$

Clearly, the second eigenvalue $\lambda_I$ in the I- direction is zero, while $\lambda_S$ is negative. Further, the eigenvector (say $Z = (z_1, z_2)^T$) corresponding to $\lambda_I$ satisfy the following:

$$JZ = \lambda Z$$

then

$$JZ = 0$$

$$Thus$$

$$\begin{pmatrix} -\mu & -\beta^* S_1 + \gamma + r \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = 0$$

From which we get that:

$$-\mu z_1 - \beta^* S_1 z_2 + (\gamma + r)z_2 = 0 \tag{8}$$

So by solving the above system of equation we get:

$$z_1 = qz_2$$

where: $q = \frac{-[\beta^* S_1 - (\gamma + r)]z_2}{\mu}$, here $z_2$ be any non zero real number. Thus

$$Z = \begin{pmatrix} qz_2 \\ z_2 \end{pmatrix}$$

Similarly the eigenvector $W = (w_1, w_2)^T$ corresponding to $\lambda_I$ of $J^T$ can be written:

$$\begin{pmatrix} -\mu & 0 \\ -\beta^* S_1 + \gamma + r & 0 \end{pmatrix} \cdot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0$$

This gives:

$$\begin{aligned} -\mu w_1 &= 0 \\ (\gamma + r - \beta^* S_1)w_1 &= 0 \end{aligned} \tag{9}$$

So by solving the above system of equation we get:

$$W = \begin{pmatrix} 0 \\ w_2 \end{pmatrix}$$

here $z_2$ be any non zero real number. Now rewrite system (3) in a vector form as:

$$\frac{dX}{dt} = f(X)$$

where, $X = (S,I)^T$ and $f(f_1, f_2)^T$ whith $f_i, i = 1,2$ given in system (3), and then determine $\frac{df}{d\beta} = f_\beta$ we get that:

$$f_\beta = \begin{pmatrix} -SI \\ SI \end{pmatrix}$$

Then

$$f_\beta(E_1,\beta^*) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Therefore:

$$W^T \cdot f_\beta(E_1,\beta^*) = 0$$

Consequently, according to Sotomayor's theorem the system has no saddle-node bifurcation near $E_1$ at $\beta^*$. Now in order to investigate the accruing of other types of bifurcation, the derivative of $f_\beta$ with respect to vector $X$, say $Df_\beta(E_1,\beta^*)$, is computed

$$Df_\beta(E_1,\beta^*) = \begin{pmatrix} 0 & -S_1 \\ 0 & S_1 \end{pmatrix}$$

So

$$W^T \cdot \left[Df_\beta(E_1,\beta^*) \cdot Z\right] = w_2 z_2 S_1 \neq 0$$

Again, according to Sotomayor theorem, if in addition to the above the following holds

$$W^T \cdot \left[D^2 f(E_1,\beta^*) \cdot (Z,Z)\right] \neq 0$$

Here $Df(E_1,\beta^*)$ is the Jacobian matrix at $E_1$ and $\beta^*$, then the system (3) possesses a transcritical bifurcation but no pitch-fork bifurcation can occur. Now since we have that:

$$\left[D^2 f(E_1,\beta^*) \cdot (Z,Z)\right] = \begin{pmatrix} 2q\beta^* z_2^2 \\ -2q\beta^* z_2^2 \end{pmatrix}$$

Therefore:

$$W^T \cdot \left[D^2 f(E_1,\beta^*) \cdot (Z,Z)\right] = -2q\beta^* w_2 z_2^2 \neq 0$$

Then the system (3) has a transcritical bifurcation at $E_1$ when the parameter $\beta$ passes through the bifurcation value $\beta^*$.

Theorem (2): The Jacobian matrix of system (3) at $(E_2)$ with parameter:

$$\alpha_\circ = \frac{\mu[\beta S_2 - (\mu+\gamma+r+\beta I_2)]}{\mu+\beta I_2} \tag{10}$$

Clearly $\alpha_\circ$ is positive provided that:

$$\beta S_2 > \mu+\gamma+r+\beta I_2 \tag{11}$$

Then the system (3) has:

1) Saddle-node bifurcation

2) No pitchfork bifurcation

3) No Transcritical bifurcation

Proof: The system (3) at the endemic equilibrium point $E_2$ has zero eigenvalue (say $\bar{\lambda}$) if and only if $det\,(J(E_2)) = 0$, therefore $\alpha_\circ = \frac{\mu[\beta S_2-(\mu+\gamma+r+\beta I_2)]}{\mu+\beta I_2}$ is taken as a candidate bifurcation parameter and the Jacobian matrix $J(E_2)$ with $\alpha = \alpha_\circ$ becomes:

$$J(\bar{\lambda}=0) = \begin{pmatrix} -(\mu+\beta I_2) & -\beta S_2 + (\gamma+r) \\ \beta I_2 & \beta S_2 - (\mu+\alpha_\circ+\gamma+r) \end{pmatrix}$$

Now, to find eigenvector (say $L = (l_1,l_2)^T$) corresponding to $\bar{\lambda} = 0$ satisfy the following:

$$J \cdot L = 0$$

Thus:

$$\begin{pmatrix} -(\mu+\beta I_2) & -\beta S_2 + (\gamma+r) \\ \beta I_2 & \beta S_2 - (\mu+\alpha_\circ+\gamma+r) \end{pmatrix} \cdot \begin{pmatrix} l_1 \\ l_2 \end{pmatrix} = 0$$

From which we get that:

$$-(\mu+\beta I_2)l_1 - (\beta S_2 - (\gamma+r))l_2 = 0 \tag{12}$$

$$\beta I_2 l_1 + (\beta S_2 - (\mu+\alpha_\circ+\gamma+r))l_2 = 0 \tag{13}$$

So by solving the above system of equations we get:

$$l_1 = z l_2$$

where:$z = \frac{-(\beta S_2-(\gamma+r))l_2}{\mu+\beta I_2}$ here $l_2$ be any non zero real number. Thus

$$L = \begin{pmatrix} z l_2 \\ l_2 \end{pmatrix}$$

Similarly the eigenvector $H = (h_1,h_2)^T$ corresponding to $\bar{\lambda} = 0$ of $J^T$ can be written:

$$\begin{pmatrix} -(\mu+\beta I_2) & \beta I_2 \\ -\beta S_2+\gamma+r & \beta S_2 - (\mu+\alpha_\circ+\gamma+r) \end{pmatrix} \cdot \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = 0$$

This gives:

$$-(\mu+\beta I_2)h_1 + \beta I_2 h_2 = 0 \tag{14}$$

$$(\gamma+r-\beta S_2)h_1 + (\beta S_1 - (\mu+\alpha_\circ+\gamma+r))h_2 = 0 \tag{15}$$

So by solving the above system of equation we get:

$$H = \begin{pmatrix} q h_2 \\ h_2 \end{pmatrix}$$

where: $q = \frac{\beta I_2}{\mu+\beta I_2}$ and $h_2$ be any non zero real number. Now rewrite system (3) in a vector form as:

$$\frac{dY}{dt} = f(Y)$$

where $Y = (S,I)^T$ and $f = (f_1,f_2)^T$ with $f_i, i = 1,2$ given in system (3), and then determine $\frac{df}{d\alpha} = f_\alpha$ we get that:

$$f_\alpha = \begin{pmatrix} 0 \\ -I \end{pmatrix}$$

Then

$$f_\alpha(E_2,\alpha_\circ) = \begin{pmatrix} 0 \\ I_2 \end{pmatrix}$$

Therefore:

$$H^T \cdot f_\alpha(E_2, \alpha_\circ) = -h_2 I_2 \neq 0$$

So, according to Sotomayor's theorem for local bifurcation the transcritical bifurcation and pitchfork bifurcation cannot occur. While the first condition of the saddle-node bifurcation is satisfied. Moreover it easy to verify that

$$D^2 f(E_2, \alpha_\circ) \cdot (L, L) = \begin{pmatrix} 2z\beta l_2^2 \\ -2z\beta l_2^2 \end{pmatrix}$$

Hence

$$H^T \cdot \left[D^2 f(E_2, \alpha_\circ) \cdot (L, L)\right] = 2z\beta h_2 l_2^2 (q-1) \neq 0$$

Consequently, according to Sotomayor's theorem the system (3) has saddle-node bifurcation near $E_2$ at $\alpha_\circ$.

## 4 The Hopf bifurcation analysis

In this section, the existence of periodic dynamic in system (3) due to changing the value of one parameter is studied in the following theorem.

Theorem (3): The system (3) has a Hopf-bifurcation around the endemic equilibrium point $E_2$ satisfy the following condition:

$$\bar{\beta} = \frac{2\mu + \alpha + \gamma + r}{S_2 + I_2} \tag{16}$$

Proof: Consider the Jacobian matrix of system (3) at $E_2$ with the characteristic equation can be written in the following form:

$$J(E_2) = \begin{pmatrix} -(\mu + \beta I_2) & -\beta S_2 + \gamma + r \\ \beta I_2 & \beta S_2 - (\mu + \alpha + \gamma + r) \end{pmatrix}$$

Then

$$\lambda^2 + T\lambda + D = 0 \tag{17}$$

Clearly, the eigenvalues of above equation can be written:

$$\lambda = \frac{1}{2}\left[-T \pm \sqrt{T^2 - 4D}\right]$$

here

$$T(trace) = \beta(S_2 - I_2) - (2\mu + \alpha + \gamma + r)$$

$$D(det) = \mu(\mu + \alpha + \gamma + r - \beta S_2 + \beta I_2(\mu + \alpha + 2(\gamma + r))$$

Obviously, system (3) dose not a Hopf-bifurcation around the endemic equilibrium point $E_2$ if and only if the trace of eigenvalues $T \neq 0$. Now, the necessary and sufficient conditions for a simple Hopf-bifurcation to occur we need to find a parameter $(\bar{\beta})$ satisfy that:

$$T(\bar{\beta}) = 0 \tag{18}$$

$$\frac{dT}{d\beta}|_{\beta=\bar{\beta}} \neq 0 \tag{19}$$

Then the system (3) has two complex conjugate eigenvalues. Clearly, the first condition (18) for the Hopf-bifurcation is satisfied at $\beta = \bar{\beta}$ if and only if provided the condition (16). Let as now check the second condition (19) in the following:

$$\frac{dT}{d\beta}|_{\beta=\bar{\beta}} = S_2 \neq 0$$

Hence, the system (3) has a Hopf-bifurcation a round the endemic equilibrium point $E_2$ at the parameter $\beta = \bar{\beta}$ and the proof is complete.

## 5 Numerical analysis

In this section, the local and Hopf bifurcation of system (3) is studied. The objectives of this study are confirming our analytical results and understand the effects of bifurcation value on the dynamics of *SIS* epidemic system. Consequently, system (3) is solved numerically for same initial condition (150,550) and different sets of parameters. It is observed that, for the following set of hypothetical parameters, system (3) is solved numerically for same set of initial value and then the trajectories of system (3) as a function of time are drawn in Figure (1).

$$\begin{aligned} \Lambda = 400; A = 100; p = 0; \beta = 0.0001 \\ \gamma = 2; \alpha = 2; r = 1; \mu = 0.1 \end{aligned} \tag{20}$$

Obviously, Figure (1) Show clearly the transcritical bifurcation accurse of system (3) near the disease free equilibrium point $E_1$, when the infection rate increases (through increasing $\beta$), the disease free equilibrium point of system (3) becomes unstable point and the trajectory of system (3) approaches asymptotically to the endemic equilibrium point.

However, for the data given equation (20) but $p = 0.01$ and $\beta = 0.01$. The trajectories of system (3) are drawn in Figure (2).

Clearly, from above figure as the infection rate decrease $(\beta)$, then the endemic equilibrium point losses it is stability and a Hopf bifurcation occurs as shown in Fig. (2a). however, as it increases the endemic equilibrium point still coexists and stable with increase in the value of infected individuals whereas the value susceptible individuals increases.

Now, in order to discuss the effect of varying the treatment rate on the dynamical behavior of system (3), the system is solved for different values of treatment rate $r = 3, 4, 5.24$ respectively, keeping other parameters fixed as given in equation (20) with $p = 0.01$ and $\beta = 0.01$, and

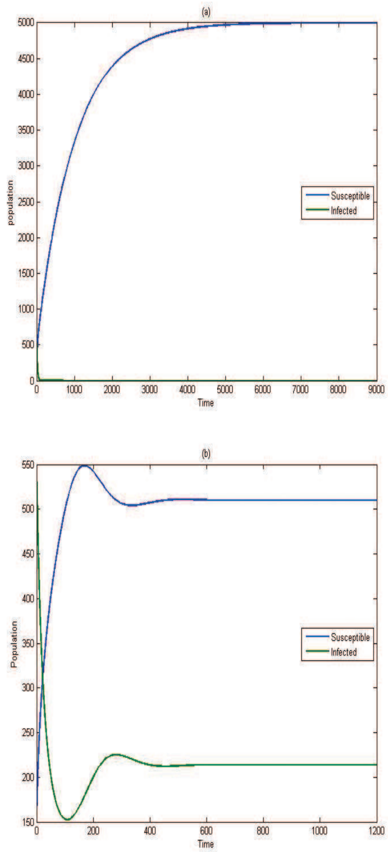


**Fig. 1:** (a) Time series of trajectories of system (3) for data given in equation (20). (b) Time series of trajectories of system (3) for data given in equation (20) with $\beta = 0.01$.

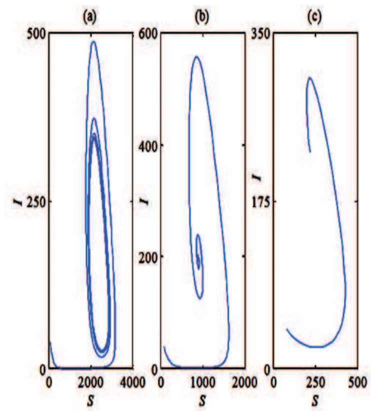


**Fig. 2:** (a) phase plots of system (3) for data given in equation (20) with $p = 0.01$ and $\beta = 0.002$. (b) Phase plots of system (3) for data given in equation (20) with $p = 0.01$ and $\beta = 0.005$. (c) Phase plots of system (3) for data given in equation (20) with $p = 0.01$ and $\beta = 0.02$.

then the solution of system (3) is drawn in Figure (3).

According to the above figure, it is clear that, as the treatment rate increases the endemic equilibrium point losses it is stability and a Hopf bifurcation occurs as shown in Figure (3c). However, as it decreases the endemic equilibrium point still coexists and stable.

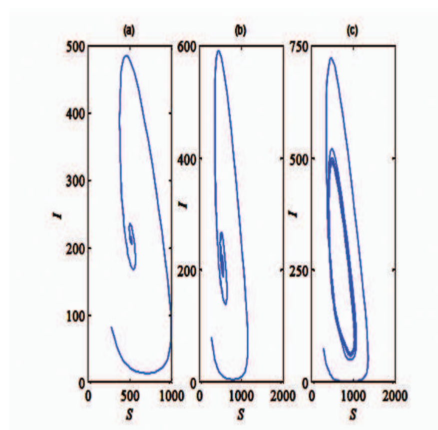


**Fig. 3:** (a) phase plots of system (3) for data given in equation (20) with $r = 3$. (b) phase plots of system (3) for data given in equation (20) with $r = 4$. (c) phase plots of system (3) for data given in equation (20) with $r = 5.24$.

## 6 Conclusions and discussion

In this paper, the system (3) has a transcritical bifurcation near the disease free equilibrium point, but neither saddle-node nor pitchfork bifurcation can accrue. And, has a Hopf bifurcation near the endemic equilibrium point. Finally according to the numerically simulation the following results are obtained:

1) As the varying the infection rate increases, the asymptotic behavior of the system transfers from approaching to disease free equilibrium point to the endemic equilibrium point.

2) As the infected rate decreases then the system (3) still approaches to the endemic equilibrium point with the system (3) has occurrence of a Hopf bifurcation around the endemic equilibrium point .

3) Finally, the treatment rate increases then the asymptotic behavior of system (3) losses stability and a Hopf bifurcation occurs around the endemic equilibrium point.

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## References

[1] W.D. Wang, S.G. Ruan, Bifurcation in an epidemic model with constant removal rate of the invectives, J. Math. Anal. Appl. **291** (2004) 775-793.
[2] Brauer, F. Backward bifurcation in simple vaccination models. J. Math. Anal. Appl., 289: 418-431 (2004).
[3] Dushoff, J., Huang, W., C-Chavez, C. Backward bifurcation and catastrophe in simple models of fatal diseases. J. Math. Biol., **36**: 227-248 (1998).
[4] Hui, J., Zhu, D.M. Global stability and periodicity on SIS epidemic models with backward bifurcation. Comp. Math. Appl., **50**: 1271-1290 (2005).
[5] Van den driessche, P., Watmough, James. A simple SIS epidemic model with a backward bifurcation. J. Math. Biol., **40**: 525-540 (2000).
[6] L. Liu, X.G. Li, K.J. Zhuang, Bifurcation analysis on a delayed SIS epidemic model with stage structure, Electron. J. Differential. Equation. **(2007)** 1-17.
[7] T.L. Zhang, J.L. Liu, Z.D. Teng, Bifurcation analysis of a delayed SIS epidemic model with stage structure, Chaos Solutions Fract. **40** (2009) 563-576.
[8] T.L. Zhang, J.L. Liu, Z.D. Teng, Stability of Hopf bifurcation of a delayed SIRS epidemic model with stage structure, Nonlinear Anal. **11** (2010) 293-306.
[9] B.D. Hassard, N.D. Kazarinoffand, Y.H. Wan, Theory and Application of Hopf Bifurcation, Cambridge University Press, Cambridge, **(1981)**. pp. 181-219.
[10] Carlo B., Massimilano F., Luca G., Hopf Bifurcation in a Delayed-Energy-Based Model of Capital Accumulation, J. Appl. Math. Inf. Sci. Lett. 7, **(2013)**. pp. 1-5.
[11] Abid A., Khalid H., Backward Bifurcation and Optimal Contrl of a Vector borne Disease, J. Appl. Math. Inf. Sci. Lett. 7, **(2013)**. pp. 301-309.
[12] R.K. Naji, Sh. K. Shafeeq, The effects of treatment and immigrants on the dynamics of SIS epidemic model, J. Dirasat, pure, **(2013)**. pp. 73-82.
[13] Li. X.Z. et. al., Stability and bifurcation of an SIS epidemic model with treatment, J. Chaos, Solution and Fractals, **42**, 2822-2832, (2009).
[14] Sotomayor, J., Generic bifurcations of dynamical systems, in dynamical systems, M. M. Peixoto, New York, academic press **(1973)**.

**Ahmed Ali** received the master's degree in applied mathematical (mathematical model) From the College of Education, Ibn al-Haytham of Pure Science, University of Baghdad, Baghdad-Iraq. His research interests are in the areas of applied mathematics and he now a assistant lecturer in the Iraqi Ministry of Education. He has published research articles in reputed international journals of Applied Mathematical.