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Certain Integral Transforms for the Incomplete Functions

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Abstract: Srivastava *et al.* [14] introduced the incomplete Pochhammer symbols that lead to a natural generalization and decomposition of a class of hypergeometric and other related functions to mainly investigate certain potentially useful closed-form representations of definite and semi-definite integrals of various special functions. Here, in this paper, we use the integral transforms like Beta transform, Laplace transform, Mellin transform, Whittaker transforms, K-transform and Hankel transform to investigate certain interesting and (potentially) useful integral transforms the incomplete hypergeometric type functions ${}_p\gamma_q[z]$ and ${}_p\Gamma_q[z]$. Relevant connections of the various results presented here with those involving simpler and earlier ones are also pointed out.

Keywords: Gamma function; Beta function; Incomplete gamma functions; Incomplete Pochhammer symbols; incomplete hypergeometric functions; Beta transform; Laplace transform; Mellin transform; Whittaker transforms; K-transform; Hankel transform

1 Introduction and Preliminaries

The incomplete Gamma type functions like $\gamma(s,x)$ and $\Gamma(s,x)$, both of which are certain generalizations of the classical Gamma function $\Gamma(z)$, given in (1) and (2), respectively, have been investigated by many authors. The incomplete Gamma functions have proved to be important for physicists and engineers as well as mathematicians. For more details, one may refer to the books [1,2,5,20, 22] and the recent papers [3,13,14,17,18] and [19] on the subject.

The familiar incomplete Gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ defined by

$$\gamma(s,x) := \int_0^x t^{s-1}e^{-t}dt \quad (\Re(s)>0;\ x\geq 0), \tag{1}$$

and

$$\Gamma(s,x) := \int_x^\infty t^{s-1}e^{-t}dt \tag{2}$$

$(x\geq 0;\ \Re(s)>0 \text{ when } x=0)$,

respectively, satisfy the following decomposition formula

$$\gamma(s,x)+\Gamma(s,x)=\Gamma(s) \quad (\Re(s)>0),$$

where $\Gamma(s)$ is the well-known Euler's Gamma function defined by

$$\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt \quad (\Re(s)>0). \tag{3}$$

We also recall the Pochhammer symbol $(\lambda)_n$ defined (for $\lambda\in\mathbb{C}$) by

$$(\lambda)_n := \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n\in\mathbb{N}:=\{1,2,\ldots\}) \end{cases} = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-), \tag{4}$$

where $\mathbb{Z}_0^-$ denotes the set of nonpositive integers (see, *e.g.*, [15, p. 2 and p. 5]).

The theory of the incomplete Gamma functions, as a part of the theory of confluent hypergeometric functions, has received its first systematic exposition by Tricomi [21] in the early 1950s. Musallam and Kalla [8,9] considered a more general incomplete gamma function involving the Gauss hypergeometric function and established a number of analytic properties including recurrence relations, asymptotic expansions and computation for special values of the parameters. Very recently, Srivastava *et al.* [14] introduced and studied some fundamental properties and characteristics of a family of two potentially useful and generalized incomplete hypergeometric functions defined as follows:

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$$
{}_p\gamma_q\left[\begin{matrix}(a_1,x),a_2,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}z\right]:=\sum_{n=0}^{\infty}\frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{z^n}{n!} \tag{5}
$$

and

$$
{}_p\Gamma_q\left[\begin{matrix}(a_1,x),a_2,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}z\right]:=\sum_{n=0}^{\infty}\frac{[a_1;x]_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{z^n}{n!}, \tag{6}
$$

where $(a_1;x)_n$ and $[a_1;x]_n$ are interesting generalization of the Pochhammer symbol $(\lambda)_n$, in terms of the incomplete gamma type functions $\gamma(s,x)$ and $\Gamma(s,x)$ defined as follows

$$
(\lambda;x)_\nu:=\frac{\gamma(\lambda+\nu,x)}{\Gamma(\lambda)}\quad(\lambda,\,\nu\in\mathbb{C};\;x\geq 0) \tag{7}
$$

and

$$
[\lambda;x]_\nu:=\frac{\Gamma(\lambda+\nu,x)}{\Gamma(\lambda)}\quad(\lambda,\,\nu\in\mathbb{C};\;x\geq 0). \tag{8}
$$

These incomplete Pochhammer symbols $(\lambda;x)_\nu$ and $[\lambda;x]_\nu$ satisfy the following decomposition relation:

$$
(\lambda;x)_\nu+[\lambda;x]_\nu=(\lambda)_\nu\quad(\lambda,\nu\in\mathbb{C};x\geq 0). \tag{9}
$$

Remark 1. We repeat the remark given by Srivastava *et al*. [14, Remark 7] for completeness and an easier reference. In (1), (2), (5), (6), (7) and (8), the argument $x\geq 0$ is independent of the argument $z\in\mathbb{C}$ which occurs in the (5) and (6) and also in the result presented in this paper. Since (see, *e.g.*, [14, p. 675])

$$
|(\lambda;x)_n|\leq|(\lambda)_n| \text{ and } |[\lambda;x]_n|\leq|(\lambda)_n| \quad (n\in\mathbb{N};\lambda\in\mathbb{C};x\geq 0), \tag{10}
$$

the precise (sufficient) conditions under which the infinite series in the definitions (5) and (6) would converge absolutely can be derived from those that are well-documented in the case of the generalized hypergeometric function ${}_pF_q$ $(p,q\in\mathbb{N})$ (see, for details, [10, pp. 73-74] and [16, p. 20]; see also [5]).

Integral transforms are widely used to solve differential equations and integral equations. So a lot of work has been done on the theory and applications of integral transforms. Most popular integral transforms are due to Euler, Laplace, Fourier, Mellin, Hankel and so on. Here, in this paper, we use the integral transforms like Beta transform, Laplace transform, Mellin transform, Whittaker transforms, K-transform and Hankel transform to investigate certain interesting and (potentially) useful integral transforms for incomplete hypergeometric type functions ${}_p\gamma_q[z]$ and ${}_p\Gamma_q[z]$.

2 Integral Transform of the Incomplete Hypergeometric functions

In this section, we shall prove six theorems, which exhibit certain connections between the integral transforms such as Euler transform, Laplace transform, Mellin transform, Whittaker transforms, K-transform and Hankel transform and the generalized incomplete hypergeometric type functions ${}_p\gamma_q[z]$ and ${}_p\Gamma_q[z]$ given by (5) and (6), respectively.

Theorem 1. *Suppose that $x\geq 0$, α, $\beta\in\mathbb{C}$ and $p,q\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$. Then we have*

$$
B\left\{{}_p\gamma_q[yz]:\alpha,\beta\right\}=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\times{}_{p+1}\gamma_{q+1}\left[\begin{matrix}\alpha,(a_1,x),a_2,\cdots,a_p;\\ \alpha+\beta,b_1,\cdots,b_q;\end{matrix}y\right] \tag{11}
$$

and

$$
B\left\{{}_p\Gamma_q[yz]:\alpha,\beta\right\}=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\times{}_{p+1}\Gamma_{q+1}\left[\begin{matrix}\alpha,(a_1,x),a_2,\cdots,a_p;\\ \alpha+\beta,b_1,\cdots,b_q;\end{matrix}y\right], \tag{12}
$$

where $B\{f(z):\alpha,\beta\}$ denotes the Euler (Beta) transform of $f(z)$ defined by (see, e.g., [12]):

$$
B\{f(z):\alpha,\beta\}=\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}f(z)dz. \tag{13}
$$

*Proof.*Applying (5) to the Euler (Beta) transform in (13), we get

$$
\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\,{}_p\gamma_q[yz]dz=\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\times\sum_{n=0}^{\infty}\frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{(yz)^n}{n!}dz. \tag{14}
$$

By changing the order of integration and summation in and using the Beta function $B(\alpha,\,\beta)$ (see, e.g., [15, p. 8]):

$$
B(\alpha,\beta)=\begin{cases}\displaystyle\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt & (\Re(\alpha)>0;\Re(\beta)>0)\\[2mm] \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} & (\alpha,\,\beta\in\mathbb{C}\setminus\mathbb{Z}_0^-),\end{cases} \tag{15}
$$

we get

$$
\int_0^1 z^{\alpha-1}(1-z)^{\beta-1}\,{}_p\gamma_q[yz]dz=\sum_{n=0}^{\infty}\frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\times\frac{\Gamma(\alpha+n)\Gamma(\beta)}{\Gamma(\alpha+\beta+n)}\frac{(y)^n}{n!}, \tag{16}
$$

which, upon using (5), yields our desired result (11).

It is easy to see that a similar argument as in the proof of (11) will establish the result (12). This completes the proof of Theorem 1.

Theorem 2. *Suppose that* $x \geq 0$, $\Re(s) > 0$, $\alpha \in \mathbb{C}$, $p,q \in \mathbb{N}_0$ *and* $|\frac{y}{s}| < 1$. *Then we obtain*

$$L\left\{z^{\alpha-1}\ {}_p\gamma_q[yz]\right\} = \frac{\Gamma(\alpha)}{s^n}\ {}_{p+1}\gamma_q\left[\begin{matrix}\alpha,(a_1,x),a_2,\cdots,a_p;\ \frac{y}{s}\\ b_1,\cdots,b_q;\ \end{matrix}\right] \tag{17}$$

and

$$L\left\{z^{\alpha-1}\ {}_p\Gamma_q[yz]\right\} = \frac{\Gamma(\alpha)}{s^n}\ {}_{p+1}\Gamma_q\left[\begin{matrix}\alpha,(a_1,x),a_2,\cdots,a_p;\ \frac{y}{s}\\ b_1,\cdots,b_q;\ \end{matrix}\right], \tag{18}$$

where $L\{f(z)\}$ denotes the Laplace transform of $f(z)$ defined by (see,[12]):

$$L\{f(z)\} = \int_0^\infty e^{-sz}f(z)dz. \tag{19}$$

Here we assume both sides of above results exist.

*Proof.*Applying (5) to the Laplace transform in (19), we get

$$\int_0^\infty z^{\alpha-1}e^{-sz}\ {}_p\gamma_q[yz]dz = \int_0^\infty z^{\alpha-1}e^{-sz} \times \sum_{n=0}^\infty \frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{(yz)^n}{n!}. \tag{20}$$

By changing the order of integration and summation and using the known Laplace transform (see, e.g., [11, Eq.(2.2)]):

$$L\{z^\nu\} = \frac{\Gamma(\nu+1)}{s^{\nu+1}} \quad (\Re(\nu) > -1; \Re(s) > 0), \tag{21}$$

we get

$$\int_0^\infty z^{\alpha-1}e^{-sz}\ {}_p\gamma_q[yz]dz = \sum_{n=0}^\infty \frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{\Gamma(\alpha+n)}{s^{\alpha+n}}\frac{(y)^n}{n!}, \tag{22}$$

which, upon using (5), yields our desired result (17).

By following the same procedure as in the proof of the result (17), one can easily establish the result (18). Therefore, we omit the details of the proof of the result (18). This completes the proof of Theorem 2.

Mellin transform is based on the preliminary assertions giving Mellin-Barnes contour integral representations of the incomplete generalized hypergeometric functions ${}_p\gamma_q[z]$ and ${}_p\Gamma_q[z]$.

Lemma 1 (see [14, Theorem 18]). *Let* $\mathfrak{L} = \mathfrak{L}_{(\sigma;\mp i\infty)}$ *be a Mellin-Barnes-type contour starting at the point* $\sigma - i\infty$ *and terminating at the end points* $\sigma + i\infty$ $(\sigma \in \mathbb{R})$ *with the usual indentations in order to separate one set of poles from the other set of poles of the integrand. Then we have*

$$\begin{aligned}{}_p\gamma_q\left[\begin{matrix}(A_p,x);\\ B_q;\end{matrix}\ z\right] &= {}_p\gamma_q\left[\begin{matrix}(a_1,x),a_2,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\ z\right]\\ &= \frac{1}{2\pi i}\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\times\\ &\int_{\mathfrak{L}}\frac{\gamma(a_1+s,x)\Gamma(a_2+s)\cdots\Gamma(a_p+s)\Gamma(-s)}{\Gamma(b_1+s)\cdots\Gamma(b_q+s)}\\ &\qquad \Gamma(-s)(-z)^s ds,\end{aligned} \tag{23}$$

and

$$\begin{aligned}{}_p\Gamma_q\left[\begin{matrix}(A_p,x);\\ B_q;\end{matrix}\ z\right] &= {}_p\Gamma_q\left[\begin{matrix}(a_1,x),a_2,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\ z\right]\\ &= \frac{1}{2\pi i}\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\times\\ &\int_{\mathfrak{L}}\frac{\Gamma(a_1+s,x)\Gamma(a_2+s)\cdots\Gamma(a_p+s)\Gamma(-s)}{\Gamma(b_1+s)\cdots\Gamma(b_q+s)}\\ &\qquad \Gamma(-s)(-z)^s ds,\end{aligned} \tag{24}$$

where $|\arg(-z)| < \pi$.

Remark 2 (see [14, Remark 7]). In their special case when $p-1 = q = 1$, the assertions (23) and (24) of Lemma 1 would immediately yield the corresponding Mellin-Barnes type contour integral representations for the incomplete Gauss hypergeometric functions ${}_2\gamma_2$ and ${}_2\Gamma_2$, respectively.

Theorem 3. *Suppose* $x \geq 0$, $\Re(s) > 0$, $w \in \mathbb{C}$ *and* $p,q \in \mathbb{N}_0$. *Then we have*

$$M\left\{{}_p\gamma_q[-wt];s\right\} = \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)} \times\frac{\gamma(a_1+s,x)\Gamma(a_2+s)\cdots\Gamma(a_p+s)\Gamma(-s)}{\Gamma(b_1+s)\cdots\Gamma(b_q+s)}\cdot w^s \tag{25}$$

and

$$M\left\{{}_p\Gamma_q[-wt];s\right\} = \frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\times \frac{\Gamma(a_1+s,x)\Gamma(a_2+s)\cdots\Gamma(a_p+s)\Gamma(-s)}{\Gamma(b_1+s)\cdots\Gamma(b_q+s)}\cdot w^s, \tag{26}$$

where $M\{f(z);s\}$ *denotes the Mellin transform of* $f(z)$ *defined by* (*see, e.g.*, [7, p.46, Eqs. (2.1) and (2.2)]:

$$M\{f(z);s\} = \int_0^\infty z^{s-1}f(z)dz = \overline{f}(s), \Re(s) > 0 \tag{27}$$

and its inverse Mellin transform M^{-1}

$$\overline{f}(s) = M^{-1}[\overline{f}(s);t] = \frac{1}{2\pi i}\int_{\mathscr{L}}\overline{f}(s)t^{-s}ds. \tag{28}$$

Here we assume that both members of the Mellin transforms exist.

*Proof.*Setting $z=-\frac{w}{t}$ in (23), we get

$$
{}_p\gamma_q[-wt]=\frac{1}{2\pi i}\frac{\Gamma(b_1)\cdots\Gamma(b_q)}{\Gamma(a_1)\cdots\Gamma(a_p)}\times \int_{\mathfrak{L}}\frac{\gamma(a_1+s,x)\Gamma(a_2+s)\cdots\Gamma(a_p+s)\Gamma(-s)}{\Gamma(b_1+s)\cdots\Gamma(b_q+s)}\times \Gamma(-s)\left(\frac{w}{t}\right)^s ds. \tag{29}
$$

Using (28) in the resulting identity, Equation (29) immediately leads to (25).

Following the same procedure, one can easily establish the result (26).

Theorem 4. *Suppose that* $x\geq 0$, $\rho,\delta,\lambda,\mu\in\mathbb{C}$ *and* $p,q\in\mathbb{N}_0$. *Then the following integral relations hold*:

$$
\int_0^\infty t^{\rho-1}e^{-\delta t/2}W_{\lambda,\mu}(\delta t)\,{}_p\gamma_q[wt]dt = \delta^{-\rho}\frac{\Gamma(\frac{1}{2}+\mu+\rho)\Gamma(\frac{1}{2}-\mu+\rho)}{\Gamma(1-\lambda+\rho)} \times {}_{p+2}\gamma_{q+1}\left[\begin{matrix}\frac{1}{2}+\mu+\rho,\frac{1}{2}-\mu+\rho,(a_1,x),a_2,\cdots,a_p;\\ (1-\lambda+\rho),b_1,\cdots,b_q;\end{matrix}\ \frac{w}{\delta}\right] \tag{30}
$$

and

$$
\int_0^\infty t^{\rho-1}e^{-\delta t/2}W_{\lambda,\mu}(\delta t)\,{}_p\Gamma_q[wt]dt = \delta^{-\rho}\frac{\Gamma(\frac{1}{2}+\mu+\rho)\Gamma(\frac{1}{2}-\mu+\rho)}{\Gamma(1-\lambda+\rho)} \times {}_{p+2}\Gamma_{q+1}\left[\begin{matrix}\frac{1}{2}+\mu+\rho,\frac{1}{2}-\mu+\rho,(a_1,x),a_2,\cdots,a_p;\\ (1-\lambda+\rho),b_1,\cdots,b_q;\end{matrix}\ \frac{w}{\delta}\right]. \tag{31}
$$

Here we assume that both members of the Whittaker transform transform exist.

*Proof.*For simplicity and convenience, let $\mathscr{L}$ be the left-hand side of (30). Setting $\delta t=v$ in $\mathscr{L}$, we have

$$
\mathscr{L}=\int_0^\infty\left(\frac{v}{\delta}\right)^{\rho-1}e^{-v/2}W_{\lambda,\mu}(v)\times \sum_{n=0}^\infty\frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{(wv)^n}{\delta^n\delta n!}dv.
$$

Changing the order of integration and summation, we get

$$
\mathscr{L}=\delta^{-\rho}\sum_{n=0}^\infty\frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{(w)^n}{\delta^n n!}\times \int_0^\infty v^{\rho+n-1}e^{-v/2}W_{\lambda,\mu}(v)dv. \tag{32}
$$

Now, if we use the integral formula involving the Whittaker function (see, *e.g.*, [7, p. 56, Eq.(2.41)]; see also [6, p. 79]):

$$
\int_0^\infty t^{\nu-1}e^{-t/2}W_{\lambda,\mu}(t)dt = \frac{\Gamma(\frac{1}{2}+\mu+\nu)\Gamma(\frac{1}{2}-\mu+\nu)}{\Gamma(1-\lambda+\nu)} \quad \left(\Re(\nu\pm\mu)>-\frac{1}{2}\right), \tag{33}
$$

we obtain

$$
\mathscr{L}=\delta^{-\rho}\sum_{n=0}^\infty\frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\times \frac{(w)^n}{\delta^n n!}\frac{\Gamma(\frac{1}{2}+\mu+\rho+n)\Gamma(\frac{1}{2}-\mu+\rho+n)}{\Gamma(1-\lambda+\rho+n)}, \tag{34}
$$

which, in view of (5), yields our desired result (30).

It is easy to see that a similar argument as in the proof of (30) will establish the result (31). This completes the proof of Theorem 4.

Theorem 5. *Suppose that* $x\geq 0$, $\Re(u)>0$, $\rho,\nu,w\in\mathbb{C}$ and $p,q\in\mathbb{N}_0$. *Then the following integral formulas hold*:

$$
\int_0^\infty t^{\rho-1}K_\nu(ut)\,{}_p\gamma_q[wt^2]dt=\frac{1}{4}\left(\frac{2}{u}\right)^\rho\Gamma(\frac{\rho\pm\nu}{2}) \times {}_{p+2}\gamma_q\left[\begin{matrix}\frac{\rho+\nu}{2},\frac{\rho-\nu}{2},(a_1,x),a_2,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\ \frac{4w}{u^2}\right] \tag{35}
$$

and

$$
\int_0^\infty t^{\rho-1}K_\nu(ut)\,{}_p\Gamma_q[wt^2]dt=\frac{1}{4}\left(\frac{2}{u}\right)^\rho\Gamma(\frac{\rho\pm\nu}{2}) \times {}_{p+2}\Gamma_q\left[\begin{matrix}\frac{\rho+\nu}{2},\frac{\rho-\nu}{2},(a_1,x),a_2,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}\ \frac{4w}{u^2}\right]. \tag{36}
$$

Here we assume that both members of the K-transform exist.

*Proof.*By applying (5) to the integrand of (35) and then interchanging the order of integral sign and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

$$
\int_0^\infty t^{\rho-1}K_\nu(ut)\,{}_p\gamma_q[wt^2]dt = \sum_{n=0}^\infty\frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{(w)^n}{n!}\times \int_0^\infty t^{\rho+2n-1}K_\nu(ut)\,dt. \tag{37}
$$

Now, if we use the integral formula involving the K-function (see, *e.g.*, [7, p. 54, Eq.(2.37)]; see also [6, p. 78]):

$$
\int_0^\infty t^{\rho-1}K_\nu(at)dt=2^{\rho-2}a^{-\rho}\Gamma\left(\frac{\rho\pm\nu}{2}\right) \quad (\Re(a)>0;\ \Re(\rho)>|\Re(\nu)|), \tag{38}
$$

we find

$$\int_0^\infty t^{\rho-1}K_\nu(ut)\,{}_p\gamma_q[wt^2]dt = 2^{\rho-2}(u)^{-\rho}$$
$$\times\sum_{n=0}^\infty \frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{(4w)^n}{u^{2n}n!}$$
$$\times\Gamma\left(\frac{\rho+\nu}{2}+n\right)\Gamma\left(\frac{\rho-\nu}{2}+n\right), \quad (39)$$

which, on using (5), yields our desired result (35).

It is easy to see that a similar argument as in the proof of (35) will establish the result (36). This completes the proof of Theorem 5.

Theorem 6. *Suppose* $x\geq 0$, $\Re(u)>0$, $\rho,\nu,w\in\mathbb{C}$ *and* $p,q\in\mathbb{N}_0$. *Then the following integral formulas hold*:

$$\int_0^\infty t^{\rho-1}J_\nu(ut)\,{}_p\gamma_q[wt^2]dt = \frac{1}{2}\left(\frac{2}{u}\right)^\rho \frac{\Gamma(\frac{\rho+\nu}{2})}{\Gamma(1+\frac{\nu-\rho}{2})}$$
$$\times\,{}_{p+2}\gamma_q\left[\begin{matrix}\frac{\rho+\nu}{2},\frac{\nu-\rho}{2},(a_1,x),a_2,\cdots,a_p; \\ b_1,\cdots,b_q;\end{matrix} -\frac{4w}{u^2}\right] \quad (40)$$

and

$$\int_0^\infty t^{\rho-1}J_\nu(ut)\,{}_p\Gamma_q[wt^2]dt = \frac{1}{2}\left(\frac{2}{u}\right)^\rho \frac{\Gamma(\frac{\rho+\nu}{2})}{\Gamma(1+\frac{\nu-\rho}{2})}$$
$$\times\,{}_{p+2}\Gamma_q\left[\begin{matrix}\frac{\rho+\nu}{2},\frac{\nu-\rho}{2},(a_1,x),a_2,\cdots,a_p; \\ b_1,\cdots,b_q;\end{matrix} -\frac{4w}{u^2}\right], \quad (41)$$

where $(-\Re(\nu)<\Re(\rho)<\frac{3}{2})$. *Here we assume that both members of the Hankel transform exist.*

Proof. By applying (5) to the integrand of (40) and then interchanging the order of integral sign and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

$$\int_0^\infty t^{\rho-1}J_\nu(ut)\,{}_p\gamma_q[wt^2]dt$$
$$=\sum_{n=0}^\infty \frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{(w)^n}{n!}\times$$
$$\int_0^\infty t^{\rho+2n-1}J_\nu(ut)\,dt. \quad (42)$$

Now, if we use the integral formula involving the K-function (see, e.g., [7, p. 57, Eq. (2.46)]):

$$\int_0^\infty t^{\rho-1}J_\nu(at)dt = 2^{\rho-1}a^{-\rho}\frac{\Gamma(\frac{\rho+\nu}{2})}{\Gamma(1+\frac{\nu-\rho}{2})} \quad (43)$$
$$\left(\Re(a)>0,-\Re(\nu)<\Re(\rho)<\frac{3}{2}\right),$$

we obtain

$$\int_0^\infty t^{\rho-1}J_\nu(ut)\,{}_p\gamma_q[wt^2]dt = 2^{\rho-1}(u)^{-\rho}\times$$
$$\sum_{n=0}^\infty \frac{(a_1;x)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\cdot\frac{(4w)^n}{u^{2n}n!}\,\frac{\Gamma(\frac{\rho+\nu}{2}+n)}{\Gamma(\frac{2+\rho+\nu}{2}-n)}, \quad (44)$$

which, on using the known identity (see, *e.g.*, [15, p. 5, Eq. (23)])

$$(\lambda)_{-n} = \frac{(-1)^n}{(1-\lambda)_n},$$

in view of (5), yields our desired result (40).

It is easy to see that a similar argument as in the proof of (40) will establish the result (41). This completes the proof of Theorem 6.

3 Special Cases and Concluding Remarks

We consider some further consequences of main results derived in the preceding sections. To do this, we recall the following two standard formulae [4, p. 79, Eqs. (14–15)]:

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}}\cos z \quad\text{and}\quad J_{1/2}(z) = \sqrt{\frac{2}{\pi z}}\sin z. \quad (45)$$

Setting $\nu=\pm\frac{1}{2}$ for the parameter values of the Bessel function of first kind in (40) and (41), we obtain two further pairs of integral formulae as follows:

Case 1. $\nu=\frac{1}{2}$; $x\geq 0$, $\Re(u)>0$, $\rho,u,w\in\mathbb{C}$ and $p,q\in\mathbb{N}_0$.

$$\int_0^\infty t^{\rho-1}\sin(ut)\,{}_p\gamma_q[wt^2]dt = \sqrt{\pi}\frac{(2)^{\rho-\frac{3}{2}}}{u^{\rho-\frac{1}{2}}}\frac{\Gamma(\frac{2\rho+1}{4})}{\Gamma(1+\frac{1-2\rho}{4})}$$
$$\times\,{}_{p+2}\gamma_q\left[\begin{matrix}\frac{2\rho+1}{4},\frac{1-2\rho}{4},(a_1,x),a_2,\cdots,a_p; \\ b_1,\cdots,b_q;\end{matrix} -\frac{4w}{u^2}\right] \quad (46)$$

and

$$\int_0^\infty t^{\rho-1}\sin(ut)\,{}_p\Gamma_q[wt^2]dt = \sqrt{\pi}\frac{(2)^{\rho-\frac{3}{2}}}{u^{\rho-\frac{1}{2}}}\frac{\Gamma(\frac{2\rho+1}{4})}{\Gamma(1+\frac{1-2\rho}{4})}$$
$$\times\,{}_{p+2}\Gamma_q\left[\begin{matrix}\frac{2\rho+1}{4},\frac{1-2\rho}{4},(a_1,x),a_2,\cdots,a_p; \\ b_1,\cdots,b_q;\end{matrix} -\frac{4w}{u^2}\right]. \quad (47)$$

Case 2. $\nu=-\frac{1}{2}$; $x\geq 0$, $\Re(u)>0$, $\rho,u,w\in\mathbb{C}$ and $p,q\in\mathbb{N}_0$.

$$\int_0^\infty t^{\rho-1}\cos(ut)\,{}_p\gamma_q[wt^2]dt = \sqrt{\pi}\frac{(2)^{\rho-\frac{3}{2}}}{u^{\rho-\frac{1}{2}}}\frac{\Gamma(\frac{2\rho+1}{4})}{\Gamma(1+\frac{1-2\rho}{4})}$$
$$\times\,{}_{p+2}\gamma_q\left[\begin{matrix}\frac{2\rho+1}{4},\frac{1-2\rho}{4},(a_1,x),a_2,\cdots,a_p; \\ b_1,\cdots,b_q;\end{matrix} -\frac{4w}{u^2}\right] \quad (48)$$

and

$$\int_0^\infty t^{\rho-1}\cos(ut)\,{}_p\Gamma_q[wt^2]dt = \sqrt{\pi}\frac{(2)^{\rho-\frac{3}{2}}}{u^{\rho-\frac{1}{2}}}\frac{\Gamma(\frac{2\rho+1}{4})}{\Gamma(1+\frac{1-2\rho}{4})}$$
$$\times\,{}_{p+2}\Gamma_q\left[\begin{matrix}\frac{2\rho+1}{4},\frac{1-2\rho}{4},(a_1,x),a_2,\cdots,a_p; \\ b_1,\cdots,b_q;\end{matrix} -\frac{4w}{u^2}\right]. \quad (49)$$

We, now, briefly consider some further consequences of the results derived in this section. Following Srivastava *et al.* [14], when $x = 0$, both ${}_p\gamma_q$ $(p,q \in \mathbb{N})$ and ${}_p\Gamma_q$ $(p,q \in \mathbb{N})$ would reduce immediately to the extensively investigated generalized hypergeometric function ${}_pF_q$ $(p,q \in \mathbb{N})$. Furthermore, as an immediate consequence of the definition (5) and (6), we have the following decomposition formula:

$$
{}_p\gamma_q\left[\begin{matrix}(a_1,x),a_2,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}z\right] + {}_p\Gamma_q\left[\begin{matrix}(a_1,x),a_2,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}z\right] = {}_pF_q\left[\begin{matrix}a_1,\cdots,a_p;\\ b_1,\cdots,b_q;\end{matrix}z\right], \tag{50}
$$

in terms of the generalized hypergeometric function ${}_pF_q$ $(p,q \in \mathbb{N})$. Therefore, if we set $x = 0$ or make use of the result (50), Theorems 1 to 6 yield the various integral transforms involving the generalized hypergeometric function ${}_pF_q$.

The generalized incomplete hypergeometric type functions defined by (5) and (6) possess the advantage that a number of incomplete gamma functions and hypergeometric function happen to be the particular cases of these functions. Further, applications of these functions in communication theory, probability theory and groundwater pumping modeling are shown by Srivastava *et al.* [14]. Therefore, we conclude this paper by noting that, the results deduced above are significant and can lead to yield numerous other integral transforms involving various special functions by suitable specializations of arbitrary parameters in the theorems. More importantly, they are expected to find some applications in probability theory and to the solutions of integral equations.

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