Digital Cohomology Operations

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Abstract: In this paper, our aim is to study the digital version of Steenrod Algebra. For this purpose, we define the digital cohomology operations and deal with main properties of digital Steenrod squares. Moreover, some related results are given for digital images. We finally explain the theory with nice examples.

Keywords: Digital cohomology group, digital cohomology operation, digital Steenrod square.

1 Introduction

Digital topology is a growing area in the mathematics and computer vision with nice applications. Since many concepts in Algebraic Topology are useful, researchers benefit from these to get important results in this field.

Cohomology operations are algebraic invariants for a space. These operations are beneficial because the cohomology and the cup product sometimes fail to seperate two spaces. We can say that if two spaces have isomorphic cohomology groups but the behaviour of the ring structure or cohomology operations is different, then they are not homeomorphic. It is known that the cohomology operations of a space are more useful invariants than its homology. Researchers firstly construct homology for their works and then since this structure is not enough, they develop the cohomology. Cohomology operations could not be directly produced from the algorithms previously mentioned for computing the homology. Finding an effective method for the construction and computation of the cohomology operations is always desirable in algebraic topology.

Steenrod squares are important classes of cohomology operations in Algebraic Topology. Steenrod Squares can be explanatory on the structure of a topological space. There are several methods for constructing Steenrod squares. It can be developed a method with using cohomology operations.

Up to now, many researchers have studied on cohomology operations. Real [25] gives the formulae to obtain an algorithm for calculating Steenrod squares.

Gonzalez-Diaz and Real [13] present a combinatorial method for computing cup-i products and Steenrod squares of a simplicial set X. Their aim is to obtain a method which gives an explicit formula for the component morphisms of a higher diagonal approximation via face operators of X. They give a generalization of the method to Steenrod reduced powers.

In [14], since there is no extensive knowledge about the algorithmic structure of cohomology operations, it is established an algorithm for computing homology which allows us the computation of the cup product and the effective evaluation of the primary and secondary cohomology operations on the cohomology of a finite simplicial complex. It is also given a program in Mathematica for cohomology computations.

Gonzalez-Diaz and Real [15] develop a software to obtain simplicial formulation. It provides a way to design an efficient algorithm for computing any Steenrod cohomology operation on any cohomology class of any degree.

Gonzalez-Diaz and Real [16] give a method to compute cohomology operations on finite simplicial complexes. They also deal with a procedure for calculating primary and secondary cohomology operations. It is given a solution to the problem of computing Steenrod squares, reduced pth powers and Adem secondary cohomology operations.

In [17], it is determined combinatorial descriptions of Steenrod kth powers in terms of face operators and developed some techniques to obtain a formula for

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cohomology operations. Gonzalez-Diaz and Real [18] use formulas which are obtained by them to compute Adem cohomology operations. They also improve an algorithm for this process.

Ege et al. [12] deal with relative and reduced homology groups of digital images. Ege and Karaca [11] propose a mathematical construction that can be used for defining the simplicial cohomology theory of digital images. They show that the Kunneth formula for cohomology doesn't hold in digital images. It is also defined the simplicial cup product and proved its some properties for digital images. Karaca and Burak [22] study the relative cohomology groups of digital images. They give a method to compute the cohomology ring of digital images and some examples about cohomology ring.

Here is a summary of the present paper. In Section 2, we introduce the necessary backgrounds on digital topology and digital cohomology theory. Section 3 is dedicated to digital cohomology operations, digital Steenrod squares and other related results. Finally, we give a conclusion about these topics.

2 Preliminaries

Let X be a subset of $\mathbb{Z}^n$ for a positive integer n where $\mathbb{Z}^n$ is the set of lattice points in the n-dimensional Euclidean space and κ be represent a specific adjacency relation for the members of X. A digital image consists of (X, κ).

Definition 2.1. [5]. Let l, n be positive integers, $1 \le l \le n$ and two distinct points

$$p = (p_1, p_2, \ldots, p_n), \ q = (q_1, q_2, \ldots, q_n) \in \mathbb{Z}^n$$

p and q are k_l-adjacent if there are at most l indices i such that $|p_i - q_i| = 1$ and for all other indices j such that

$$|p_j - q_j| \neq 1 \quad \text{and} \quad p_j = q_j.$$

The following statements can be obtained from Definition 2.1:
• Two points p and q in $\mathbb{Z}$ are 2-adjacent if $|p - q| = 1$.
• Two points p and q in $\mathbb{Z}^2$ are 8-adjacent if they are distinct and differ by at most 1 in each coordinate.
• Two points p and q in $\mathbb{Z}^2$ are 4-adjacent if they are 8-adjacent and differ in exactly one coordinate.
• Two points p and q in $\mathbb{Z}^3$ are 26-adjacent if they are distinct and differ by at most 1 in each coordinate.
• Two points p and q in $\mathbb{Z}^3$ are 18-adjacent if they are 26-adjacent and differ at most two coordinates.
• Two points p and q in $\mathbb{Z}^3$ are 6-adjacent if they are 18-adjacent and differ in exactly one coordinate.

A κ-*neighbor* [5] of $p \in \mathbb{Z}^n$ is a point of $\mathbb{Z}^n$ that is κ-adjacent to p where $\kappa \in \{2, 4, 8, 6, 18, 26\}$. The set

$$N_\kappa(p) = \{q \mid q \text{ is } \kappa - \text{adjacent to } p\}$$

is called the κ-*neighborhood* of p. A *digital interval* [4] is defined by

$$[a, b]_{\mathbb{Z}} = \{z \in \mathbb{Z} \mid a \le z \le b\}$$

where $a, b \in \mathbb{Z}$ and $a < b$.

A digital image $X \subset \mathbb{Z}^n$ is κ-*connected* [21] if and only if for every pair of different points $x, y \in X$, there is a set $\{x_0, x_1, \ldots, x_r\}$ of points of a digital image X such that $x = x_0$, $y = x_r$ and x_i and x_{i+1} are κ-neighbors where $i = 0, 1, \ldots, r - 1$.

Definition 2.2. [5]. Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$, $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images and $f : X \longrightarrow Y$ be a function.
• If for every κ_0-connected subset U of X, $f(U)$ is a κ_1-connected subset of Y, then f is said to be (κ_0, κ_1)-continuous.
• f is (κ_0, κ_1)-continuous $\Leftrightarrow$ for every κ_0-adjacent points $\{x_0, x_1\}$ of X, either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are a κ_1-adjacent in Y.

A $(2, \kappa)$-continuous function $f : [0, m]_{\mathbb{Z}} \longrightarrow X$ such that $f(0) = x$ and $f(m) = y$ is called a digital κ-path [5] from x to y in a digital image X. In a digital image (X, κ), for every two points, if there is a κ-path, then X is called κ-path connected. A simple closed κ-curve of $m \ge 4$ points [9] in a digital image X is a sequence $\{f(0), f(1), \ldots, f(m-1)\}$ of images of the κ-path $f : [0, m-1]_{\mathbb{Z}} \longrightarrow X$ such that $f(i)$ and $f(j)$ are κ-adjacent if and only if $j = i \pm \mod m$.

Let $(X, \kappa_0) \subset \mathbb{Z}^{n_0}$, $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images and $f : X \longrightarrow Y$ be a function. If f is (κ_0, κ_1)-continuous, bijective and f^{-1} is (κ_1, κ_0)-continuous, then f is called (κ_0, κ_1)-*isomorphism* [8] and denoted by $X \cong_{(\kappa_0, \kappa_1)} Y$.

A point $x \in X$ is called a κ-corner [3] if x is κ-adjacent to two and only two points $y, z \in X$ such that y and z are κ-adjacent to each other. If y, z are not κ-corners and if x is the only point κ-adjacent to both y, z, then we say that the κ-corner x is simple [2]. X is called a generalized simple closed κ-curve [23] if what is obtained by removing all simple κ-corners of X is a simple closed κ-curve. For a κ-connected digital image (X, κ) in $\mathbb{Z}^n$, there is a following statement:

$$|X|^x = N_{3^n - 1}(x) \cap X.$$

$$\kappa \in \left\{ 2n \ (n \ge 1), 3^n - 1 \ (n \ge 2), 3^n - \sum_{t=0}^{r-2} C_t^n 2^{n-t} - 1 \right.$$
$$\left. (2 \le r \le n - 1, \ n \ge 3) \right\}, \tag{1}$$

where $C_t^n = \frac{n!}{(n-t)! t!}$.

Definition 2.3. [19]. Let (X, κ) be a digital image in $\mathbb{Z}^n$, $n \ge 3$ and $\overline{X} = \mathbb{Z}^n - X$. Then X is called a *closed κ-surface* if it satisfies the following.
1. In case that $(\kappa, \overline{\kappa}) \in \{(\kappa, 2n), (2n, 3^n - 1)\}$, where the κ-adjacency is taken from (1) with $\kappa \neq 3^n - 2^n - 1$ and $\overline{\kappa}$ is the adjacency on $\overline{X}$, then
• for each point $x \in X$, $|X|^x$ has exactly one κ-component κ-adjacent to x;

• $|\overline{X}|^x$ has exactly two $\overline{\kappa}$-components $\overline{\kappa}$-adjacent to x; we denote by C^{xx} and D^{xx} these two components; and

• for any point $y \in N_\kappa(x) \cap X$, $N_{\overline{\kappa}}(y) \cap C^{xx} \neq \emptyset$ and $N_{\overline{\kappa}}(y) \cap D^{xx} \neq \emptyset$, where $N_\kappa(x)$ means the κ-neighbors of x. Further, if a closed κ-surface X does not have a simple κ-point, then X is called simple.

2. In case that $(\kappa, \overline{\kappa}) = (3^n - 2^n - 1, 2n)$, then

• X is κ-connected,

• for each point $x \in X$, $|X|^x$ is a generalized simple closed κ-curve.

Moreover, if the image $|X|^x$ is a simple closed κ-curve, then the closed κ-surface X is called simple.

Example 2.4.[19] $MSS_{18}' = \{c_0 = (1,1,0), c_1 = (0,2,0), c_2 = (-1,1,0), c_3 = (0,0,0), c_4 = (0,1,-1), c_5 = (0,1,1)\} \subset \mathbb{Z}^3$ is a minimal simple closed 18-surface (see figure 1).
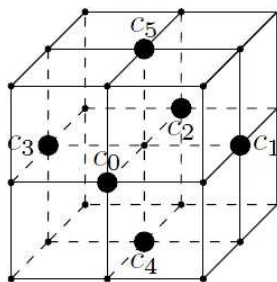

Fig. 1: MSS_{18}'

Definition 2.5. [5]. Let $(X, \kappa_1) \subset \mathbb{Z}^{n_1}$ and $(Y, \kappa_2) \subset \mathbb{Z}^{n_2}$ be digital images and $f, g : X \longrightarrow Y$ be two (κ_1, κ_2)-continuous functions. f and g are called digitally (κ_1, κ_2)-homotopic in Y if there is a positive integer m and a function $H : X \times [0, m]_\mathbb{Z} \longrightarrow Y$ such that

• for all $x \in X$, $H(x, 0) = f(x)$ and $H(x, m) = g(x)$,

• for all $x \in X$, $H_x : [0, m]_\mathbb{Z} \longrightarrow Y$ defined by

$$H_x(t) = H(x, t)$$

for all $t \in [0, m]_\mathbb{Z}$ is $(2, \kappa_2)$-continuous,

• for all $t \in [0, m]_\mathbb{Z}$, $H_t : X \longrightarrow Y$ defined by $H_t(x) = H(x, t)$ for all $x \in X$ is (κ_1, κ_2)-continuous.

The function H is called a digital (κ_1, κ_2)-homotopy between f and g.

We notice that a digital image (X, κ) is said to be κ-*contractible* [4] if its identity map is (κ, κ)-homotopic to a constant function $\overline{c}$ for some $c \in X$ which is defined by $\overline{c}(x) = c$ for all $x \in X$.

Let (X, κ) be a digital image and its subset be (A, κ). (X, A) is called a digital image pair with κ-adjacency and when A is a singleton set $\{x_0\}$, then (X, x_0) is called a pointed digital image.

Definition 2.6. [26]. Let S be a set of nonempty subset of digital image (X, κ). Let the following statements hold:

• If p and q are distinct points of $s \in S$, then p and q are κ-adjacent,

• If $s \in S$ and $\emptyset \neq t \subset s$, then $t \in S$.

Then the members of S are called simplexes of (X, κ).

An m-simplex is a simplex S such that $|S| = m + 1$. Let P be a digital m-simplex. If P' is a nonempty proper subset of P, then P' is called a face of P. We write $Vert(P)$ to denote the vertex set of P. A digital subcomplex A of a digital simplicial complex X with κ-adjacency is a digital simplicial complex contained in X with $Vert(A) \subset Vert(X)$.

Definition 2.7. [1]. Let (X, κ) be a finite collection of digital m-simplices, $0 \leq m \leq d$ for some non-negative integer d. If the followings hold, then (X, κ) is called a finite digital simplicial complex:

• If P belongs to X, then every face of P also belongs to X.

• If $P, Q \in X$, then $P \cap Q$ is either empty or a common face of P and Q.

Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital simplicial complex. If there is an ordering on the vertex set of (X, κ), then it is called *oriented simplicial complex* [1].

Definition 2.8. [1]. Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital oriented simplicial complex with m-dimension. $C_q^\kappa(X)$ is a free abelian group with basis all digital (κ, q)-simplices in X. A homomorphism

$$\partial_q : C_q^\kappa(X) \longrightarrow C_{q-1}^\kappa(X)$$

called the boundary operator. If $\sigma = [v_0, \ldots, v_q]$ is an oriented simplex with $0 < q \leq m$, ∂_q is defined by

$$\partial_q \sigma = \partial_q [v_0, \ldots, v_q] = \sum_{i=0}^{q} (-1)^i [v_0, \ldots, \hat{v}_i, \ldots, v_q]$$

where $\hat{v}_i$ means the vertex v_i is to be deleted from the array. We remark that for $q < 0, m < q$, since $C_q^\kappa(X)$ is the trivial group, the operator ∂_q is the trivial homomorphism for $q \leq 0, m < q$.

We notice that $\partial_{q-1} \circ \partial_q = 0$ [1] for $q \geq 0$.

Definition 2.9. [1]. Let $(X, \kappa) \subset \mathbb{Z}^n$ be a digital oriented simplicial complex with m-dimension.

• $Z_q^\kappa(X) = \text{Ker } \partial_q$ is called the group of digital simplicial q-cycles.

• $B_q^\kappa(X) = \text{Im } \partial_{q+1}$ is called the group of digital simplicial q-boundaries.

• $H_q^\kappa(X) = Z_q^\kappa(X) / B_q^\kappa(X)$ is called the qth digital simplicial homology group.

Let $\varphi : (X, \kappa_1) \longrightarrow (Y, \kappa_2)$ be a function between digital images. If for every digital (κ_1, m)-simplex P determined by the adjacency relation κ_1 in X, $\varphi(P)$ is a (κ_1, n)-simplex in Y for some $n \leq m$, then φ is called *a digital simplicial map* [10].

Definition 2.10. [11]. Let (X, κ) be a digital simplicial complex and R be an abelian group. The digital simplicial cochain complex $(\mathscr{C}(X), \delta)$ is defined as follows. For any

$q \in \mathbb{Z}$, the q-dimensional digital cochain group with coefficients in R is the group

$$C^{q,\kappa}(X,R) = Hom(C_q^{\kappa}(X),R).$$

If $R = \mathbb{Z}$, then R is omitted from the notation. Elements of $C^{q,\kappa}(X)$ are called digital cochains and denoted by c^q. The value of a digital cochain c^q on a chain d_q is denoted by $< c^q, d_q >$. The q-th coboundary map $\delta^q : C^{q,\kappa}(X) \longrightarrow C^{q+1,\kappa}(X)$ is the dual of the boundary operator ∂_{q+1} defined by

$$< \delta^q c^q, d_{q+1} > = < c^q, \partial_{q+1} d_{q+1} > .$$

We have the following statements from [11]:
- $Z^{q,\kappa}(X) = \operatorname{Ker} \delta^q$ is the group of digital simplicial q-cocycles.
- $B^{q,\kappa}(X) = \operatorname{Im} \delta^{q-1}$ is the group of digital simplicial q-coboundaries.
- $H^{q,\kappa}(X) = Z^{q,\kappa}(X)/B^{q,\kappa}(X)$ is the qth digital simplicial cohomology group.

Theorem 2.11. [11]. If (X,κ) is a single vertex, then

$$H^{q,\kappa}(X) = \begin{cases} \mathbb{Z}, & q = 0 \\ 0, & q \neq 0. \end{cases}$$

Ege and Karaca [11] give the definition of digital simplicial cup product for digital images. They define the digital simplicial cup product

$$\smile : C^{p,\kappa}(X,R) \times C^{q,\kappa}(X,R) \longrightarrow C^{p+q,\kappa}(X,R)$$

of cochains c^p and c^q by the formula

$$< c^p \smile c^q, [v_0,\ldots,v_{p+q}] > = < c^p, [v_0,\ldots,v_p] > .$$
$$< c^q, [v_p,\ldots,v_{p+q}] >$$

where $v_0 < \ldots < v_{p+q}$ in the given ordering and . is the product in R. They also show that the digital simplicial cup product is bilinear. It is shown that there exists the following equality:

$$\delta(c^p \smile c^q) = \delta c^p \smile c^q + (-1)^p c^p \smile \delta c^q.$$

It is obtained the cup product on digital simplicial cochains is associative. If $c^p \in H^{p,\kappa}(X,G_1)$ and $c^q \in H^{q,\kappa}(X,G_2)$ are digital cocycles, then $c^p \smile c^q = (-1)^{pq} c^q \smile c^p$. Let $(X,\kappa_1) \subset \mathbb{Z}^{n_1}$ and $(Y,\kappa_2) \subset \mathbb{Z}^{n_2}$ be digital images. If $f : (X,\kappa_1) \longrightarrow (Y,\kappa_2)$ is a digitally continuous map and $c^p \in H^{p,\kappa}(X,G_1)$ and $c^q \in H^{q,\kappa}(X,G_2)$ are digital cocycles, then $f^*(c^p \smile c^q) = f^*(c^p) \smile f^*(c^q)$. They conclude that the digital simplicial cohomology ring of X is the graded abelian group $H^{*,\kappa}(X)$ with the graded multiplication given by the digital simplicial cup product.

3 Digital cohomology operations

In this section, we define a digital cohomology operation, digital Steenrod square and give some theorems with examples.

Definition 3.1. Let (X,κ) and (Y,κ') be digital images. A digital cohomology operation of type $(n,A;q,B)$ is a transformation

$$\theta : H^{n,\kappa}(X,A) \longrightarrow H^{q,\kappa}(X,B)$$

defined for all digital images (X,κ), with fixed positive integers n,m and abelian groups G_1, G_2 and satisfying the property $f^*\theta = \theta f^*$; that is, the following diagram

$$
\begin{array}{ccc}
H^{n,\kappa'}(Y;G_1) & \xrightarrow{\quad\theta\quad} & H^{q,\kappa'}(Y;G_2) \\
\downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
H^{n,\kappa}(X;G_1) & \xrightarrow{\quad\theta\quad} & H^{q,\kappa}(X;G_2)
\end{array}
$$

is commutative for all digital simplicial maps $f : X \longrightarrow Y$.

Let us give some important examples on digital cohomology operations.

Example 3.2. Let (A,κ) and (B,κ') be digital images. If $f : A \longrightarrow B$ is a digital homomorphism, then f induces digital homomorphisms

$$f^* : H^{n,\kappa}(X,A) \longrightarrow H^{n,\kappa}(X,B)$$

for all n. Thus f defines a digital cohomology operation of type $(n,A;n,B)$ for any n.

Example 3.3. Let R be a ring. If $\theta^n : H^{n,\kappa}(X,R) \longrightarrow H^{2n,\kappa}(X,R)$ is a digital map defined by

$$\theta^n(x) = x \smile x,$$

where $\smile$ is the digital simplicial cup product, then θ_n is a natural transformation since

$$f^*(x \smile x) = f^*(x) \smile f^*(x).$$

As a result, θ^n is a digital cohomology operation of type $(n,R;2n,R)$ for any n.

We remark that θ is not always a digital homomorphism, since $\theta(x+y) \neq \theta(x) + \theta(y)$ in general. So we can give the following.

Corollary 3.4. A digital cohomology operation does not require it to be a digital homomorphism.

We deal with a digital Steenrod square and its properties as a special digital cohomology operation.

Definition 3.5. The ith digital Steenrod square, $i \geq 0$, is a digital cohomology operation

$$Sq^i : H^{n,\kappa}(X;\mathbb{Z}_2) \longrightarrow H^{n+i,\kappa}(X;\mathbb{Z}_2)$$

which is a digital homomorphism satisfying following axioms:

Axiom-1: $Sq^0 = 1$.
Axiom-2: If $deg\, x = i$, then $Sq^i(x) = x^2$.
Axiom-3: If $i > deg\, x$, then $Sq^i(x) = 0$.
Axiom-4: Sq^1 is a Bockstein homomorphism for the coefficient sequence

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0.$$

Axiom-5: (Adem relations) If $0 < a < 2b$, then

$$Sq^a Sq^b = \sum_{j=0}^{[a/2]} \binom{b-1-j}{a-2j} Sq^{a+b-j} Sq^j.$$

Theorem 3.6. (Cartan formula) For $x, y \in H^{*, \kappa}(X, \mathbb{Z}_2)$ and $x \times y \in H^{*, \kappa}(X, \mathbb{Z}_2)$, we have

$$Sq^n(x \times y) = \sum_{i=0}^{n} Sq^i(x) \times Sq^{n-i}(y).$$

Proof. The proof is the same as in Algebraic Topology.□
    We now give a formula which exists in Algebraic Topology.

Proposition 3.7. For $x, y \in H^{*, \kappa}(X, \mathbb{Z}_2)$ and $x \smile y \in H^{*, \kappa}(X, \mathbb{Z}_2)$, we have

$$Sq^n(x \smile y) = \sum_{i=0}^{n} Sq^i(x) \smile Sq^{n-i}(y).$$

The following example shows that the Proposition 3.7 doesn't hold in digital images.

Example 3.8. Let $X = [0,1]_{\mathbb{Z}} \times [0,1]_{\mathbb{Z}} \times [0,1]_{\mathbb{Z}}$ be the digital image with 6-adjacency (see figure 2).
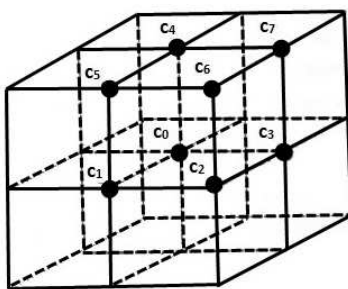

Fig. 2: X

From [11], we have

$$H^{n,6}(X; \mathbb{Z}_2) = \begin{cases} \mathbb{Z}_2, & n = 0 \\ \mathbb{Z}_2^5, & n = 1 \\ 0, & n \neq 0, 1. \end{cases}$$

Let $x, y \in H^{0,2}(X; \mathbb{Z}_2)$. Since

$$0 = Sq^2(x \smile y) = \sum_{i=0}^{2} Sq^i(x) \smile Sq^{n-i}(y)$$
$$= Sq^0(x) \smile Sq^2(y) + Sq^1(x) \smile Sq^1(y) +$$
$$Sq^2(x) \smile Sq^0(y)$$
$$= x \smile 0 + Sq^1(x) \smile Sq^1(y) + 0 \smile y$$
$$= Sq^1(x) \smile Sq^1(y) \in \mathbb{Z}_2^5,$$

we conclude that the Proposition 3.7 doesn't hold.

Proposition 3.9. Sq^i commutes with δ, that is, the following diagram is commutative:

$$\begin{array}{ccc} H^{n,\kappa}(X \times Y; \mathbb{Z}_2) & \xrightarrow{\delta} & H^{n+1,\kappa}(X \times Y; \mathbb{Z}_2) \\ {\scriptstyle Sq^i} \downarrow & & \downarrow {\scriptstyle Sq^i} \\ H^{n+i,\kappa}(X \times Y; \mathbb{Z}_2) & \xrightarrow{\delta} & H^{n+i+1,\kappa}(X \times Y; \mathbb{Z}_2) \end{array}$$

Proof. Let $x \times y \in H^{n,\kappa}(X; \mathbb{Z}_2)$. Since

$$Sq^i(\delta(x \times y)) = Sq^i(x \times \delta(y))$$
$$= Sq^i(x) \times Sq^0(\delta(y)) \quad (by\ Theorem\ 3.6)$$
$$= Sq^i(x) \times \delta(y) \quad (by\ Axiom\ 1)$$
$$= \delta(Sq^i(x) \times y)$$
$$= \delta(Sq^i(x \times y))$$
$$\quad (by\ Theorem\ 3.6, Axioms\ 1\ and\ 3),$$

we get the required result.□

Theorem 3.10. If $f : (Y, \kappa') \longrightarrow (X, \kappa)$ is a digital simplicial map, then $f^* Sq^i = Sq^i f^*$ for every i.
Proof. Let $\alpha \in H^{q,\kappa}(X)$ and consider the map $f^* : H^{q,\kappa}(X) \longrightarrow H^{q,\kappa'}(Y)$. Since

$$f^* Sq^i(\alpha) = f^*(\alpha \cup \alpha)$$
$$= f^*(\alpha) \cup f^*(\alpha)$$
$$= Sq^i(f^*(\alpha))$$
$$= Sq^i f^*(\alpha),$$

we obtain the required result.□

The following theorem exists in Algebraic Topology and thus we don't prove it but we give an example.

Theorem 3.11. Let k be a natural number and x be a generator with $deg(x) = 1$.
a) $Sq^i(x^k) = \binom{k}{i} x^{k+i}$.
b)

$$Sq^i(x^{2^k}) = \begin{cases} x^{2^k}, & i = 0 \\ x^{2^{k+1}}, & i = 2^k \\ 0, & i \neq 0, 2^k. \end{cases}$$

Example 3.12. Let

$$MSS_{18} = \{p_0 = (0,0,1), p_1 = (1,1,1), p_2 = (1,2,1),$$
$$p_3 = (0,3,1), p_4 = (-1,2,1), p_5 = (0,0,1),$$
$$p_6 = (0,0,1), p_7 = (0,0,1), p_8 = (0,0,1),$$
$$p_9 = (0,1,2)\}$$

with 18-adjacency. It is a minimal simple closed 18-surface [19] (see Figure 3).
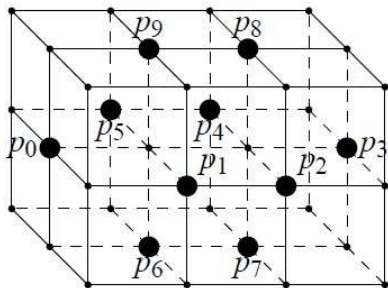

Fig. 3: MSS_{18}.

Karaca and Burak [22] compute the cohomology groups of MSS_{18} as follow:

$$H^{q,18}(MSS_{18}) = \begin{cases} \mathbb{Z}, & q = 0 \\ \mathbb{Z}^3, & q = 1 \\ 0, & q \geq 2. \end{cases}$$

Let $x \in H^{0,18}(MSS_{18})$ with $deg(x) = 1$. If we take $k = 1$ and $i = 2$, from the Theorem 3.11 a), it is easy to see that

$$Sq^2(x) = \binom{1}{2} x^{1+2} = 0.$$

In fact, we have $Sq^2(x) \in H^{2,18}(MSS_{18}) = 0$.

We recall that a cohomology operation θ is said to be stable operation if it commutes with the suspension isomorphism; notationally, $\sigma \circ \theta = \theta \circ \sigma$ where $\sigma : H^{n,\kappa}(X;\mathbb{Z}_2) \to H^{n+1,\kappa'}(SX;\mathbb{Z}_2)$.

Now let's define suspension for digital images.

Definition 3.13. A suspension SX of a digital image is defined by

$$SX = X \times [0,m]_{\mathbb{Z}}/X \times \{0,m\}$$

where $m > 0$ is an integer and $[0,m]_{\mathbb{Z}}$ is a digital interval.

Theorem 3.14. The digital Steenrod square operation is stable.

Proof. Let (X,κ) be a digital image and (SX,κ') is the suspension of X. Consider the following diagram.

$$\begin{array}{ccc} H^{n,\kappa}(X;\mathbb{Z}_2) & \xrightarrow{\sigma} & H^{n+1,\kappa'}(SX;\mathbb{Z}_2) \\ {\scriptstyle Sq^i} \downarrow & & \downarrow {\scriptstyle Sq^i} \\ H^{n+i,\kappa}(X;\mathbb{Z}_2) & \xrightarrow{\sigma} & H^{n+i+1,\kappa'}(SX;\mathbb{Z}_2) \end{array}$$

Since σ is a kind of coboundary operation, by the Proposition 3.9, we conclude that

$$\sigma \circ Sq^i(x) = Sq^i \circ \sigma(x)$$

for all $x \in H^{n,\kappa}(X;\mathbb{Z}_2)$. As a consequence, we get the required result. $\square$

In the Euclidean space, the suspension of the unit circle S^1 is the unit sphere S^2 but this is not valid in digital images. Boxer [7] defines sphere-like digital image as follows.

$$S_n = [-1,1]_{\mathbb{Z}}^{n+1} \setminus \{0_{n+1}\} \subset \mathbb{Z}^{n+1},$$

where 0_n denotes the origin of $\mathbb{Z}^n$.

Although there is a relation $\sum S^n \approx S^{n+1}$ where $\sum$ is a suspension and S^n is the n-sphere in Algebraic Topology, this relation doesn't hold in digital images. Let's show it by the following example.

Example 3.15. Consider the digital 1-sphere S_1 in $\mathbb{Z}^2$ which is defined by

$$S_1 = \{c_0 = (1,0), c_1 = (1,1), c_2 = (0,1), c_3 = (-1,1),$$
$$c_4 = (-1,0), c_5 = (-1,-1), c_6 = (0,-1), c_7 = (1,-1)\}$$
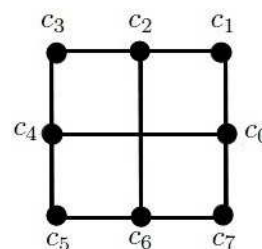
with 4-adjacency (see Figure 4).

Fig. 4: Digital 1-sphere S_1

The suspension of S_1 is defined by

$$SS_1 = S_1 \times [0,4]_{\mathbb{Z}}/S_1 \times \{0,4\}.$$

So we have the following digital image:

But the digital 2-sphere S_2 with 6-adjacency is given as follows:

As a result, we get the suspension the digital 1-sphere S_1 is not the digital 2-sphere S_2, i.e. $SS_1 \not\approx S_2$.
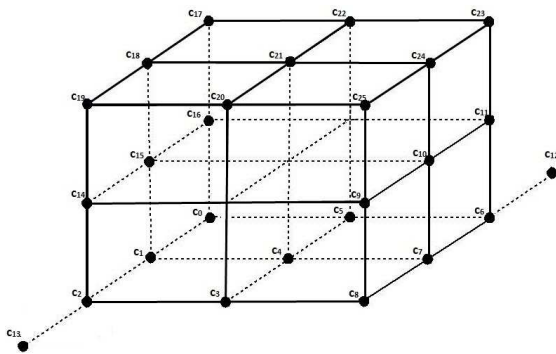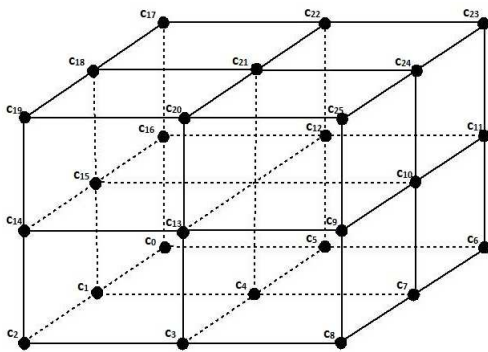
Fig. 5: SS_1

Fig. 6: Digital 2-sphere S_2

4 Conclusion

The aim of this paper is to give characteristic properties of digital Steenrod squares and digital cohomology operations. Since these operations are useful invariant for digital images, we first deal with digital cohomology groups. Then we define the digital Steenrod squares and give their axioms. We finally make some examples about digital Steenrod squares.

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