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Two-Step Methods for Variational Inequalities on Hadamard Manifolds

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Abstract: In this paper, we suggest and analyze a two-step method for solving the variational inequalities on Hadamard manifold using the auxiliary principle technique. The convergence of this new method requires only the partially relaxed strongly monotonicity, which is a weaker condition than monotonicity. Results can be viewed as refinement and improvement of previously known results.

Keywords: Variational inequalities, Hadamard manifold, Implicit methods, convergence
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1 Introduction

In recent years, much attention have been given to study the variational inequalities and related problems on Riemannian and Hadamard manifolds. This framework is a useful for the developments of various fields on nonlinear setting. Several ideas and techniques form the Euclidean space have been extended and generalized to this nonlinear framework. Hadamard manifolds are examples of hyperbolic spaces and geodesics, see [1,2,3,4,19,21,22] and the references therein. Nemeth [7], Tang et al [28] and Colao et al [2] have considered the variational inequalities and equilibrium problems on Hadamard manifolds. In particular, Colao et al. [2] and Tang et al [28] studied the existence of a solution solution of the equilibrium problems under some suitable conditions. To the best of our knowledge, no one has considered the auxiliary principle technique for solving the variational inequalities on Hadamard manifolds. In this paper, we use the auxiliary principle technique to suggest and analyze a two-step iterative method for solving the variational inequalities on Hadamard manifolds. We show that the convergence of this new method requires only the partially relaxed strongly monotonicity which is a weaker condition than monotonicity. Our results represent the refinement of previously known results for the variational inequalities. We hope that the technique and idea of this paper may stimulate further research in this area.

2 Preliminaries

We recall some fundamental and basic concepts need for a reading of this paper. These results and concepts can be found in the books on Riemannian geometry [2,3,25].

Let M be a simply connected m-dimensional manifold. Given $x \in M$, the tangent space of M at x is denoted by T_xM and the tangent bundle of M by $TM = \cup_{x\in M} T_xM$, which is a naturally manifold. A vector field A on M is a mapping of M into TM which associates to each point $x \in M$, a vector $A(x) \in T_xM$. We always assume that M can be endowed with a Riemannian metric to become a Riemannian manifold.. We denote by $\langle\cdot,\cdot\rangle$, the scalar product on T_xM with the associated norm $\|.\|_x$, where the subscript x will be omitted. Given a piecewise smooth curve $\gamma : [a,b] \longrightarrow M$ joining x to y (that is,$\gamma(a) = x$ and $\gamma(b) = y$,) by using the metric, we can define the length of γ as $L(\gamma) = \int_a^b \|\gamma'(t)\|dt$. Then, for any $x,y \in M$, the Riemannian distance $d(x,y)$, which includes the original topology on M, is defined by minimizing this length over the set of all such curves joining x to y.

Let Δ be the Levi-Civita connection with $(M,\langle.,.\rangle)$. Let γ be a smooth curve in M. A vector field A is said to be parallel along γ, if $\Delta_{\gamma'}A = 0$. If γ' itself is parallel along γ, then we say that γ is a geodesic and in this case $\|\gamma'\|$ is constant. When $\|\gamma'\| = 1$, γ is said to be normalized. A geodesic joining x to y in M is said to be minimal, if its length equals $d(x,y)$.

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A Riemannian manifold is complete, if for any $x \in M$ all geodesics emanating from x are defined for all $t \in R$. By the Hopf-Rinow Theorem, we know that if M is complete, then any pair of points in M can be joined by a minimal geodesic. Moreover, (M,d) is a complete metric space and bounded closed subsets are compact.

Let M be complete. Then the exponential map $\exp_x : T_xM \longrightarrow M$ at x is defined by $\exp_x v = \gamma_v(1,x)$ for each $v \in T_xM$, where $\gamma(.) = \gamma_v(.,x)$ is the geodesic starting at x with velocity $v(i.e., \gamma(0) = x$ and $\gamma'(0) = v)$ Thus $\exp_x tv = \gamma_v(t,x)$ for each real number t.

A complete simply connected Riemannian manifold of nonpositive sectional curvature is called a *Hadamard manifold.* Throughout the remainder of this paper, we always assume that M is a Hadamard m- manifold.

We also recall the following well known results, which are essential for our work.

Lemma 2.1 [25]. Let $x \in M$. Then $\exp_x : T_xM \longrightarrow M$ is a diffeomorphism, and for any two points $x, y \in M$, there exists a unique normalized geodesic joining x to $y, \gamma_{x,y}$, which is minimal.

So from now on, when referring to the geodesic joining two points, we mean the unique minimal normalized one. Lemma 2.1 says that M is diffeomorphic to the Euclidean space R^m. Thus M has the same topology and differential structure as R^m. It is also known that Hadamard manifolds and euclidean spaces have similar geometrical properties. Recall that a geodesic triangle $\triangle(x_1,x_2,x_3)$ of a Riemannian manifold is a set consisting of three points x_1,x_2,x_3 and three minimal geodesics joining these points.

Lemma 2.2 [2,3,25]. (*Comparison theorem for triangles).* Let $\triangle(x_1,x_2,x_3)$ be a geodesic triangle. Denote, for each $i = 1,2,3(mod3)$, by $\gamma_i : [0,l_i] \longrightarrow M$, the geodesic joining x_i to x_{i+1}, and $\alpha_i; = L(\gamma_i'(0), -\gamma_l'(i-1)(li-1))$, the angle between the vectors $\gamma_i'(0)$ and $-\gamma_{i-1}'(l_{i-1})$, and $l_i; = L(\gamma_i)$. Then

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi \tag{1}$$

$$l_l^2 + l_{i+1}^2 - 2L_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2. \tag{2}$$

In terms of the distance and the exponential map, the inequality (2) can be rewritten as:

$$d^2(x_i,x_{i+1}) + d^2(x_{i+1},x_{i+2}) - 2\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2}\rangle \leq d^2(x_{i-1},x_i), \tag{3}$$

since

$$\langle \exp_{x_{i+1}}^{-1} x_i, \exp_{x_{i+1}}^{-1} x_{i+2}\rangle = d(x_i,x_{i+1})d(x_{i+1},x_{i+2})\cos \alpha_{i+1}.$$

Lemma 2.3 [25]. Let $\triangle(x,y,z)$ be a geodesic triangle in a Hadamard manifold M. Then, there exist $x',y',z' \in R^2$ such that

$$d(x,y) = \|x'-y'\|, \quad d(y,z) = \|y'-z'\|, \quad d(z,x) = \|z'-x'\|.$$

The triangle $\triangle(x',y',z')$ is called the comparison triangle of the geodesic triangle $\triangle(x,y,z)$, which is unique up to isometry of M.

From the law of cosines in inequality (3), we have the following inequality, which is a general characteristic of the spaces with nonpositive curvature [25]:

$$\langle \exp_x^{-1} y, \exp_x^{-1} z\rangle + \langle \exp_y^{-1} x, \exp_y^{-1} z\rangle \geq d^2(x,y). \tag{4}$$

From the properties of the exponential map, we have the following known result.

Lemma 2.4 [25]. Let $x_0 \in M$ and $\{x_n\} \subset M$ such that $x_n \longrightarrow x_0$. Then the following assertions hold.

(i). For any $y \in M$,

$$\exp_{x_n}^{-1} y \longrightarrow \exp_{x_o}^{-1} y \quad \text{and} \quad \exp_y^{-1} x_n \longrightarrow \exp_y^{-1} x_o.$$

(ii). If $\{v_n\}$ is a sequence such that $v_n \in T_{x_n}M$ and $v_n \longrightarrow v_0$, then $v_0 \in T_{x_0}M$.

(iii). Given the sequences $\{u_n\}$ and $\{v_n\}$ satisfying $u_n, v_n \in T_{x_n}M$. If $u_n \longrightarrow u_0$ and $v_n \longrightarrow v_0$ with $u_0, v_0 \in T_{x_0}M$, then

$$\langle u_n, v_n\rangle \longrightarrow \langle u_0, v_0\rangle.$$

A subset $K \subseteq M$ is said to be convex if for any two points $x, y \in K$, the geodesic joining x and y is contained in K, that is, if, $\gamma : [a,b] \longrightarrow M$ is a geodesic such that $x = \gamma(a)$ and $y = \gamma(b)$, then $\gamma((1-t)a+tb) \in K, \quad \forall t \in [0,1]$.

From now onward $K \subseteq M$ will denote a nonempty, closed and convex set, unless explicitly stated otherwise.

A real-valued function f defined on K is said to be a convex function, if for any geodesic γ of M, the composition function $f \circ \gamma : R \longrightarrow R$ is convex, that is,

$$(f \circ \gamma)(ta + (1-t)b) \leq t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b), \quad \forall a, b \in R, \quad t \in [0,1].$$

The subdifferential of a function $f : M \longrightarrow R$ is the set-valued mapping $\partial f : M \longrightarrow 2^{TM}$ defined as:

$$\partial f(x) = \{u \in T_xM : \langle u, \exp_x^{-1} y\rangle \leq f(y) - f(x), \forall y \in M\}, \quad \forall x \in M,$$

and its elements are called subgradients. The subdifferential $\partial f(x)$ at a point $x \in M$ is a closed and convex (possibly empty) set. Let $D(\partial f)$ denote the domain of ∂f defined by

$$D(\partial f) = \{x \in M : \partial f(x) \neq \emptyset\}.$$

The existence of subgradients for convex functions is guaranteed by the following proposition, see [29].

Lemma 2.4 [25,29]. Let M be a Hadamard manifold and $f : M \longrightarrow R$ be convex. Then, for any $x \in M$, the subdifferential $\partial f(x)$ of f at x is nonempty. That is, $D(\partial f) = M$.

For a given single-valued vector field $T : M \longrightarrow TM$, we consider the problem of finding $u \in K$ such that

$$\langle Tu, \exp_u^{-1} v\rangle \geq 0, \quad \forall v \in K, \tag{5}$$

which is called the variational inequality. This problem was considered by Nemeth [7], Colao et al [2], Tang et al

[27,28] and Noor and Noor [18,19]. Colao et al [2] proved the existence of a solution of Problem (5) by proving and using the KKM principle on Hadamard manifolds. In the linear setting, variational inequalities have been studied extensively, see [5,7,8,9,10,11,12,13, 14,15,16,17,18,19,24,26,30]and the references therein.

Definition 2.1. An operator T is said to be partially relaxed strongly monotonicity, if and only if, there exists a constant $\alpha > 0$ such that

$$\langle Tu, \exp_v^{-1} z\rangle + \langle Tv, \exp_z^{-1} v\rangle \le \alpha d^2(z,u), \quad \forall u,v,z \in M.$$

We note, if $z = u$, then partially relaxed strongly monotonicity reduces to monotonicity, but the converse is not true.

3 Main Results

We now use the auxiliary principle technique of Glowinski et al [5] to suggest and analyze an explicit iterative method for solving the variational inequality (5) on the Hadamard manifolds. See also Noor and Noor [18, 19] for recent applications of the auxiliary principle technique.

For a given $u \in K$ satisfying (5), consider the problem of finding $w \in K$ such that

$$\langle \rho Tu + (\exp_u^{-1} w), \exp_w^{-1} v\rangle \ge 0, \quad \forall v \in K, \tag{6}$$

which is called the auxiliary variational inequality on Hadamard manifolds. Here $\rho > 0$ is a constant. We note that, if $w = u$, then w is a solution of the variational inequality (5). This observation enables to suggest and analyze the following two-step iterative method for solving the variational inequality (2.5).

Algorithm 3.1. For a given $u_0 \in K$, compute the approximate solution by the iterative scheme

$$\langle \rho Tu_n + \exp_{u_n}^{-1} y_n, \exp_{y_n}^{-1} v\rangle \ge 0, \tag{7}$$

$$\langle \rho Ty_n + \exp_{y_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} v\rangle \ge 0, \quad \forall v \in K. \tag{8}$$

Algorithm 3.1 is called the two-step iterative method for solving the variational inequality on the Hadamard manifolds. Using the technique of Tang et al [28], one can show that Algorithm 3.1 is well-defined.

If $y_n = u_n$, then Algorithm 3.1 reduces to the explicit iterative method for solving the variational inequalities on Hadamard manifolds.

Algorithm 3.2. For a given $u_0 \in K$, compute the approximate solution by the iterative scheme

$$\langle \rho Tu_n + \exp_{u_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} v\rangle \ge 0, \forall v \in K.$$

If $M = R^n$, then Algorithm 3.1 collapses to:

Algorithm 3.3. For a given $u_0 \in K$, find the approximate solution u_{n+1} by the iterative scheme.

$$\langle \rho Tu_n + y_n - u_n, v - y_n\rangle \ge 0, \forall v \in K$$
$$\langle \rho Ty_n + u_{n+1} - y_n, v - u_{n+1}\rangle \ge 0, \forall v \in K,$$

which is known as the two-step method method for solving the variational inequalities. For the convergence analysis of Algorithm 3.2, see [9,10].

Using the projection technique, Algorithm 3.3 is equivalent to the following two-step projection iterative method for solving the variational inequalities in the linear setting.

Algorithm 3.4. For a given $u_0 \in R^n$, find the approximate solution u_{n+1} by the iterative schemes

$$y_n = P_K[u_n - \rho Tu_n]$$
$$u_{n+1} = P_K[y_n - \rho Ty_n], \quad n = 0,1,2,\ldots$$

Algorithm 3.4 is also called the double projection method. For the convergence analysis and numerical aspects of Algorithm 3.4, see Noor [10].

In a similar way, one can obtains various iterative methods for solving the variational inequalities and related problems.

We now consider the convergence analysis of Algorithm 3.1 and this is the main motivation of our next result.

Theorem 3.1. Let T be a partially relaxed strongly monotone vector field with a constant $\alpha > 0$. Let u_n be a approximate solution of the variational inequality (2.5)obtained from Algorithm 3.1. Then

$$d^2(u_{n+1},u) \le d^2(y_n,u) - (1-2\alpha\rho)d^2(u_{n+1},y_n) \tag{9}$$

$$d^2(y_n,u) \le d^2(u_n,u) - (1-2\rho\alpha)d^2(y_n,u_n), \tag{10}$$

where $u \in K$ is a solution of the variational inequality (5).

Proof. Let $u \in K$ be a solution of the variational inequality (5). Then,

$$\langle \rho T(u), \exp_u^{-1} v\rangle \ge 0, \forall v \in K. \tag{11}$$

Taking $v = y_n$ in (6), we have

$$\langle \rho T(u), \exp_u^{-1} y_n\rangle \ge 0. \tag{12}$$

Taking $v = u$ in (7), we have

$$\langle \rho Tu_n + \exp_{u_n}^{-1} y_n, \exp_{y_n}^{-1} u\rangle \ge 0. \tag{13}$$

From (12) and (13), we have

$$-\langle \exp_{y_n}^{-1} u_n, \exp_{y_n}^{-1} u\rangle$$
$$\ge -\alpha\rho\{\langle T(u), \exp_u^{-1} y_n\rangle + \langle Tu_n, \exp_{y_n}^{-1} u\rangle\}$$
$$\ge -\rho\alpha d^2(y_n,u_n),$$

which implies that

$$\langle \exp_{y_n}^{-1} u_n, \exp_{y_n}^{-1} u\rangle \le \rho\alpha d^2(y_n,u_n). \tag{14}$$

For the geodesic triangle $\triangle(u_n,y_n,u)$, the inequality (3) can be written as:

$$d^2(y_n,u) + d^2(y_n,u_n) - \langle \exp_{y_n}^{-1} u_n, \exp_{y_n}^{-1} u\rangle \le d^2(u_n,u) \tag{15}$$

Thus, from (14) and (15), we obtain

$$d^2(y_n,u) \le d^2(u_n,u) - (1-2\rho\alpha)d^2(y_n,u_n), \tag{16}$$

the required inequality (10).

Take $v = y_n$ in (6), we have

$$\langle \rho T(u), \exp_u^{-1} y_n \rangle \geq 0. \tag{17}$$

and take $v = u$ in (7) to have

$$\langle \rho T u_n + \exp_{y_n}^{-1} u_{n+1}, \exp_{u_{n+1}}^{-1} u \rangle \geq 0. \tag{18}$$

From (17) and (18), we have

$$\begin{aligned} & -\langle \exp_{u_{n+1}}^{-1} y_n, \exp_{u_{n+1}}^{-1} u \rangle \\ & \geq -\alpha\rho \{ \langle T(u), \exp_u^{-1} u_{n+1} \rangle + \langle T y_n, \exp_{u_{n+1}}^{-1} u \rangle \}. \\ & \geq -\rho\alpha d^2(u_{n+1}, y_n). \end{aligned} \tag{19}$$

For the geodesic triangle $\triangle(y_n, u_{n+1}, u)$, the inequality (3) can be written using (19) as:

$$\begin{aligned} d^2(u_{n+1}, u) + d^2(u_{n+1}, y_n) & \leq \langle \exp_{y_n}^{-1} u_n, \exp_{y_n}^{-1} u \rangle + d^2(u_n, u) \\ & \leq 2\rho\alpha d^2(u_{n+1}, y_n) + d^2(u_n, u), \end{aligned}$$

which implies (9). □

Theorem 3.2. Let $u \in K$ be a solution of (5) and let u_{n+1} be the approximate solution obtained from Algorithm 3.1. If $\rho < \frac{1}{2\alpha}$, then $\lim_{n \longrightarrow} u_{n+1} = u$.

Proof. Let $u \in K$ be a solution of (2.5). Then, from (9) and (10), it follows that the sequence $\{u_n\}$ is bounded and

$$\sum_{n=0}^{\infty} (1 - 2\alpha\rho) d^2(u_{n+1}, y_n) \leq d^2(y_0, u),$$

$$\sum_{n=0}^{\infty} (1 - 2\alpha\rho) d^2(y_n, u_n) \leq d^2(u_0, u),$$

which implies

$$\lim_{n \longrightarrow \infty} d(u_{n+1}, y_n) = 0, \quad \lim_{n \longrightarrow \infty} d(y_n, u_n) = 0.$$

Thus

$$\begin{aligned} \lim_{n \longrightarrow \infty} d(u_{n+1}, u_n) & = \lim_{n \longrightarrow \infty} d(u_{n+1}, y_n) + \lim_{n \longrightarrow \infty} d(y_n, u_n) \\ & = 0. \end{aligned} \tag{20}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$. Then there exits a subsequence $\{u_{n_i}\}$ such that $\{u_{u_i}\}$ converges to $\hat{u}$. Replacing u_{n+1} by u_{n_i} in (9), y_n by u_{n_i} in (10), taking the limit, and using (20), we have

$$\langle T\hat{u}, \exp_{\hat{u}}^{-1} v \rangle \geq 0, \quad \forall v \in K,$$

This shows that $\hat{u} \in K$ solves (5) and

$$d^2(u_{n+1}, \hat{u}) \leq d^2(u_n, \hat{u}),$$

which implies that the sequence $\{u_n\}$ has unique cluster point and $\lim_{n \longrightarrow \infty} u_n = \hat{u}$ is a solution of (5), the required result. □

Conclusion. We have used the auxiliary principle technique to suggest and analyze a two-step iterative method solving the variational inequalities on Hadamard manifolds. Some special cases are also discussed. Convergence analysis of the new two-step method is proved under weaker conditions. Results obtained in this paper may stimulate further research in this area. The implementation of the new method and its comparison with other methods is an open problem. The ideas and techniques of this paper may be extended for other related optimization problems.

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