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Existence and Uniqueness for a Solution of Pseudohyperbolic equation with Nonlocal Boundary Condition

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Abstract: Motivated by a number of recent investigations, we define and investigate the various properties of a class of pseudohyperbolic equation defined on purely integral (nonlocal) conditions. We derive useful results involving this class including (for example) existence, uniqueness and continuous arising from the Laplace transform method. In addition, we make use of obtaining such a problem to solve the using a numerical technique (Stehfest algorithm) which provides to show the accuracy of the proposed method.

Keywords: Pseudohyperbolic equation; Laplace transform method; Nonlocal boundary condition.**2010 Mathematics Subject Classification.** 35B45, 35L82, 44A10.

1 Introduction

Throughout of this paper, we make use of the following notations:

$$\mathbb{N} := \{1,2,3,\cdots\} \text{ and } \mathbb{N}^* = \mathbb{N} \cup \{0\}.$$

Here $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real numbers and $\mathbb{C}$ denotes the set of complex numbers. Certain problems of modern physics and technology have been studied by many mathematicians for a long time *cf*. [3-13, 16-23]. Recent investigations on the nonlocal conditions include the data on the boundary which can not be measured directly. In [25], [26], a large number of physical phenomena reduce to a work derived by initial-boundary value problem, as follows: For $(x,t) \in D = \Omega \times I$ with the bounded intervals in $\mathbb{R}_+$ as $\Omega = (0,1)$, $I = (0,T)$

$$\frac{\partial^2 v}{\partial v^2} - \alpha \frac{\partial^2 v}{\partial x^2} - \beta \frac{\partial^3 v}{\partial t \partial x^2} = g(x,t) \tag{1}$$

where α and β are positive constants.

Let us consider the following function $v = v(x,t)$ satisfying the Eq. (1) in D

$$v(x,0) = \Phi(x), \ \frac{\partial v(x,0)}{\partial t} = \Psi(x) \quad (x \in \Omega),$$
$$\int_\Omega v(x,t)\,dx = \mu(t), \ \int_\Omega xv(x,t)\,dx = m(t) \quad (t \in I). \tag{2}$$

It follows from (2) that

$$g \in C\left(\overline{D}\right), \ \Phi, \Psi \in C^1\left(\overline{\Omega}\right), \mu \text{ and } m \in C^2\left(\overline{I}\right)$$

and the suitable conditions are as follows:

$$\textstyle\int_\Omega \Phi(x)\,dx = \mu(0), \ \int_\Omega x\Phi(x)\,dx = m(0), \ \int_\Omega \Psi(x)\,dx = \mu'(0), \ \int_\Omega x\Psi(x)\,dx = m'(0).$$

2 Reformulation of the problem

Since nonlocal (integral) boundary conditions are inhomogeneous, it is applicable to convert the problem (1)-(2) into an equivalent problem with homogeneous nonlocal conditions. Now, we introduce an unknown

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function $u = u(x,t)$ subtracting the function $v = v(x,t)$ from the function $w = w(x,t)$ known in [8], as follows:

$$u(x,t) = v(x,t) - w(x,t). \tag{3}$$

The problem (1)-(2) can be equivalently reduced to the problem for finding the function u satisfying the following

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial^3 u}{\partial t \partial x^2} = f(x,t), \; ((x,t) \in D),\\
&u(x,0) = \varphi(x), \; \frac{\partial u(x,0)}{\partial t} = \psi(x), \; (x \in \Omega),\\
&\int_\Omega u(x,t)\,dx = 0, \; \int_\Omega xu(x,t)\,dx = 0, \; (t \in I),
\end{aligned} \tag{4}$$

where

$$\begin{aligned}
&\varphi'(0) = 0, \; \int_\Omega \varphi(x)\,dx = 0, \; \psi'(0) = 0, \; \int_\Omega \psi(x)\,dx = 0,\\
&f(x,t) = g(x,t) - \left(\frac{\partial w}{\partial t} - \alpha \frac{\partial^2 w}{\partial x^2}\right),\\
&\varphi(x) = \Phi(x) - w(x,0),\\
&\psi(x) = \Psi(x) - \frac{w(x,0)}{\partial t}.
\end{aligned}$$

Hence, the solution of problem (1)-(2) will be obtained by the Eq. (3).

3 Notations and Preliminary results

3.1 Hilbert Space

We are now in a position to state the following definition which will be useful for sequel of this paper.

Definition 1. *Let $H = L^2(\Omega)$ be a Hilbert space with a norm $\|.\|_H$ with the $\Omega = (0,1)$.*

(i) Let $L^2(0,T,H)$ be the set of all square measurable abstract functions $u(.,t)$ from $(0,T)$ into H with the following norm

$$\|u\|_{L^2(0,T,H)} = \left(\int_0^T \|u(.,t)\|_H^2\,dt\right)^{1/2} < \infty$$

(ii) Let $C(0,T,H)$ be the set of all continuous functions $u(.,t) : (0,T) \longrightarrow H$ defined by means of

$$\|u\|_{C(0,T,H)} = \max_{0 \le t \le T} \|u(.,t)\|_H < \infty$$

(iii) We make use of $C_0(\Omega)$ that stands for the vector space of continuous functions with compact support in Ω. Since such functions are Lebesgue integrable with respect to x, we can define on $C_0(\Omega)$ the bilinear form given by

$$((u,w)) = \int_\Omega (J_x^m u)(J_x^m w)dx \qquad (m \ge 1) \tag{5}$$

where we have used the following

$$J_x^m u = \int_0^x \frac{(x-\zeta)^{m-1}}{(m-1)!} u(\zeta,t)\,d\zeta \quad (m \ge 1). \tag{6}$$

The bilinear form (5) is considered as a scalar product on $C_0(\Omega)$ is not complete.

3.2 Bouziani Space

Definition 2. *Let $B_2^m(\Omega)$ (for $m \ge 1$) be the completion of $C_0(\Omega)$ for the scalar product (5) denoted by $(.,.)_{B_2^m(\Omega)}$ (see [4]). Via the norm of function u including $B_2^m(\Omega)$ (for $m \ge 1$), we have*

$$\|u\|_{B_2^m(\Omega)} = \left(\int_\Omega (J_x^m u)^2\,dx\right)^{1/2} = \|J_x^m u\|. \tag{7}$$

Lemma 1. *For all $m \in \mathbb{N}$, the following inequality holds true:*

$$\|u\|^2_{B_2^m(\Omega)} \le \frac{1}{2} \|u\|^2_{B_2^{m-1}(\Omega)} \tag{8}$$

Proof. The proof of this lemma can be easily done by the Reference [4]. So we omit it.

Corollary 1. *For all $m \in \mathbb{N}$, we obtain*

$$\|u\|^2_{B_2^m(\Omega)} \le \left(\frac{1}{2}\right)^m \|u\|^2_{L^2(\Omega)}. \tag{9}$$

Definition 3. *Let $L^2(0,T;B_2^m(\Omega))$ be the space of functions to be square integrable in the Bochner sense with the scalar product*

$$(u,w)_{L^2(0,T;B_2^m(\Omega))} = \int_0^T (u(.,t), w(.,t))_{B_2^m(\Omega)}\,dt$$

Since the space $B_2^m(\Omega)$ is a Hilbert space, it can be shown that $L^2(0,T;B_2^m(\Omega))$ is a Hilbert space as well. The set of all continuous abstract functions in $[0,T]$ with the following norm

$$\sup_{0 \le t \le T} \|u(.,t)\|_{B_2^m(\Omega)}$$

is denoted by $C(0,T;B_2^m(0,1))$.

Corollary 2. *We deduce the continuity of the $L^2(\Omega) \longrightarrow B_2^m(\Omega)$ for $m \ge 1$.*

Lemma 2. (**Gronwall Lemma**) *Let $f_1(t)$, $f_2(t) \ge 0$ be two integrable functions on $[0,T]$ and $f_2(t)$ be nondecreasing. If*

$$f_1(\tau) \le f_2(\tau) + c\int_0^\tau f_1(t)\,dt, \; \forall \tau \in [0,T],$$

where $c \in \mathbb{R}^+$, then

$$f_1(t) \le f_2(t)\exp(ct), \; \forall t \in [0,T]. \tag{10}$$

Lemma 3.(**Cauchy inequality with** ε) *For all* $\varepsilon > 0$ *and arbitrary variables a, b in* $\mathbb{R}$, *we have the following inequality*

$$|ab| \leq \frac{\varepsilon}{2}|a|^2 + \frac{1}{2\varepsilon}|b|^2. \tag{11}$$

4 Uniqueness and continuous dependence of the solution

We first establish a priori estimates. In addition, the uniqueness and continuous based on the solution with respect to the data are immediate consequences.

Theorem 1.*If* $u(x,t)$ *is a solution of problem (4) and* $f \in C(\overline{D})$, *then we have a priori estimates:*

$$\begin{aligned} \|u(.,\tau)\|^2_{L^2(\Omega)} &\leq c_1 \left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(\Omega)}dt + \|\varphi\|^2_{L^2(\Omega)} + \|\psi\|^2_{B_2^1(\Omega)}\right) \\ \left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(\Omega)} &\leq c_2 \left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(\Omega)}dt + \|\varphi\|^2_{L^2(\Omega)} + \|\psi\|^2_{B_2^1(\Omega)}\right) \end{aligned} \tag{12}$$

where $c_1 = \frac{1}{\alpha}\max\left(1, \alpha, \frac{1}{4\beta}\right)$, $c_2 = \max\left(1, \alpha, \frac{1}{4\beta}\right)$ *and* $0 \leq \tau \leq T$.

*Proof.*It is proved by taking the scalar product in $B_2^1(\Omega)$ of the pseudohyperbolic eqaution in the Eq. (4), $\frac{\partial u}{\partial t}$ and integrating over $(0,\tau)$, it becomes

$$\begin{aligned} &\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial t^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt - \alpha\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial x^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt \\ &-\beta\int_0^\tau \left(\frac{\partial^3 u(.,t)}{\partial t\partial x^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt \\ &= \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt. \end{aligned} \tag{13}$$

The integration by parts of the left-hand side of the Eq. (13) gives

$$\begin{aligned} &\alpha\|u(.,\tau)\|^2_{L^2(\Omega)} + \left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(\Omega)} + 2\beta\int_0^\tau \left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{L^2(\Omega)} dt \\ &= 2\int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt + \alpha\|\varphi\|^2_{L^2(\Omega)} + \|\psi\|^2_{B_2^1(\Omega)}. \end{aligned} \tag{14}$$

It follows from the Eq. (11) and the Eq. (9) that

$$\int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(\Omega)} dt \leq \frac{\varepsilon}{2}\int_0^\tau \|f(.,t)\|^2_{B_2^1(\Omega)}dt + \frac{1}{4\varepsilon}\int_0^\tau \left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{L^2(\Omega)} dt.$$

We choose $\varepsilon = \frac{1}{4\beta}$ on that it yields to

$$\begin{aligned} &\alpha\|u(.,\tau)\|^2_{L^2(\Omega)} + \left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(\Omega)} \\ &\leq \frac{1}{4\beta}\int_0^\tau \|f(.,t)\|^2_{B_2^1(\Omega)}dt + \alpha\|\varphi\|^2_{L^2(\Omega)} + \|\psi\|^2_{B_2^1(\Omega)} \end{aligned} \tag{15}$$

Finally, it follows from (15) that we obtain estimates (12).

5 Existence of Solution

5.1 Laplace transform technique

Laplace transform is widely used in the area of engineering technology and mathematical science. There are many problems whose solutions may be found in terms of the Laplace transform. In fact, it is an efficient method for solving many differential equations and partial differential equations. The main difficult of the method of the Laplace transform is in inverting the solution of the Laplace domain into the real domain. Hence we apply the technique of the Laplace transform [2,16,17,18,19,21] to find solutions of the problem (1)-(2).

Suppose that $v(x,t)$ is defined and is of the exponential order for $t \geq 0$, i.e. there exists A, $\gamma > 0$ and $t_0 > 0$ such that $|v(x,t)| \leq A\exp(\gamma t)$ for $t \geq t_0$. Then the Laplace transform $V(x,s)$ including the function $v(x,t)$ is introduced by

$$V(x,s) = \{v(x,t); t \longrightarrow s\} = \int_0^\infty v(x,t)\exp(-st)\,dt \tag{16}$$

where s is known as a Laplace variable. A capital letter V represents Laplace transform of function v, i.e., V is a function in the Laplace domain.

If we start at this approximation and apply Laplace transform on the both sides of the problem (1)-(2), with respect to t, then we discover

$$\begin{aligned} &\frac{d^2}{dx^2}V(x;s) - \frac{s^2}{(\alpha+\beta s)}V(x;s) = \\ &-\frac{1}{(\alpha+\beta s)}\left[G(x;s) + \frac{\partial V(x;0)}{\partial t} + sV(x;0) - \beta\frac{d^2V(x;0)}{dx^2}\right], \end{aligned} \tag{17}$$

$$\begin{aligned} &V(x;0) = \Phi(x), \frac{\partial V(x;0)}{\partial t} = \Psi(x), \\ &\int_0^1 V(x;s)\,dx = A(s), \int_0^1 xV(x;s)\,dx = B(s). \end{aligned} \tag{18}$$

Using the initial conditions, it becomes

$$\begin{aligned} &\frac{d^2}{dx^2}V(x;s) - \frac{s^2}{(\alpha+\beta s)}V(x;s) \\ &= -\frac{1}{(\alpha+\beta s)}\left[G(x;s) + \Psi(x) + s\Phi(x) - \beta\frac{d^2\Phi(x)}{dx^2}\right], \\ &\int_0^1 V(x;s)\,dx = A(s), \int_0^1 xV(x;s)\,dx = B(s) \end{aligned} \tag{19}$$

Notice that

$$\begin{aligned} V(x;t) &= \{v(x,t); t \longrightarrow s\} \\ G(x;t) &= \{g(x,t); t \longrightarrow s\} \\ A(s) &= \{\mu(t); t \longrightarrow s\} \\ B(s) &= \{m(t); t \longrightarrow s\}. \end{aligned}$$

Hence, it is reduced to the boundary value problem by the inhomogeneous ordinary differential equation of second order. From this, we obtain a general solution of the Eq. (19), as follows:

$$V(x;t) = -\frac{\sqrt{\alpha+\beta s}}{s}\int_0^x \left[G(\tau;s)+\Psi(\tau)+s\Phi(\tau)-\beta\frac{d^2\Phi(\tau)}{dx^2}\right]\times \sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)d\tau + C_1(s)\exp\left(-\frac{s}{\sqrt{\alpha+\beta s}}x\right)+ C_2(s)\exp\left(\frac{s}{\sqrt{\alpha+\beta s}}x\right), \quad (20)$$

where C_1 and C_2 are arbitrary functions of s. Substituting the Eq. (20) into the integral boundary conditions in the Eq. (19), we have

$$C_1(s)\int_0^1 \exp\left(-\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx + C_2(s)\int_0^1 \exp\left(\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx =$$

$$\frac{\sqrt{\alpha+\beta s}}{s}\int_0^1\left[\left[G(\tau;s)+\Psi(\tau)+s\Phi(\tau)-\beta\frac{d^2\Phi(\tau)}{dx^2}\right]\times \int_\tau^1 \sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)dx\right]d\tau + A(s),$$

$$C_1(s)\int_0^1 x\exp\left(-\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx + C_2(s)\int_0^1 x\exp\left(\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx =$$

$$\frac{\sqrt{\alpha+\beta s}}{s}\int_0^1\left[\left[G(\tau;s)+\Psi(\tau)+s\Phi(\tau)-\beta\frac{d^2\Phi(\tau)}{dx^2}\right]\times \int_\tau^1 x\sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)dx\right]d\tau + B(s).$$

which in turn yields to

$$\begin{pmatrix} C_1(s) \\ C_2(s)\end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s)\end{pmatrix}^{-1}\times\begin{pmatrix} b_1(s) \\ b_2(s)\end{pmatrix}$$

where

$$a_{11}(s) = \int_0^1 \exp\left(-\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx,\ a_{12}(s) = \int_0^1 \exp\left(\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx,$$
$$a_{21}(s) = \int_0^1 x\exp\left(-\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx,\ a_{22}(s) = \int_0^1 x\exp\left(\frac{s}{\sqrt{\alpha+\beta s}}x\right)dx,$$

$$b_1(s) = \frac{\sqrt{\alpha+\beta s}}{s}\int_0^1\left[\left[G(\tau;s)+\Psi(\tau)+s\Phi(\tau)-\beta\frac{d^2\Phi(\tau)}{dx^2}\right]\times \int_\tau^1 \sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)dx\right]d\tau + A(s),$$

$$b_2(s) = \frac{\sqrt{\alpha+\beta 1}}{s}\int_0^1\left[\left[G(\tau;s)+\Psi(\tau)+s\Phi(\tau)-\beta\frac{d^2\Phi(\tau)}{dx^2}\right]\times \int_\tau^1 x\sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)dx\right]d\tau + B(s), \quad (21)$$

Thus, by evaluating all integrals appeared in the Eq. (20) and the Eq. (21), we find out the solution of the Laplace domain. This can be done for known functions G, Ψ, Φ, A, B; however, in many cases, the results of the functions are not easy to show exactly. Therefore, it is needed to numerical approximations of the integrals. As it is known, Gaussian Quadrature formula exists for computing integrals numerically (see [1]). Using this formula, we have approximate of the above integrals, as follows:

$$\int_0^1 \left(\tfrac{1}{x}\right)\exp\left(\pm\tfrac{s}{\sqrt{\alpha+\beta s}}x\right)dx \simeq \tfrac{1}{2}\textstyle\sum_{i=1}^n \omega_i \left(\tfrac{1}{\frac{1}{2}(x_i+1)}\right)\exp\left(\pm\tfrac{s}{\sqrt{\alpha+\beta s}}\left(\tfrac{1}{2}(x_i+1)\right)\right),$$

$$\int_0^x \left[G(\tau;s)+\Psi(\tau)+s\Phi(\tau)-\beta\tfrac{d^2\Phi(\tau)}{dx^2}\right]\sinh\left(\tfrac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)d\tau \simeq$$

$$\frac{x}{2}\sum_{i=1}^n \omega_i\left[G\left(\frac{x}{2}(x_i+1);s\right)+\Psi\left(\frac{x}{2}(x_i+1)\right)+s\Phi\left(\frac{x}{2}(x_i+1)\right)-\beta\frac{d^2\Phi\left(\frac{x}{2}(x_i+1)\right)}{d\overline{\tau}^2}\right]\times \sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}\left(x-\frac{x}{2}(x_i+1)\right)\right),$$

where we have used $\overline{\tau} = \frac{x}{2}(x_i+1)$.

$$\int_0^1 \left[\left[G(\tau;s)+\Psi(\tau)+s\Phi(\tau)-\beta\tfrac{d^2\Phi(\tau)}{dx^2}\right]\int_\tau^1 \left(\tfrac{1}{x}\right)\sinh\left(\tfrac{s}{\sqrt{\alpha+\beta s}}(x-\tau)\right)dx\right]d\tau \simeq$$

$$\tfrac{1}{2}\textstyle\sum_{i=1}^n \omega_i\left[G\left(\tfrac{1}{2}(x_i+1);s\right)+\Psi\left(\tfrac{1}{2}(x_i+1)\right)+s\Phi\left(\tfrac{1}{2}(x_i+1)\right)-\beta\tfrac{d^2\Phi\left(\frac{1}{2}(x_i+1)\right)}{d\overline{\tau}^2}\right]\times$$
$$\left(\frac{1-\frac{1}{2}(x_i+1)}{2}\right)\sum_{j=1}^n \omega_j\left(\frac{1}{\left(\frac{1-\frac{1}{2}(x_i+1)}{2}\right)x_j+\left(\frac{1+\frac{1}{2}(x_i+1)}{2}\right)}\right)\times$$
$$\sinh\left(\frac{s}{\sqrt{\alpha+\beta s}}\left(\left(\frac{1-\frac{1}{2}(x_i+1)}{2}\right)x_j+\frac{1+\frac{1}{2}(x_i+1)}{2}-\frac{1}{2}(x_i+1)\right)\right), \quad (22)$$

where $\overline{\tau} = \frac{1}{2}(x_i+1)$, x_i and ω_i are defined by

$$x_i:\ i^{th}\text{ zero of } P_n(x),\ \omega_i = \frac{2}{\left(1-x_i^2\right)\left[P_n'(x_i)\right]^2}.$$

Their values can be found in [1] for different values of n.

5.2 Numerical inversion of Laplace transform

The domain of the Laplace transformation is given by the Eq. (16). So we expect to obtain a solution of original problem by means of inverting the Laplace transform. Simple transformations can often be inverted using readily in the available table. More complex functions can be analytically inverted through the complex inversion formula

$$v(t) = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\exp(st)V(s)\,ds,$$

where c is a positive real number such that all the poles of the function $V(s)$ lie at the left of the line $Re(s) = c$.

Sometimes, analytical inversion of the domain of the Laplace solution is difficult to obtain. Therefore, a numerical inversion method must be used. A variety of different methods for numerically inverting the Laplace transform are available that can be employed. There exists no universal method but different types of methods work well for different classes of functions. A nice comparision of four frequently used numerical Laplace inversion algorithms is given by [14]. We use the Stehfest algorithm [20] in this work which provides to show the accuracy of the proposed method. This numerical technique was first introduced by Graver [13] this algorithm offered by Stehfest. Stehfest's algorithm approximates the time domain of the solution as follows:

$$v(x,t) \approx \frac{\ln 2}{t} \sum_{k=1}^{2m} \lambda_k V\left(x; \frac{\ln 2}{t} k\right) \tag{23}$$

where m is the positive integer and

$$\lambda_k = (-1)^{m+k} \sum_{l=\left[\frac{k+1}{2}\right]}^{\min(k,m)} \frac{l^m (2l)!}{(m-l)!l!(l-1)!(k-l)!(2l-k)!}.$$

Here $[r]$ denotes the integer part of r. The parameter m is an arbitrary variable that should be optimized by trial and error. It was seen that increasing m accuracy of results increases up to a point and then owing to the rounding errors it decreases [20]. Thus, for choosing optimum m, it is beneficial to apply an algorithm repeatedly for different values of m and study its effect on the solution. The other way to choose optimal value of m could be applied the Stehfest's algorithm for inverting the Laplace transform of known some elementry functions.

*Remark.*Stehfest's method gives accurate results for many problems including diffusion problem, fractional functions in the Laplace domain. However, it fails to predict $\exp(t)$ type functions or those with oscillatory behavior such as sine and wave function (see [20]).

*Remark.*Note that more than one numerical inversion algorithm can also be performed to check the accuracy of the results.

6 Numerical Examples

In this section, we perform some results of numerical computations using Laplace transform method proposed in the previous section. This technique is applied to solve the problem defined by the problem (1)-(2). The method of solution is easily applicable via the computer, is used Matlab 7.9.3 program.

*Example 1.*We take

$$\begin{aligned}
g(x,t) &= -\frac{8\tanh(x+t)\left(2+\sinh^2(x+t)\right)}{\cosh^4(x+t)},\ 0<x<1,\ 0<t\le T,\\
\Phi(x) &= \frac{1}{\cosh^2(x)},\ 0<x<1,\\
\Psi(x) &= \frac{-2\tanh(x+t)}{\cosh^2(x+t)},\ 0<x<1,\\
\mu(t) &= \tanh(1+t)-\tanh t,\ 0<t\le T,\\
m(t) &= \tanh(1+t)-\ln\cosh(1+t)+\ln\cosh(t),\ 0<t\le T,
\end{aligned}$$

in this case exact solution is given by

$$v(x,t) = \frac{1}{\cosh^2(x+t)},\ 0<x<1,\ 0<t\le T.$$

The method of solution is easily implemented on the computer, numerical results obtained by $n=8$ in (22) and $m=5$ in (23), then we compared the exact solution with numerical solution. For $t=0.10$ and $x\in[0.10,0.90]$, we calculate u numerically using the proposed method of solution and compare it with the exact solution in Table 1.

*Example 2.*We take

$$\begin{aligned}
g(x,t) &= -\frac{4\cosh(x+t)\left(\sinh^2(x+t)-2\right)}{\sinh(x+t)},\ 0<x<1,\ 0<t\le T,\\
\Phi(x) &= \coth^2(x),\ 0<x<1,\\
\Psi(x) &= \frac{-2\cosh(x)}{\sinh^2(x)},\ 0<x<1,\\
\mu(t) &= 1-\coth(1+t)+\coth(t),\ 0<t\le T,\\
m(t) &= \frac{1}{2}-\coth(1+t)-\ln\sinh(1+t)-\ln\sinh(t),\ 0<t\le T,
\end{aligned}$$

in this case exact solution given by

$$v(x,t) = \coth^2(x+t),\ 0<x<1,\ 0<t\le T.$$

For $t=0.10$ and $x\in[0.10,0.90]$, we calculate u numerically using the proposed method of the solution and compare it with the exact solution in Table 2:

*Example 3.*We take

$$\begin{aligned}
g(x,t) &= \frac{4\tanh(x+t)}{\coth^2(x+t)},\ 0<x<1,\ 0<t\le T,\\
\Phi(x) &= \frac{1}{\coth^2(x)},\ 0<x<1,\\
\Psi(x) &= 2\tanh(x),\ 0<x<1,\\
\mu(t) &= 1-\tanh(x+t)+\tanh(t),\ 0<t\le T,\\
m(t) &= \frac{1}{2}-\tanh(t)+\ln\cosh(t)-\ln\cosh(t),\ 0<t\le T,
\end{aligned}$$

in this case exact solution is given by

$$v(x,t) = \frac{1}{\coth^2(x+t)},\ 0<x<1,\ 0<t\le T.$$

For $t=0.10$ and $x\in[0.10,0.90]$, we calculate u numerically using the proposed method of the solution and compare it with the exact solution in Table 3:

Table 1: Results of Example 1

x	0.10	0.30	0.50	0.70	0.90
$u\ exact$	$0,0782785$	$0,0879214$	$0,1057214$	$0,1345645$	$0,1791275$
$u\ numerical$	$0,0778127$	$0,0878643$	$0,1057002$	$0,1340946$	$0,1790532$
$absolute\ error$	$0,0004658$	$0,0000517$	$0,0000212$	$0,0004699$	$0,0000743$

Table 2: Results of Example 2

x	0.10	0.30	0.50	0.70	0.90
$u\ exact$	$25,6693160$	$6,9270684$	$3,4671390$	$2,2678574$	$1,7240617$
$u\ numerical$	$25,6692268$	$6,9269887$	$3,4670924$	$2,2678555$	$1,724061657$
$absolute\ error$	$0,0000892$	$0,0000797$	$0,0000466$	$0,0000019$	$0,0000043$

Table 3: Results of Example 3

x	0.10	0.30	0.50	0.70	0.90
$u\ exact$	$0,0353069$	$0,0095278$	$0,0047689$	$0,0031193$	$0,0023714$
$u\ numerical$	$0,0352700$	$0,0093904$	$0,0044651$	$0,0030620$	$0,0023015$
$absolute\ error$	$0,0000369$	$-0,0001374$	$0,0003038$	$0,0000573$	$0,0000699$

References

[1] Abramowitz, M., Stegun. I.A.: Handbook of Mathematical Functions, Dover, New York, 1972.

[2] Ang, W.T.: A Method of Solution for the One-Dimentional Heat Equation Subject to Nonlocal Conditions, Southeast Asian Bulletin of Mathematics **26**,185-191, 2002.

[3] Bouziani, A., Temsi, M. S.: On a Pseudohyperbolic Equation with Nonlocal Boundary Condition, Kobe J. Math., 21, 15-31, 2004.

[4] Bouziani, A.: Mixed problem with boundary integral conditions for a certain parabolic equation, J. Appl. Math. Stochastic Anal. **9,** No. 3, 323-330, 1996.

[5] Bouziani, A.: Solution forte d'un problème mixte avec une condition non locale pour une classe d'équations hyperbolique, Acad. Roy. Belg. Bull. Cl. Sci. **8,** 53-70, 1997.

[6] Bouziani, A.: Strong solution for a mixed problem with nonlocal condition for certain pluriparabolic equations, Hiroshima Math. J. **27,** No. 3, 373-390, 1997.

[7] Bouziani, A.: On the solvability of nonlocal pluriparabolic problems, Electronic J. Differential Equations**,** 1-16, 2001.

[8] Bouziani, A.: Initial-boundary value problem with nonlocal condition for a viscosity equation, Int. J. Math. & Math. Sci.**30**, no. 6, 327-338, 2002.

[9] Bouziani, A.: On the solvabiliy of parabolic and hyperbolic problems with a boundary integral condition, Internat. J. Math. & Math. Sci. **31** , 435-447, 2002.

[10] Bouziani, A., On the solvability of a class of singular parabolic equations with nonlocal boundary conditions in nonclassical function spaces, Internat. J. Math. & Math. Sci. **30** , 435-447, 2002.

[11] Bouziani, A.: On a classe of nonclassical hyperbolic equations with nonlocal conditions. J. Appl. Math. Stochastic Anal. **15,** No. 2, 136-153, 2002.

[12] Bouziani, A., Benouar, N.E. :Mixed problem with integral conditions for a third order parabolic equation, kobe J. Math. **15,** No. 1, 47-58, 1998.

[13] Cannon, J. R :The solution of the diffusion equation subject to the specification of energy, Quart. Appl. Math., 21 ,155-160, 1963.

[14] Graver D. P.: Observing stochastic processes and aproximate transform inversion, Oper. Res. 14, 444-459, 1966.

[15] Hassanzadeh, H., Pooladi, M. D.: Comparision of different numerical Laplace inversion methods for engineering applications, Appl. Math. Comp. 189 1966-1981, 2007.

[16] Merad, A., Marhoune, A. L.: Strong solution for a high order boundary value problem with integral condition, Turk. J. Math. 37, 299-307, 2013.

[17] Merad, A., Bouziani, A.: A Method of solution of integro-differential parabolic equation with purely integral conditions, Advances in applied mathematics and approximation theory, Springer proceeding in mathematics and statistics, 41, 2013.

[18] Merad, A., Bouziani, A.: Solvability the Telegraph equation with purely integral conditions, TWMS J. App. Eng. Math., V. 3, N. 2, pp. 117-125, 2013.

[19] Merad, A., Bouziani, A.: Lapalce transform for pseudoparabolic equation with nonlocal conditions, TJMM, V. 5, N. 1, pp. 59-64 (2013).

[20] Pulkina, L. S., A non-local problem with integral conditions for hyperbolic equations, Electron. J. Differential Equations 1999 , no. 45, 1–6, 1999.

[21] Pulkina, L. S., On the solvability in L2 of a nonlocal problem with integral conditions for a hyperbolic equation, Differ. Equ. 36 , no. 2, 316–318, 2000.

[22] Pulkina, L. S., A mixed problem with integral condition for the hyperbolic equation, Matematicheskie Zametki, vol. 74, no. 3, , pp. 435–445, 2003.

[23] Stehfest, H.: Numerical Inversion of the Laplace Transforme, Comm. ACM **13**, 47-49, 1970

[24] Shruti, A.D.: Numerical Solution for Nonlocal Sobolev-type Differential Equations, Electronic Journal of Differential Equations, Conf. 19, pp. 75-83, 2010.

[25] Tikhonov, A.I. Samarskii, A., A.: Equations of Mathematical physics, Edit. Mir, Moscow, 1984.

[26] Yurchuk, N. I.: Mixed problem with an integral condition for certain parabolic equations, Differents. Uravn., **22** , 2117-2126, 1986.

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