Applied Mathematics & Information Sciences
An International Journal

http://dx.doi.org/10.12785/amis/090418

The Pitch, the Angle of Pitch, and the Distribution Parameter of a Closed Ruled Surface

Nemat Abazari[1,*], *Ilgin Sağer*[2] *and Hasan Hilmi Hacisalihoğlu*[3]

[1] Department of Mathematics, Faculty of Mathematical Sciences, University of Mohaghegh Ardabili, 56199-11367, Ardabil, Iran
[2] Department of Mathematics, Faculty of Science, Ankara University, Ankara, Turkey
[3] Department of Mathematics, Bilecik Seyh Edebali University, Bilecik, Turkey

Received: 19 Oct. 2014, Revised: 19 Jan. 2015, Accepted: 20 Jan. 2015
Published online: 1 Jul. 2015

Abstract: In this paper, the pitch, the angle of pitch, and the distribution parameter of the closed ruled surfaces, generated by tangent, bi-normal, normal vectors, and unit Darboux vector of a unit dual spherical motion, are studied on the Frenet frame.

Keywords: Dual Steiner vector, dual spherical motion, pitch of the motion, Pfaffian vector, ruled surface.

1 Introduction

For the analysis of spatial motions in differential geometry [2, 6] and in kinematics of the spatial mechanisms [11–13], the use of dual vectors, dual quaternion, and dual matrix algebra over the ring of dual numbers is a very direct method. Important properties of a real vector analysis of real matrix algebra are valid for the dual vectors and dual matrices. The principal part of this method is based on work by E. Study [12]. The essential idea is to replace points by straight lines as fundamental building blocks of geometric structure. The set of oriented lines in Euclidean three-dimensional space E^3 is in one-to-one correspondence with the points of a unit dual sphere in the dual space D^3 of triples of dual numbers.

The definition of the Steiner vector for the real unit sphere [1] is extended [2] to the definition of dual Steiner vector. In [2], an expression of the pitch of a closed ruled surface is derived in terms of the elements of the dual Steiner vector. Using this expression, the spatial extensions for planar [2] and spherical [1, 4] theorems called Steiner theorems and Holditch theorems are given.

The motion corresponding to the dual spherical closed motion K/K' on D-module is the one-parameter motion H/H' on the line-space. The closed ruled surface (X), which is drawn by the line $\overrightarrow{X}$ of H on the fixed space H', the pitch, and the angle of pitch of the closed ruled surface (X) are studied in [3].

In this paper, the pitch, the angle of pitch, and distribution parameter of the closed ruled surface each of which generated by tangent, bi-normal, normal vectors, and unit Darboux vector on unit dual sphere, respectively, are investigated.

2 The Pitch of a Closed Ruled Surface

Let H' and H denote the fixed and moving line-spaces, respectively. According to the result of E. Study, unit dual spheres K' and K centered at a point M correspond to these spaces on D-module, respectively. Also, dual-spherical motion K/K' corresponds to one-parameter spatial motion H/H'. Let us take a line $\overrightarrow{X}$ on H. That is to say, we consider a fixed point X of the unit dual sphere K. During the motion H/H', the line $\overrightarrow{X}$ traces a ruled surface (X) which is called the orbit surface on H'. The variation of the point X according to the fixed sphere K', i.e., the variation of the line $\overrightarrow{X}$ on H' is

$$d_f\overrightarrow{X} = \overrightarrow{\Psi} \wedge \overrightarrow{X},$$

where the vector $\overrightarrow{\Psi} = \overrightarrow{\psi} + \varepsilon\overrightarrow{\psi^*} = (\Psi_1, \Psi_2, \Psi_3)$ with $\Psi_i = \psi_i + \varepsilon\psi_i^*$, $i = 1,2,3$, is called the instantaneous Pfaffian vector of the motion H/H'.

The ruled surface (X) is given by $\overrightarrow{X} = \overrightarrow{X}(t) = \overrightarrow{x}(t) + \varepsilon\overrightarrow{x^*}(t)$, where $\overrightarrow{X} = \overrightarrow{X}(t)$ is the unit dual vectorial function

* Corresponding author e-mail: abazari@uma.ac.ir

parameterized by $t \in \mathbb{R}$. The dual curve (X) is the dual spherical formation of the ruled surface. For the dual arc element $d\Phi = d\varphi + \varepsilon d\varphi^*$ on the dual spherical curve $\overrightarrow{X} = \overrightarrow{X}(t)$,

$$d\Phi^2 = \left\langle d\overrightarrow{X}, d\overrightarrow{X} \right\rangle = \langle d\overrightarrow{x}, d\overrightarrow{x} \rangle + 2\varepsilon \left\langle d\overrightarrow{x}, d\overrightarrow{x^*} \right\rangle$$

is valid. The distribution parameter of the ruled surface is defined [5] by

$$\frac{1}{d} = \frac{\langle d\overrightarrow{x}, d\overrightarrow{x} \rangle}{\left\langle d\overrightarrow{x}, d\overrightarrow{x^*} \right\rangle} = \frac{d\varphi \cdot d\varphi^*}{d\varphi^2} = \frac{d\varphi^*}{d\varphi}.$$

Definition 1. *Let K be the moving unit dual sphere. Subject to the condition that the pitch of the motion is nonvanishing, a new coordinate system is introduced by $\{\overrightarrow{R_1}, \overrightarrow{R_2}, \overrightarrow{P} = \overrightarrow{R_3}\}$. This frame is called a canonical coordinate frame. For this case, $\overrightarrow{\Psi} = \Psi_3 \overrightarrow{R_3} = \Psi_3 \overrightarrow{P}$ is the instantaneous Pfaffian vector [5].*

The declaration of the variation of a point of $X \in K$ according to the canonical coordinate frame on one-parameter motion K/K' is given [5] by

$$\begin{aligned} d_f \overrightarrow{X} &= \overrightarrow{\Psi} \wedge \overrightarrow{X} = \psi_3 \left(\overrightarrow{P} \wedge \overrightarrow{X} \right) \\ &= \psi_3 \left(\overrightarrow{R_3} \wedge \overrightarrow{X} \right) = \psi_3 \left(X_1 \overrightarrow{R_2} - X_2 \overrightarrow{R_1} \right), \end{aligned}$$

where $\overrightarrow{\Psi} = \Psi_3 \overrightarrow{R_3}$, $\Psi_3 = \psi_3 + \varepsilon \psi_3^*$, $\overrightarrow{X} = X_1 \overrightarrow{R_1} + X_2 \overrightarrow{R_2} + X_3 \overrightarrow{R_3}$, and $X_i = x_i + \varepsilon x_i^*$, $i = 1,2,3$. The distribution parameter of the ruled surface is

$$\frac{1}{d} = p - \frac{x_2 x_3^*}{1 - x_3^2},$$

where $p = \frac{\psi_3^*}{\psi_3}$ is the pitch [5] of the motion H/H'.

3 The Distribution Parameter of a Closed Ruled Surface

On the one-parameter dual spherical motion, the fixed point $X \in K$ constructs a dual curve on K'. The tangent, bi-normal, and normal of the dual curve (X) at a point X are

$$\begin{aligned} \overrightarrow{T} &= \frac{d_f \overrightarrow{X}}{\left\| d_f \overrightarrow{X} \right\|} = \frac{\overrightarrow{\Psi} \wedge \overrightarrow{X}}{\left\| \overrightarrow{\Psi} \wedge \overrightarrow{X} \right\|}, \\ \overrightarrow{B} &= \frac{d_f \overrightarrow{X} \wedge d_f^2 \overrightarrow{X}}{\left\| d_f \overrightarrow{X} \wedge d_f^2 \overrightarrow{X} \right\|}, \\ \overrightarrow{N} &= \overrightarrow{B} \wedge \overrightarrow{T}, \end{aligned}$$

respectively.

Theorem 1. *On the one-parameter dual motion K/K', the tangent, bi-normal, and normal vectors of a dual curve (X) at a point X are given by*

$$\begin{aligned} \overrightarrow{T} &= \frac{1}{\sqrt{1 - X_3^2}} \left(-X_2 \overrightarrow{R_1} + X_1 \overrightarrow{R_2} \right), \\ \overrightarrow{N} &= \frac{-1}{\sqrt{1 - X_3^2}} \left(X_1 \overrightarrow{R_1} + X_2 \overrightarrow{R_2} \right), \\ \overrightarrow{B} &= \overrightarrow{R_3}, \end{aligned}$$

respectively, where $X = (X_1, X_2, X_3)$.

Proof. Suppose that $R = \begin{pmatrix} \overrightarrow{R}_1 \\ \overrightarrow{R}_2 \\ \overrightarrow{R}_3 \end{pmatrix}$, where $\{\overrightarrow{R}_1, \overrightarrow{R}_2, \overrightarrow{R}_3\}$ is a unit dual orthogonal frame for 3-space D^3. Since X is on the unit dual sphere K, we may write that

$$\begin{aligned} \overrightarrow{X} &= X_1 \overrightarrow{R}_1 + X_2 \overrightarrow{R}_2 + X_3 \overrightarrow{R}_3 = X^T R, \\ \left\| \overrightarrow{X} \right\|^2 &= X_1^2 + X_2^2 + X_3^2 = \mathbf{1} = (1,0), \end{aligned}$$

where $X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$, and $\overrightarrow{X}$ is the dual vector corresponding to X. The displacements of R with respect to K and K', the dual moving, and fixed sphere, respectively, are given by

$$dR = \Omega R, \quad d'R = \Omega' R,$$

$$\Omega = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ -\Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix}$$

and

$$\Omega' = \begin{pmatrix} 0 & \Omega_3' & -\Omega_2' \\ -\Omega_3' & 0 & \Omega_1' \\ \Omega_2' & -\Omega_1' & 0 \end{pmatrix}.$$

Then the displacements of $\overrightarrow{X}$ with respect to K and K' are given by

$$d\overrightarrow{X} = dX^T R + X^T dR = (dX^T + X^T \Omega) R \qquad (1)$$

and

$$d'\overrightarrow{X} = d'^T R + X^T d'R = (d'^T + X^T \Omega') R, \qquad (2)$$

respectively. Since Ω and Ω' are anti-symmetric matrixes, we have

$$\Omega^T = -\Omega, \quad \Omega'^T = -\Omega'.$$

For any fixed vector $\overrightarrow{X}$, we get

$$d\overrightarrow{X} = 0, \quad d'\overrightarrow{X} = 0.$$

Therefore from equations (1) and (2), we get

$$\begin{aligned} dX^T + X^T \Omega &= 0, \\ dX^T &= -X^T \Omega = X^T \Omega^T \end{aligned} \qquad (3)$$

and

$$\begin{aligned} d'X^T + X^T\Omega' &= 0, \\ d'X^T &= -X^T\Omega' = X^T\Omega'^T. \end{aligned} \tag{4}$$

Now, suppose that X is fixed in K and let us calculate its velocity $d_f X$ with respect to K'. Then we obtain that

$$\begin{aligned} d_f\overrightarrow{X} &= d'\overrightarrow{X} - d\overrightarrow{X} \\ &= X^T(\Omega' - \Omega)R. \end{aligned}$$

If we define a new dual vector whose components in the relative system are $\Psi_i = \Omega_i' - \Omega_i$, where $i = 1,2,3$, and

$$\begin{aligned} \overrightarrow{\Psi} &= (\Psi_1, \Psi_2, \Psi_3) \\ &= \Psi_1\overrightarrow{R}_1 + \Psi_2\overrightarrow{R}_2 + \Psi_3\overrightarrow{R}_3, \end{aligned}$$

then we get

$$d_f\overrightarrow{X} = \overrightarrow{\Psi} \wedge \overrightarrow{X}, \tag{5}$$

where $\overrightarrow{\Psi}$ is the Pfaffian vector corresponding to the dual spherical motion K/K'. To calculate the acceleration $J = d_f^2\overrightarrow{X}$ of X, we have

$$\begin{aligned} J = d_f^2\overrightarrow{X} &= \overrightarrow{\Psi} \wedge (\overrightarrow{\Psi} \wedge \overrightarrow{X}) + \overrightarrow{\Psi} \wedge \overrightarrow{X} \\ &= -\left\langle \overrightarrow{\Psi}, \overrightarrow{\Psi} \right\rangle \overrightarrow{X} + \left\langle \overrightarrow{\Psi}, \overrightarrow{X} \right\rangle \overrightarrow{\Psi} + \overrightarrow{\Psi} \wedge \overrightarrow{X} \\ &= -\left\| \overrightarrow{\Psi} \right\|^2 \overrightarrow{X} + \left\langle \overrightarrow{\Psi}, \overrightarrow{X} \right\rangle \overrightarrow{\Psi} + \overrightarrow{\Psi} \wedge \overrightarrow{X} \\ &= -\Psi^2\overrightarrow{X} + \left\langle \overrightarrow{\Psi}, \overrightarrow{X} \right\rangle \overrightarrow{\Psi} + \overrightarrow{\Psi} \wedge \overrightarrow{X}. \end{aligned}$$

By using the matrix form, the relations (3) and (4) yield

$$d_f X = MX$$

and

$$J = d_f^2 X = (M^2 + \dot{M})X,$$

where

$$M = \begin{pmatrix} 0 & -\Psi_3 & \Psi_2 \\ \Psi_3 & 0 & -\Psi_1 \\ -\Psi_2 & \Psi_1 & 0 \end{pmatrix}$$

and

$$\dot{M} = dM = \begin{pmatrix} 0 & -\dot{\psi}_3 & \dot{\Psi}_2 \\ \dot{\Psi}_3 & 0 & -\dot{\Psi}_1 \\ -\dot{\Psi}_2 & \dot{\Psi}_1 & 0 \end{pmatrix}.$$

Hence we obtain

$$\begin{aligned} M^2 &= \begin{pmatrix} -\Psi_3^2 - \Psi_2^2 & \Psi_1\Psi_2 & \Psi_1\Psi_3 \\ \Psi_1\Psi_2 & -\Psi_3^2 - \Psi_1^2 & \Psi_2\Psi_3 \\ \Psi_1\Psi_3 & \Psi_2\Psi_3 & -\Psi_2^2 - \Psi_1^2 \end{pmatrix} \\ &= \begin{pmatrix} -\Psi^2 - \Psi_1^2 & \Psi_1\Psi_2 & \Psi_1\Psi_3 \\ \Psi_1\Psi_2 & \Psi^2 + \Psi_2^2 & \Psi_2\Psi_3 \\ \Psi_1\Psi_3 & \Psi_2\Psi_3 & -\Psi^2 - \Psi_3^2 \end{pmatrix}. \end{aligned}$$

If $\overrightarrow{\Psi} = \Psi_3\overrightarrow{R}_3$, then $\Psi_1 = \Psi_2 = 0$. Thus,

$$M = \begin{pmatrix} 0 & -\Psi_3 & 0 \\ \Psi_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and hence the velocity vector of $X \in K$ is given by

$$\begin{aligned} d_f\overrightarrow{X} = MX &= \begin{pmatrix} -\Psi_3 X_2 \\ \Psi_3 X_1 \\ 0 \end{pmatrix} \\ &= \Psi_3(-X_2\overrightarrow{R_1} + X_1\overrightarrow{R_2}) \end{aligned}$$

and

$$\begin{aligned} \overrightarrow{T} &= \frac{d_f\overrightarrow{X}}{\left\| d_f\overrightarrow{X} \right\|} \\ &= \frac{\Psi_3(-X_2\overrightarrow{R_1} + X_1\overrightarrow{R_2})}{\sqrt{\Psi_3^2(X_2^2 + X_1^2)}} \\ &= \frac{-1}{\sqrt{1 - X_3^2}}(X_2\overrightarrow{R_1} - X_1\overrightarrow{R_2}). \end{aligned}$$

By using the same arguments, it is easy to see that $\overrightarrow{B} = \overrightarrow{R}_3$. Therefore

$$\begin{aligned} \overrightarrow{N} = \overrightarrow{B} \wedge \overrightarrow{T} &= \overrightarrow{R}_3 \wedge \frac{-1}{\sqrt{1 - X_3^2}}\left(X_2\overrightarrow{R_1} - X_1\overrightarrow{R_2}\right) \\ &\frac{-1}{\sqrt{1 - X_3^2}}\left(X_2\overrightarrow{R}_3 \wedge \overrightarrow{R}_1 - X_1\overrightarrow{R}_3 \wedge \overrightarrow{R}_2\right) \\ &= \frac{-1}{\sqrt{1 - X_3^2}}\left(X_2\overrightarrow{R}_2 + X_1\overrightarrow{R}_1\right). \end{aligned}$$

This completes the proof.

From Theorem 1, the following corollary can be obtained.

Corollary 1. *On the one-parameter motion K/K', the unit Darboux vector is*

$$\begin{aligned} \overrightarrow{D_0} &= \frac{\tau\overrightarrow{T} + \kappa\overrightarrow{B}}{\sqrt{\tau^2 + \kappa^2}} \\ &= \frac{1}{\sqrt{\tau^2 + \kappa^2}}\left[\frac{\tau}{\sqrt{1 - X_3^2}}\left(-X_2\overrightarrow{R_1} + X_1\overrightarrow{R_2}\right) + \kappa\overrightarrow{R_3}\right], \end{aligned}$$

where $\kappa = k_1 + \varepsilon k_1^*$ and $\tau = k_2 + \varepsilon k_2^*$ are the first and second dual curvatures, respectively, and $\overrightarrow{X} = (X_1, X_2, X_3)$.

Theorem 2. *Consider a point T on the unit moving dual sphere K of the dual-spherical motion K/K', corresponding to one-parameter spatial motion H/H',* *such that during the motion H/H', the line $\overrightarrow{T}$ draws a ruled surface (T) on the fixed space H'. Then the distribution parameter of this ruled surface is*

$$\left(\frac{1}{d}\right)_T = p = \frac{\psi_3^*}{\psi_3}.$$

Proof. The declaration of the variation of a point T according to the canonical coordinate frame of the one-parameter motion K/K' is

$$d_f\overrightarrow{T} = \overrightarrow{\Psi}\wedge\overrightarrow{T} = \Psi_3\left(\overrightarrow{R_3}\wedge\overrightarrow{T}\right) = \frac{-\Psi_3}{\sqrt{1-X_3^2}}\left(X_2\overrightarrow{R_1}+X_1\overrightarrow{R_2}\right).$$

The distribution parameter of the ruled surface is

$$\left(\frac{1}{d}\right)_T = \frac{d\varphi\cdot d\varphi^*}{d\varphi^2} = \frac{\psi_3^*\cdot\psi_3}{\psi_3^2} = \frac{\psi_3^*}{\psi_3} = p,$$

where $d\Phi = d\varphi+\varepsilon d\varphi^*$ denotes the dual arc element of the ruled surface (T).

Theorem 3. *Consider a point B on the unit moving dual sphere K such that during the motion H/H', the line $\overrightarrow{B}$ draws a ruled surface (B) on the fixed space H'. Then the distribution parameter of this ruled surface is undefined.*

Proof. The declaration of the variation of the point B according to the canonical coordinate frame on the one-parameter motion K/K' is

$$d_f\overrightarrow{B} = \overrightarrow{\Psi}\wedge\overrightarrow{B} = \Psi_3\left(\overrightarrow{R_3}\wedge\overrightarrow{B}\right) = 0,$$

since $\overrightarrow{B} = \overrightarrow{R_3}$. So on the motion K/K', the point B is fixed and the distribution parameter of the ruled surface is undefined.

Theorem 4. *Consider a point N on the unit moving dual sphere K such that during the motion H/H', the line $\overrightarrow{N}$ draws a ruled surface (N) on the fixed space H'. Then the distribution parameter of this ruled surface is*

$$\left(\frac{1}{d}\right)_N = \frac{\psi_3^*}{\psi_3} = p.$$

Proof. The declaration of the variation of the point N according to the canonical coordinate frame on the one-parameter motion K/K' is

$$d_f\overrightarrow{N} = \overrightarrow{\Psi}\wedge\overrightarrow{N} = \psi_3\left(\overrightarrow{R_3}\wedge\overrightarrow{N}\right) = \frac{\psi_3}{\sqrt{1-X_3^2}}\left(X_2\overrightarrow{R_1}-X_1\overrightarrow{R_2}\right).$$

The distribution parameter of the ruled surface is

$$\left(\frac{1}{d}\right)_N = \frac{\psi_3^*}{\psi_3} = p,$$

where $d\Phi = d\varphi+\varepsilon d\varphi^*$ is the dual arc element of the ruled surface (N) and p is the pitch of the motion.

Theorem 5. *Consider a point D_0 on the unit moving dual sphere K such that during the motion H/H', the line $\overrightarrow{D_0}$ draws a ruled surface (D_0) on the fixed space H'. Then the distribution parameter of this ruled surface is*

$$\left(\frac{1}{d}\right)_{D_0} = \frac{k_1k_1^*+k_2k_2^*}{k_1^2+k_2^2}+\frac{k_2^*}{k_2}+p,$$

where p is the pitch of the motion and k_1, k_2 are the first and second dual curvatures, respectively.

Proof. The declaration of the variation of the point D_0 according to the canonical coordinate frame on the one-parameter motion K/K' is

$$d_f\overrightarrow{D_0} = \overrightarrow{\Psi}\wedge\overrightarrow{D_0} = \frac{-\tau\Psi_3}{\sqrt{\tau^2+k^2}\sqrt{1-X_3^2}}\left(X_1\overrightarrow{R_1}+X_2\overrightarrow{R_2}\right).$$

The distribution parameter of the ruled surface is

$$\left(\frac{1}{d}\right)_{D_0} = \frac{k_1k_1^*+k_2k_2^*}{k_1^2+k_2^2}+\frac{k_2^*}{k_2}+p,$$

where $d\Phi = d\varphi+\varepsilon d\varphi^*$ is the dual arc element of the ruled surface (D_0).

4 The Pitch and the Angle of Pitch

There are four cases of the pitch and angle of pitch of the closed ruled surfaces (T), (N), (B), and (D_0) on the canonical coordinate frame.

Theorem 6. *The pitch, the angle of pitch, and the dual angle of pitch of the closed ruled surface (T) on the canonical coordinate frame are vanishing.*

Proof. We know that

$$\overrightarrow{T} = \frac{1}{\sqrt{1-X_3^2}}\left(-X_2\overrightarrow{R_1}+X_1\overrightarrow{R_2}\right)$$

is valid, where $\overrightarrow{T} = \overrightarrow{t}+\varepsilon\overrightarrow{t^*}$ and $X_i = \overrightarrow{x_i}+\varepsilon\overrightarrow{x_i^*}$, $\overrightarrow{R_i} = \overrightarrow{r_i}+\varepsilon\overrightarrow{r_i^*}$ for $i=1,2,3$, and

$$\overrightarrow{t} = \frac{1}{\sqrt{1-x_3^2}}\left(-x_2\overrightarrow{r_1}+x_1\overrightarrow{r_2}\right),$$

$$\overrightarrow{t^*} = \frac{1}{\sqrt{1-x_3^2}}\left(\lambda_1\overrightarrow{r_1}+\lambda_2\overrightarrow{r_2}-x_2\overrightarrow{r_1^*}+x_1\overrightarrow{r_2^*}\right),$$

where $\lambda_1 = \frac{x_3x_3^*x_2}{x_3^2-1}-x_2^*$ and $\lambda_2 = -\frac{x_3x_3^*x_1}{x_3^2-1}+x_1^*$. The dual vector $\overrightarrow{D} = \overrightarrow{d}+\varepsilon\overrightarrow{d^*} = \oint\overrightarrow{\Psi}$ for $\overrightarrow{\Psi} = \Psi_3\overrightarrow{R_3}$, $\Psi_3 = \psi_3+\varepsilon\psi_3^*$ is the Steiner vector, where $\overrightarrow{d} = (\oint\psi_3)\overrightarrow{r_3}$ and $\overrightarrow{d^*} =$

$(\oint \psi_3^*)\,\vec{r_3} + (\oint \psi_3)\,\vec{r_3^*}$. Consequently, the angle of pitch is vanishing, i.e.,

$$\lambda_T = \left\langle \vec{d}, \vec{t} \right\rangle = \left\langle \left(\oint \psi_3\right)\vec{r_3}, \frac{1}{\sqrt{1-x_3^2}}\left(-x_2\vec{r_1} + x_1\vec{r_2}\right)\right\rangle = 0,$$

and the pitch is defined as

$$l_T = \left\langle \vec{d^*}, \vec{t} \right\rangle + \left\langle \vec{d}, \vec{t^*} \right\rangle.$$

From the equations

$$\left\langle \vec{d}, \vec{t^*} \right\rangle = \left\langle \left(\oint \psi_3\right)\vec{r_3}, \frac{1}{\sqrt{1-x_3^2}}\left(\lambda_1\vec{r_1} + \lambda_2\vec{r_2} - x_2\vec{r_1^*} + x_1\vec{r_2^*}\right)\right\rangle$$
$$= -\frac{1}{\sqrt{1-x_3^2}}\left(\oint \psi_3\right)x_2\left\langle \vec{r_3}, \vec{r_1^*}\right\rangle + \frac{1}{\sqrt{1-x_3^2}}\left(\oint \psi_3\right)x_1\left\langle \vec{r_3}, \vec{r_2^*}\right\rangle$$

and

$$\left\langle \vec{d^*}, \vec{t} \right\rangle = \left\langle \left(\oint \psi_3^*\right)\vec{r_3} + \left(\oint \psi_3\right)\vec{r_3^*}, \frac{1}{\sqrt{1-x_3^2}}\left(-x_2\vec{r_1} + x_1\vec{r_2}\right)\right\rangle$$
$$= -\frac{1}{\sqrt{1-x_3^2}}\left(\oint \psi_3\right)x_2\left\langle \vec{r_3^*}, \vec{r_1}\right\rangle + \frac{1}{\sqrt{1-x_3^2}}\left(\oint \psi_3\right)x_1\left\langle \vec{r_3^*}, \vec{r_2}\right\rangle,$$

and since we have

$$\left\langle \vec{r_3}, \vec{r_1^*}\right\rangle = -\left\langle \vec{r_3^*}, \vec{r_1}\right\rangle$$

and

$$\left\langle \vec{r_3}, \vec{r_2^*}\right\rangle = -\left\langle \vec{r_3^*}, \vec{r_2}\right\rangle,$$

the pitch is

$$l_T = 0.$$

The dual angle of pitch is

$$\Lambda_T = l_T - \varepsilon\lambda_T = 0.$$

Hence (T) is an extendable ruled surface.

Theorem 7. *The pitch, the angle of pitch, and the dual angle of pitch of the closed ruled surface (N) on the canonical coordinate frame are vanishing.*

Proof. We know that

$$\vec{N} = \frac{-1}{\sqrt{1-X_3^2}}\left(X_1\vec{R_1} + X_2\vec{R_2}\right)$$

is valid, where $\vec{N} = \vec{n} + \varepsilon\vec{n^*}$ and $X_i = \vec{x_i} + \varepsilon\vec{x_i^*}$, $\vec{R_i} = \vec{r_i} + \varepsilon\vec{r_i^*}$, $i = 1,2,3$, and

$$\vec{n} = \frac{1}{\sqrt{1-x_3^2}}\left(x_1\vec{r_1} + x_2\vec{r_2}\right),$$
$$\vec{n^*} = \frac{1}{\sqrt{1-x_3^2}}\left(\left(x_1^* - \frac{x_3x_3^*x_1}{x_3^2-1}\right)\vec{r_1} + \left(x_2 - \frac{x_3x_3^*x_2}{x_3^2-1}\right)\vec{r_2} + x_1\vec{r_1^*} + x_2\vec{r_2^*}\right).$$

Since the Steiner vector is defined as $\vec{D} = \vec{d} + \varepsilon\vec{d^*} = \oint\vec{\Psi}$ for $\vec{\Psi} = \Psi_3\vec{R_3}$, $\Psi_3 = \psi_3 + \varepsilon\psi_3^*$, where $\vec{d} = (\oint\psi_3)\,\vec{r_3}$ and $\vec{d^*} = (\oint\psi_3^*)\,\vec{r_3} + (\oint\psi_3)\,\vec{r_3^*}$, the angle of pitch is vanishing, i.e.,

$$\lambda_N = \left\langle \vec{d}, \vec{n} \right\rangle = \left\langle \left(\oint \psi_3\right)\vec{r_3}, \frac{-1}{\sqrt{1-x_3^2}}\left(x_1\vec{r_1} + x_2\vec{r_2}\right)\right\rangle = 0,$$

and the pitch is defined as

$$l_N = \left\langle \vec{d^*}, \vec{n} \right\rangle + \left\langle \vec{d}, \vec{n^*} \right\rangle.$$

From the equations

$$\left\langle \vec{d}, \vec{n^*} \right\rangle = \left\langle \left(\oint \psi_3\right)\vec{r_3}, \frac{-1}{\sqrt{1-x_3^2}}\left(\left(x_1^* - \frac{x_3x_3^*x_1}{x_3^2-1}\right)\vec{r_1} + \left(x_2 - \frac{x_3x_3^*x_2}{x_3^2-1}\right)\vec{r_2} + x_1\vec{r_1^*} + x_2\vec{r_2^*}\right)\right\rangle$$
$$= \frac{-x_1}{\sqrt{1-x_3^2}}\left(\oint \psi_3\right)\left\langle \vec{r_3}, \vec{r_1^*}\right\rangle - \frac{x_2}{\sqrt{1-x_3^2}}\left(\oint \psi_3\right)\left\langle \vec{r_3}, \vec{r_2^*}\right\rangle$$

and

$$\left\langle \vec{d^*}, \vec{n} \right\rangle = \left\langle \left(\oint \psi_3^*\right)\vec{r_3} + \left(\oint \psi_3\right)\vec{r_3^*}, \frac{-1}{\sqrt{1-x_3^2}}\left(x_1\vec{r_1} + x_2\vec{r_2}\right)\right\rangle$$
$$= \frac{-x_1}{\sqrt{1-x_3^2}}\left(\oint \psi_3\right)\left\langle \vec{r_3^*}, \vec{r_1}\right\rangle$$

$$-\frac{x_2}{\sqrt{1-x_3^2}}\left(\oint\psi_3\right)\left\langle\overrightarrow{r_3^*},\overrightarrow{r_2}\right\rangle,$$

and since we have

$$\left\langle\overrightarrow{r_3},\overrightarrow{r_1^*}\right\rangle=-\left\langle\overrightarrow{r_3^*},\overrightarrow{r_1}\right\rangle$$

and

$$\left\langle\overrightarrow{r_3},\overrightarrow{r_2^*}\right\rangle=-\left\langle\overrightarrow{r_3^*},\overrightarrow{r_2}\right\rangle,$$

the pitch is

$$l_N=0.$$

The dual angle of pitch is

$$\Lambda_N=l_N-\varepsilon\lambda_N=0.$$

So (N) is an extendable ruled surface.

Theorem 8. *The pitch, the angle of pitch, and the dual angle of pitch of the closed ruled surface* (B) *on the canonical coordinate frame are vanishing.*

Proof. This is clear since we have $B=\mathbb{R}^3$, so this ruled surface is extendable.

Theorem 9. *The pitch, the angle of pitch, and the dual angle of pitch of the closed ruled surface* (D_0) *on the canonical coordinate frame are given by*

$$l_{D_0}=\frac{1}{\sqrt{k_2^2+k_1^2}}\left[k_1^*\left(\oint\psi_3\right)+k_1\left(\oint\psi_3^*\right)\right],$$

$$\lambda_{D_0}=\frac{k_1}{\sqrt{k_1^2+k_2^2}}\left(\oint\psi_3\right),$$

and

$$\Lambda_{D_0}=\frac{1}{\sqrt{k_1^2+k_2^2}}\left[k_1^*\left(\oint\psi_3\right)+k_1\left(\oint\psi_3^*\right)\right]-\varepsilon\frac{k_1}{\sqrt{k_1^2+k_2^2}}\left(\oint\psi_3\right),$$

respectively.

Proof. We know that

$$\overrightarrow{D_0}=\frac{\tau\overrightarrow{T}+\kappa\overrightarrow{B}}{\sqrt{\tau^2+\kappa^2}}=\frac{1}{\sqrt{\tau^2+\kappa^2}}\left[\frac{\tau}{\sqrt{1-X_3^2}}\left(-X_2\overrightarrow{R_1}+X_1\overrightarrow{R_2}\right)+\kappa\overrightarrow{R_3}\right]$$

is valid, where $\overrightarrow{D_0}=\overrightarrow{d}+\varepsilon\overrightarrow{d^*}$, $\kappa=k_1+\varepsilon k_1^*$, $\tau=k_2+\varepsilon k_2^*$, $X_i=\overrightarrow{x_i}+\varepsilon\overrightarrow{x_i^*}$ and $\overrightarrow{R_i}=\overrightarrow{r_i}+\varepsilon\overrightarrow{r_i^*}$ for $i=1,2,3$. We have

$$\overrightarrow{d_0}=\frac{1}{\sqrt{k_1^2+k_2^2}}\left[\frac{k_2}{\sqrt{1-x_3^2}}(-x_2\overrightarrow{r_1}+x_1\overrightarrow{r_2})+k_1\overrightarrow{r_3}\right],$$

$$\overrightarrow{d_0^*}=\frac{1}{\sqrt{k_1^2+k_2^2}}\Bigg(-x_2^*\overrightarrow{r_1}+k_2x_1^*\overrightarrow{r_2}-k_2x_2\overrightarrow{r_1^*}+k_2x_1\overrightarrow{r_2^*}+k_1\overrightarrow{r_3^*}+k_1^*\overrightarrow{r_3}-\left(k_2\frac{k_2^*k_2+k_1^*k_1}{k_1^2+k_2^2}+k_2\frac{x_3x_3^*}{x_3^2-1}+k_2^*\right)(x_2\overrightarrow{r_1}+x_1\overrightarrow{r_2})\Bigg).$$

From the definition of the Steiner vector, the angle of pitch is

$$\lambda_{D_0}=\left\langle\overrightarrow{d},\overrightarrow{d_0}\right\rangle=\left\langle\left(\oint\psi_3\right)\overrightarrow{r_3},\frac{1}{\sqrt{k_1^2+k_2^2}}\left(\frac{k_2}{\sqrt{1-x_3^2}}(-x_2\overrightarrow{r_1}+x_1\overrightarrow{r_2})+k_1\overrightarrow{r_3}\right)\right\rangle=\frac{k_1}{\sqrt{k_1^2+k_2^2}}\oint\psi_3,$$

and the pitch of (D_0) is given by

$$l_{D_0}=\left\langle\overrightarrow{d^*},\overrightarrow{d_0}\right\rangle+\left\langle\overrightarrow{d},\overrightarrow{d_0^*}\right\rangle.$$

From

$$\left\langle\overrightarrow{d},\overrightarrow{d_0^*}\right\rangle=\frac{1}{\sqrt{k_1^2+k_2^2}}\left[\frac{-k_2x_2}{\sqrt{1-x_3^2}}\left(\oint\psi_3\right)\left\langle\overrightarrow{r_3},\overrightarrow{r_1^*}\right\rangle+\frac{k_2x_1}{\sqrt{1-x_3^2}}\left(\oint\psi_3\right)\left\langle\overrightarrow{r_3},\overrightarrow{r_2^*}\right\rangle+k_1^*\left(\oint\psi_3\right)\right]$$

and

$$\left\langle\overrightarrow{d^*},\overrightarrow{d_0}\right\rangle=\left\langle\left(\oint\psi_3^*\right)\overrightarrow{r_3}+\left(\oint\psi_3\right)\overrightarrow{r_3^*},\frac{1}{\sqrt{k_1^2+k_2^2}}\left[\frac{k_2}{\sqrt{1-x_3^2}}(-x_2\overrightarrow{r_1}+x_1\overrightarrow{r_2})+k_1\overrightarrow{r_3}\right]\right\rangle$$
$$=\frac{1}{\sqrt{k_1^2+k_2^2}}\left[\frac{-k_2x_2}{\sqrt{1-x_3^2}}\left(\oint\psi_3\right)\left\langle\overrightarrow{r_3^*},\overrightarrow{r_1}\right\rangle+\frac{k_2x_1}{\sqrt{1-x_3^2}}\left(\oint\psi_3\right)\left\langle\overrightarrow{r_3^*},\overrightarrow{r_2}\right\rangle+k_1\left(\oint\psi_3^*\right)\right]$$

and since we have

$$\left\langle\overrightarrow{r_3},\overrightarrow{r_1^*}\right\rangle=-\left\langle\overrightarrow{r_3^*},\overrightarrow{r_1}\right\rangle$$

and

$$\left\langle\overrightarrow{r_3},\overrightarrow{r_2^*}\right\rangle=-\left\langle\overrightarrow{r_3^*},\overrightarrow{r_2}\right\rangle,$$

the pitch is

$$l_{D_0} = \frac{1}{\sqrt{k_2^2 + k_1^2}} \left[k_1^* \left(\oint \psi_3 \right) + k_1 \left(\oint \psi_3^* \right) \right].$$

The dual angle of pitch of (D_0) is

$$\begin{aligned} \Lambda_{D_0} &= l_{D_0} - \varepsilon \lambda_{D_0} \\ &= \frac{1}{\sqrt{k_1^2 + k_2^2}} \left[k_1^* \left(\oint \psi_3 \right) + k_1 \left(\oint \psi_3^* \right) \right] \\ &\quad - \varepsilon \frac{k_1}{\sqrt{k_1^2 + k_2^2}} \left(\oint \psi_3 \right). \end{aligned}$$

This completes the proof.

References

[1] Wilhelm Blaschke. Zur Bewegungsgeometrie auf der Kugel. *S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl.*, 2(3):31–37, 1948.

[2] Wilhelm Blaschke and Hans Robert Müller. *Ebene Kinematik*. Verlag von R. Oldenbourg, München, 1956.

[3] Hasan Hilmi Hacısalihoğlu. On the pitch of a closed ruled surface. *Mech. Mach. Theory*, 7:291–305, 1970.

[4] Hasan Hilmi Hacısalihoğlu. On closed spherical motions. *Quart. Appl. Math.*, 29:269–276, 1971.

[5] Hasan Hilmi Hacısalihoğlu. *Geometry of motions and the theory of quaternions*. Ankara University, 1983. In Turkish.

[6] Ömer Köse. Contributions to the theory of integral invariants of a closed ruled surface. *Mech. Mach. Math.*, 32:269–277, 1997.

[7] Ömer Köse. Kinematic differential geometry of a rigid body in spatial motion using dual vector calculus. I. *Appl. Math. Comput.*, 183(1):17–29, 2006.

[8] Ömer Köse, Celal Cem Sarıoğlu, B. Karabey, and İlhan Karakılıc. Kinematic differential geometry of a rigid body in spatial motion using dual vector calculus. II. *Appl. Math. Comput.*, 182(1):333–358, 2006.

[9] Mehmet Önder, H. Hüseyin Uğurlu, and Ali Çalışkan. The Euler-Savary analogue equations of a point trajectory in Lorentzian spatial motion. *Proc. Nat. Acad. Sci. India Sect. A*, 83(2):119–127, 2013.

[10] Maxim V. Shamolin. Spatial motion of a rigid body in a resisting medium. *Internat. Appl. Mech.*, 46(7):835–846, 2010.

[11] Hellmuth Stachel. Instantaneous spatial kinematics and invariants of the axodes. In *Proceedings Ball 2000 Symposium*, number 23, Cambridge, 2000.

[12] Eduard Study. *Geometrie der Dynamen*. B. G. Teubner, Leipzig, 1903.

[13] An Tzu Yang. *Application of quaternion algebra and dual numbers to the analysis of spatial mechanisms*. PhD thesis, Columbia University, 1963.

Nemat Abazari was born in Ardabil, Iran, in 1972. He received the B. S. degree from the University of Tabriz, Iran, in 1994, M. S. degree from the Valiasr University of Rafsanjan, Rafsanjan, Iran, 2001, and Ph. D. degree in Geometry, Department of Mathematics, Ankara University, Ankara, Turkey, in 2011. He is currently an Associate Professor in University of Mohaghegh Ardabili, Ardabil, Iran, since 2011.

Ilgin Sağer received the M. S. degree in differential geometry from Ankara University, Turkey, in 2005. She was working as a lecturer in Izmir University of Economics, Turkey, between 2003 and 2012. Currently she is a doctoral student in Ankara University, Turkey. Her research interests are symplectic geometry and the theory of elastic curves in Lorentz–Minkowski spaces.

H. Hilmi Hacisalihoğlu received the Ph. D. degree in differential geometry from Ankara University, Turkey, in 1966. He is the author of 33 books in the area of differential geometry, written in his native language, and more than 80 national and international publications, Editor-in-Chief of many national journals. Professor Hacisalihoğlu is the Vice President of the Balkan Mathematicians Union and a member of the organization commitee of the Balkan Mathematical Olympiad. He is a member of Turkish and European Mathematicians Associations. He has been on duty in administrative and academic positions in many universities in Turkey and he is the department head of Bilecik Seyh Edebali University, Turkey since 2007.