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Existence of Solutions for Second-Order Impulsive q-Difference Equations with Integral Boundary Conditions

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Abstract: Recently the authors introduced in [1] the notions of q_k-derivative and q_k-integral of a function on finite intervals. As applications existence and uniqueness results for initial value problems for first and second order impulsive q_k-difference equations was proved. In this paper, we study the existence and uniqueness of solutions for a boundary value problem of nonlinear second-order impulsive q_k-difference equations with integral boundary conditions. Two results are obtained by applying Banach contraction principle and Leray-Schauder's Nonlinear Alternative. Some examples are presented to illustrate the results.

Keywords: q_k-derivative; q_k-integral; impulsive q_k-difference equation; existence; uniqueness; fixed point theorems

1 Introduction and Preliminaries

In this article, we investigate the following nonlinear second-order impulsive q_k-difference equation with integral boundary conditions

$$\begin{cases} D_{q_k}^2 u(t) = f(t,u(t)), \quad t \in J := [0,T],\ t \neq t_k, \\ \Delta u(t_k) = I_k\left(u(t_k)\right), \quad k = 1,2,\dots,m, \\ D_{q_k} u(t_k^+) - D_{q_{k-1}} u(t_k) = I_k^*\left(u(t_k)\right), \\ \qquad k = 1,2,\dots,m, \\ u(0) = 0, \quad u(T) = \displaystyle\sum_{i=0}^{m} \alpha_i \int_{t_i}^{t_{i+1}} u(s) d_{q_i} s, \end{cases} \tag{1}$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $I_k, I_k^* \in C(\mathbb{R},\mathbb{R})$, $\Delta u(t_k) = u(t_k^+) - u(t_k)$, $u(t_k^+) = \lim_{h\to 0} u(t_k + h)$ for $k = 1,2,\dots,m$, and $0 < q_k < 1$, $\alpha_k \in \mathbb{R}$ for $k = 0,1,2,\dots,m$ are constants.

The notions of q_k-derivative and q_k-integral on finite intervals were introduced in [1]. For a fixed $k \in \mathbb{N} \cup \{0\}$ let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and $0 < q_k < 1$ be a constant. We define q_k-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows:

Definition 1.1. Assume $f : J_k \to \mathbb{R}$ is a continuous function and let $t \in J_k$. Then the expression

$$\begin{aligned} D_{q_k} f(t) &= \frac{f(t) - f(q_k t + (1-q_k)t_k)}{(1-q_k)(t-t_k)}, t \neq t_k, \\ D_{q_k} f(t_k) &= \lim_{t\to t_k} D_{q_k} f(t), \end{aligned} \tag{2}$$

is called the q_k-derivative of function f at t. We say that f is q_k-differentiable on J_k provided $D_{q_k} f(t)$ exists for all $t \in J_k$. Note that if $t_k = 0$ and $q_k = q$ in (2), then $D_{q_k} f = D_q f$, where D_q is the well-known q-derivative of the function $f(t)$ defined by

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}. \tag{3}$$

In addition, we should define the higher q_k-derivative of functions.

Definition 1.2. Let $f : J_k \to \mathbb{R}$ is a continuous function, we call the second-order q_k-derivative $D_{q_k}^2 f$ provided $D_{q_k} f$ is q_k-differentiable on J_k with $D_{q_k}^2 f = D_{q_k}(D_{q_k} f) : J_k \to \mathbb{R}$. Similarly, we define higher order q_k-derivative $D_{q_k}^n : J_k \to \mathbb{R}$.

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The q_k-integral is defined as follows:

Definition 1.3. Assume $f : J_k \to \mathbb{R}$ is a continuous function. Then the q_k-integral is defined by

$$\int_{t_k}^{t} f(s) d_{q_k} s = (1-q_k)(t-t_k) \sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1-q_k^n)t_k), \tag{4}$$

for $t \in J_k$. Moreover, if $a \in (t_k, t)$ then the definite q_k-integral is defined by

$$\begin{aligned}\int_{a}^{t} f(s) d_{q_k} s &= \int_{t_k}^{t} f(s) d_{q_k} s - \int_{t_k}^{a} f(s) d_{q_k} s \\ &= (1-q_k)(t-t_k) \sum_{n=0}^{\infty} q_k^n f(q_k^n t + (1-q_k^n)t_k) \\ &\quad -(1-q_k)(a-t_k) \sum_{n=0}^{\infty} q_k^n f(q_k^n a + (1-q_k^n)t_k).\end{aligned}$$

Note that if $t_k = 0$ and $q_k = q$, then (4) reduces to q-integral of a function $f(t)$, defined by $\int_0^t f(s) d_q s = (1-q)t \sum_{n=0}^{\infty} q^n f(q^n t)$ for $t \in [0, \infty)$.

For the basic properties of q_k-derivative and q_k-integral we refer to [1].

The book by Kac and Cheung [2] covers many of the fundamental aspects of the quantum calculus. In recent years, the topic of q-calculus has attracted the attention of several researchers and a variety of new results can be found in the papers [3]-[15] and the references cited therein.

Impulsive differential equations serve as basic models to study the dynamics of processes that are subject to sudden changes in their states. Recent development in this field has been motivated by many applied problems, such as control theory, population dynamics and medicine. For some recent works on the theory of impulsive differential equations, we refer the interested reader to the monographs [16]-[18].

In this paper we prove existence and uniqueness results for the impulsive boundary value problem (1) by using Banach's contraction mapping principle and Leray-Schauder's nonlinear alternative. The rest of this paper is organized as follows: In Section 2 we present an auxiliary lemma which is used to convert the impulsive boundary value problem (1) into an equivalent integral equation. The main results are given in Section 3, while examples illustrating the results are presented in Section 4.

2 An auxiliary lemma

Let $J = [0,T]$, $J_0 = [t_0, t_1]$, $J_k = (t_k, t_{k+1}]$ for $k = 1, 2, \ldots, m$. Let $PC(J, \mathbb{R}) = \{x : J \to \mathbb{R} : x(t)$ is continuous everywhere except for some t_k at which $x(t_k^+)$ and $x(t_k^-)$ exist and $x(t_k^-) = x(t_k)$, $k = 1, 2, \ldots, m\}$. $PC(J, \mathbb{R})$ is a Banach space with the norm $\|x\|_{PC} = \sup\{|x(t)|; t \in J\}$.

Lemma 2.1. Let $T \neq \sum_{i=0}^{m} \frac{\alpha_i (t_{i+1} - t_i)(t_{i+1} + q_i t_i)}{1+q_i}$. The unique solution of problem (1) is given by

$$\begin{aligned} u(t) &= \frac{t}{\Lambda} \sum_{i=0}^{m} \alpha_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \int_{t_i}^{\tau} f(r, u(r)) d_{q_i} r d_{q_i} \tau d_{q_i} s \\ &+ \frac{t}{\Lambda} \sum_{i=1}^{m} \sum_{k=1}^{i} \left(\int_{t_{k-1}}^{t_k} f(s, u(s)) d_{q_{k-1}} s + I_k^*(u(t_k)) \right) H_{ik} \\ &+ \frac{t}{\Lambda} \sum_{i=1}^{m} \sum_{k=1}^{i} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} f(\tau, u(\tau)) d_{q_{k-1}} \tau d_{q_{k-1}} s \right. \\ &\left. + I_k(u(t_k)) \right) \alpha_i (t_{i+1} - t_i) \\ &- \frac{t}{\Lambda} \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} \int_{t_{t_{k-1}}}^{s} f(\tau, u(\tau)) d_{q_{k-1}} \tau d_{q_{k-1}} s + I_k(u(t_k)) \right) \\ &- \frac{t}{\Lambda} \sum_{k=1}^{m} \left(\int_{t_{k-1}}^{t_k} f(s, u(s)) d_{q_{k-1}} s + I_k^*(u(t_k)) \right) (T - t_k) \\ &- \frac{t}{\Lambda} \int_{t_m}^{T} \int_{t_m}^{s} f(\tau, u(\tau)) d_{q_m} \tau d_{q_m} s \\ &+ \int_{t_k}^{t} \int_{t_k}^{s} f(\tau, u(\tau)) d_{q_k} \tau d_{q_k} s \\ &+ \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{s} f(\tau, u(\tau)) d_{q_{k-1}} \tau d_{q_{k-1}} s + I_k(u(t_k)) \right) \\ &+ \sum_{0<t_k<t} \left(\int_{t_{k-1}}^{t_k} f(s, u(s)) d_{q_{k-1}} s + I_k^*(u(t_k)) \right) (t - t_k), \end{aligned} \tag{5}$$

with $\sum_{0<0}(\cdot) = 0$, where constants Λ, H_{ik} are defined by

$$\Lambda = T - \sum_{i=0}^{m} \frac{\alpha_i (t_{i+1} - t_i)(t_{i+1} + q_i t_i)}{1+q_i}, \tag{6}$$

and

$$H_{ik} = \frac{\alpha_i (t_{i+1} - t_i)(t_{i+1} + q_i t_i - t_k(1+q_i))}{1+q_i}, \quad i = 0, \ldots, m, \ k = 1, \ldots, m. \tag{7}$$

Proof. Taking q_0-integral for (1) from 0 to t, it follows that

$$D_{q_0} u(t) = D_{q_0} u(0) + \int_0^t f(s, u(s)) d_{q_0} s. \tag{8}$$

For $t = t_1$, we have

$$D_{q_0} u(t_1) = D_{q_0} u(0) + \int_0^{t_1} f(s, u(s)) d_{q_0} s. \tag{9}$$

Using q_0-integral for (8) and setting $u(0) = A$ and $D_{q_0} u(0) = B$, we get

$$\begin{aligned} u(t) &= u(0) + D_{q_0} u(0) t + \int_0^t \int_0^s f(\tau, u(\tau)) d_{q_0} \tau d_{q_0} s \\ &= A + Bt + \int_0^t \int_0^s f(\tau, u(\tau)) d_{q_0} \tau d_{q_0} s. \end{aligned}$$

In particular, for $t = t_1$

$$u(t_1) = A + Bt_1 + \int_0^{t_1}\int_0^s f(\tau,u(\tau))d_{q_0}\tau d_{q_0}s. \tag{10}$$

For $t \in J_1 = (t_1,t_2]$, q_1-integrating (1), we have

$$D_{q_1}u(t) = D_{q_1}u(t_1^+) + \int_{t_1}^t f(s,u(s))d_{q_1}s.$$

Using the second impulsive condition of (1) with (9), one has

$$\begin{aligned} D_{q_1}u(t) &= B + \int_0^{t_1} f(s,u(s))d_{q_0}s + I_1^*(u(t_1)) \\ &\quad + \int_{t_1}^t f(s,u(s))d_{q_1}s. \end{aligned} \tag{11}$$

Applying q_1-integral to (11) for $t \in J_1$, we obtain

$$\begin{aligned} u(t) &= u(t_1^+) + \left[B + \int_0^{t_1} f(s,u(s))d_{q_0}s + I_1^*(u(t_1))\right](t - t_1) \\ &\quad + \int_{t_1}^t\int_{t_1}^s f(\tau,u(\tau))d_{q_1}\tau d_{q_1}s. \end{aligned} \tag{12}$$

The first impulsive condition of (1) with (10) and (12) implies

$$\begin{aligned} u(t) &= A + Bt_1 + \int_0^{t_1}\int_0^s f(\tau,u(\tau))d_{q_0}\tau d_{q_0}s + I_1(u(t_1)) \\ &\quad + \left[B + \int_0^{t_1} f(s,u(s))d_{q_0}s + I_1^*(u(t_1))\right](t - t_1) \\ &\quad + \int_{t_1}^t\int_{t_1}^s f(\tau,u(\tau))d_{q_1}\tau d_{q_1}s \\ &= A + Bt + \int_0^{t_1}\int_0^s f(\tau,u(\tau))d_{q_0}\tau d_{q_0}s + I_1(u(t_1)) \\ &\quad + \left[\int_0^{t_1} f(s,u(s))d_{q_0}s + I_1^*(u(t_1))\right](t - t_1) \\ &\quad + \int_{t_1}^t\int_{t_1}^s f(\tau,u(\tau))d_{q_1}\tau d_{q_1}s. \end{aligned}$$

Repeating the above process, for $t \in J$, we obtain

$$\begin{aligned} &u(t) = A + Bt \\ &+ \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^s f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s + I_k(u(t_k))\right) \\ &+ \sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k} f(s,u(s))d_{q_{k-1}}s + I_k^*(u(t_k))\right)(t - t_k) \\ &+ \int_{t_k}^t\int_{t_k}^s f(\tau,u(\tau))d_{q_k}\tau d_{q_k}s. \end{aligned} \tag{13}$$

For $t = T$, we have

$$\begin{aligned} u(T) &= A + BT \\ &\quad + \sum_{k=1}^m\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^s f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s + I_k(u(t_k))\right) \\ &\quad + \sum_{k=1}^m\left(\int_{t_{k-1}}^{t_k} f(s,u(s))d_{q_{k-1}}s + I_k^*(u(t_k))\right)(T - t_k) \\ &\quad + \int_{t_m}^T\int_{t_m}^s f(\tau,u(\tau))d_{q_m}\tau d_{q_m}s. \end{aligned}$$

From the first boundary condition $u(0) = 0$, it follows that $A = 0$.
On the other hand, we have

$$\begin{aligned} &\sum_{i=0}^m \alpha_i \int_{t_i}^{t_{i+1}} u(s)d_{q_i}s \\ &= B\sum_{i=0}^m \frac{\alpha_i(t_{i+1}-t_i)(t_{i+1}+q_it_i)}{1+q_i} \\ &\quad + \sum_{i=1}^m\sum_{k=1}^i\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^s f(s,u(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s \right. \\ &\quad \left. + I_k(u(t_k))\right)\alpha_i(t_{i+1}-t_i) \\ &\quad + \sum_{i=1}^m\sum_{k=1}^i\left(\int_{t_{k-1}}^{t_k} f(s,u(s))d_{q_{k-1}}s + I_k^*(u(t_k))\right)H_{ik} \\ &\quad + \sum_{i=0}^m \alpha_i\int_{t_i}^{t_{i+1}}\int_{t_i}^s\int_{t_i}^\tau f(r,u(r))d_{q_i}rd_{q_i}\tau d_{q_i}s, \end{aligned}$$

with $\sum_{0<0}(\cdot) = 0$.
It follows from the second boundary condition that

$$\begin{aligned} B &= \frac{1}{\Lambda}\sum_{i=1}^m\sum_{k=1}^i\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^s f(s,u(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s \right. \\ &\quad \left. + I_k(u(t_k))\right)\alpha_i(t_{i+1}-t_i) \\ &\quad + \frac{1}{\Lambda}\sum_{i=1}^m\sum_{k=1}^i\left(\int_{t_{k-1}}^{t_k} f(s,u(s))d_{q_{k-1}}s + I_k^*(u(t_k))\right)H_{ik} \\ &\quad + \frac{1}{\Lambda}\sum_{i=0}^m \alpha_i\int_{t_i}^{t_{i+1}}\int_{t_i}^s\int_{t_i}^\tau f(r,u(r))d_{q_i}rd_{q_i}\tau d_{q_i}s \\ &\quad - \frac{1}{\Lambda}\sum_{k=1}^m\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^s f(\tau,x(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s + I_k(u(t_k))\right) \\ &\quad - \frac{1}{\Lambda}\sum_{k=1}^m\left(\int_{t_{k-1}}^{t_k} f(s,u(s))d_{q_{k-1}}s + I_k^*(u(t_k))\right)(T - t_k) \\ &\quad - \frac{1}{\Lambda}\int_{t_m}^T\int_{t_m}^s f(\tau,u(\tau))d_{q_m}\tau d_{q_m}s. \end{aligned}$$

Substituting the values of constants A and B into (13), we obtain (5) as requested. □

3 Main Results

In view of Lemma 2.1, we define an operator $\mathscr{A} : PC(J,\mathbb{R}) \to PC(J,\mathbb{R})$ by

$$\begin{aligned} &(\mathscr{A}u)(t) \\ &= \frac{t}{\Lambda}\sum_{i=0}^m \alpha_i\int_{t_i}^{t_{i+1}}\int_{t_i}^s\int_{t_i}^\tau f(r,u(r))d_{q_i}rd_{q_i}\tau d_{q_i}s \\ &+ \frac{t}{\Lambda}\sum_{i=1}^m\sum_{k=1}^i\left(\int_{t_{k-1}}^{t_k} f(s,u(s))d_{q_{k-1}}s + I_k^*(u(t_k))\right)H_{ik} \\ &+ \frac{t}{\Lambda}\sum_{i=1}^m\sum_{k=1}^i\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^s f(\tau,u(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s\right. \end{aligned}$$

$$
\begin{aligned}
&+I_k\left(u(t_k)\right)\right)\alpha_i(t_{i+1}-t_i)\\
&-\frac{t}{\Lambda}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}f(\tau,u(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s\right.\\
&\left.+I_k(u(t_k))\right) \qquad (14)\\
&-\frac{t}{\Lambda}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}f(s,u(s))d_{q_{k-1}}s+I_k^*(u(t_k))\right)(T-t_k)\\
&-\frac{t}{\Lambda}\int_{t_m}^{T}\int_{t_m}^{s}f(\tau,u(\tau))d_{q_m}\tau d_{q_m}s\\
&+\int_{t_k}^{t}\int_{t_k}^{s}f(\tau,u(\tau))d_{q_k}\tau d_{q_k}s\\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}f(\tau,u(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k\left(u(t_k)\right)\right)\\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}f(s,u(s))d_{q_{k-1}}s+I_k^*\left(u(t_k)\right)\right)(t-t_k).
\end{aligned}
$$

It should be noticed that problem (1) has solutions if and only if the operator $\mathscr{A}$ has fixed points.

Our first result is an existence and uniqueness result for the impulsive boundary value problem (1) by using Banach's contraction mapping principle.

For convenience, we set:

$$
\begin{aligned}
\Phi_1 =& \frac{TL_1}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\frac{(t_{i+1}-t_i)^3}{1+q_i+q_i^2}\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(L_1(t_k-t_{k-1})+L_3\right)|H_{ik}|\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(L_1\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+L_2\right)|\alpha_i|(t_{i+1}-t_i)\\
&+\frac{T+|\Lambda|}{|\Lambda|}L_1\frac{(T-t_m)^2}{1+q_m} \qquad (15)\\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(L_1\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+L_2\right)\\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(L_1(t_k-t_{k-1})+L_3\right)(T-t_k),
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_2 =& \frac{TM_1}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\frac{(t_{i+1}-t_i)^3}{1+q_i+q_i^2}\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(M_1(t_k-t_{k-1})+M_3\right)|H_{ik}|\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(M_1\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+M_2\right)|\alpha_i|(t_{i+1}-t_i)\\
&+\frac{T+|\Lambda|}{|\Lambda|}M_1\frac{(T-t_m)^2}{1+q_m} \qquad (16)\\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(M_1\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+M_2\right)\\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(M_1(t_k-t_{k-1})+M_3\right)(T-t_k).
\end{aligned}
$$

Theorem 3.1. Assume that the following conditions hold:

(H_1) The function $f : J\times\mathbb{R}\to\mathbb{R}$ is continuous and there exists a constant $L_1>0$ such that

$$|f(t,x)-f(t,y)|\le L_1|x-y|,$$

for each $t\in J$ and $x,y\in\mathbb{R}$.

(H_2) The functions $I_k, I_k^*:\mathbb{R}\to\mathbb{R}$ are continuous and there exist constants L_2, $L_3>0$ such that

$$|I_k(x)-I_k(y)|\le L_2|x-y| \text{ and } |I_k^*(x)-I_k^*(y)|\le L_3|x-y|,$$

for each $x,y\in\mathbb{R}$.

If

$$\Phi_1\le\delta<1, \qquad (17)$$

where Φ_1 is defined by (15), then the impulsive q_k-difference boundary value problem (1) has a unique solution on J.

Proof. By transforming the boundary value problem (1) into a fixed point problem, $u=\mathscr{A}u$, where the operator $\mathscr{A}$ is defined by (14) and by using the Banach's contraction mapping principle, we shall show that $\mathscr{A}$ has a fixed point, which is the unique solution of the boundary value problem (1).

We define the following constants as $M_1=\sup_{t\in J}|f(t,0)|$, $M_2=\sup\{I_k(0);k=1,2,\dots,m\}$ and $M_3=\sup\{I_k^*(0);k=1,2,\dots,m\}$. By choosing

$$\rho\ge\frac{\Phi_2}{1-\varepsilon},$$

where $\delta\le\varepsilon<1$ and Φ_2 is defined by (16), we shall show that $\mathscr{A}B_\rho\subset B_\rho$, where the set $B_\rho=\{u\in PC(J,\mathbb{R}):\|u\|\le\rho\}$. For any $u\in B_\rho$, we have

$$
\begin{aligned}
&\|\mathscr{A}u\|\\
&\le\sup_{t\in J}\left\{\frac{t}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\int_{t_i}^{\tau}|f(r,u(r))|d_{q_i}rd_{q_i}\tau d_{q_i}s\right.\\
&+\frac{t}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s+|I_k^*(u(t_k))|\right)|H_{ik}|\\
&+\frac{t}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right.\\
&\left.+|I_k(u(t_k))|\right)|\alpha_i|(t_{i+1}-t_i)\\
&+\frac{t}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right.\\
&\left.+|I_k(u(t_k))|\right)\\
&+\frac{t}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s+|I_k^*(u(t_k))|\right)(T-t_k)\\
&+\frac{t}{|\Lambda|}\int_{t_m}^{T}\int_{t_m}^{s}|f(\tau,u(\tau))|d_{q_m}\tau d_{q_m}s
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t_k}^{t}\int_{t_k}^{s}|f(\tau,u(\tau))|d_{q_k}\tau d_{q_k}s \\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s \right.\\
&\left.+|I_k(u(t_k))|\right)\\
&+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s+|I_k^*(u(t_k))|\right)(t-t_k)\Bigg\}\\
\le&\ \frac{T}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\int_{t_i}^{\tau}|f(r,u(r))|d_{q_i}rd_{q_i}\tau d_{q_i}s\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s+|I_k^*(u(t_k))|\right)|H_{ik}|\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right.\\
&\left.+|I_k(u(t_k))|\right)|\alpha_i|(t_{i+1}-t_i)\\
&+\frac{T}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right.\\
&\left.+|I_k(u(t_k))|\right)\\
&+\frac{T}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s+|I_k^*(u(t_k))|\right)(T-t_k)\\
&+\frac{T}{|\Lambda|}\int_{t_m}^{T}\int_{t_m}^{s}|f(\tau,u(\tau))|d_{q_m}\tau d_{q_m}s\\
&+\int_{t_m}^{T}\int_{t_m}^{s}|f(\tau,u(\tau))|d_{q_m}\tau d_{q_m}s\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s+|I_k(u(t_k))|\right)\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s+|I_k^*(u(t_k))|\right)(T-t_k).
\end{aligned}
$$

Applying the following inequalities

$$
\begin{aligned}
&|f(t,u)|\le|f(t,u)-f(t,0)|+|f(t,0)|\le\rho L_1+M_1,\\
&|I_k(u)|\le|I_k(u)-I_k(0)|+|I_k(0)|\le\rho L_2+M_2,\\
&|I_k^*(u)|\le|I_k^*(u)-I_k^*(0)|+|I_k^*(0)|\le\rho L_3+M_3,
\end{aligned}
$$

we obtain

$$
\begin{aligned}
&\|\mathscr{A}u\|\\
\le&\ \frac{T}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\int_{t_i}^{\tau}(\rho L_1+M_1)d_{q_i}rd_{q_i}\tau d_{q_i}s\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}(\rho L_1+M_1)d_{q_{k-1}}s+(\rho L_3+M_3)\right)|H_{ik}|\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(\rho L_1+M_1)d_{q_{k-1}}\tau d_{q_{k-1}}s\right.\\
&\left.+(\rho L_2+M_2)\right)|\alpha_i|(t_{i+1}-t_i)\\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}(\rho L_1+M_1)d_{q_{k-1}}\tau d_{q_{k-1}}s\right.\\
&\left.+(\rho L_2+M_2)\right)\\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}(\rho L_1+M_1)d_{q_{k-1}}s\right.\\
&\left.+(\rho L_3+M_3)\right)(T-t_k)\\
&+\frac{T+|\Lambda|}{|\Lambda|}\int_{t_m}^{T}\int_{t_m}^{s}(\rho L_1+M_1)d_{q_m}\tau d_{q_m}s\\
=&\ \rho\Phi_1+\Phi_2\le\rho.
\end{aligned}
$$

This shows that $\mathscr{A}B_\rho\subset B_\rho$.
For $u,v\in PC(J,\mathbb{R})$ and for each $t\in J$, we have

$$
\begin{aligned}
&|\mathscr{A}u(t)-\mathscr{A}v(t)|\\
\le&\ \frac{T}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\int_{t_i}^{\tau}|f(r,u(r))-f(r,v(r))|d_{q_i}rd_{q_i}\tau d_{q_i}s\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))-f(s,v(s))|d_{q_{k-1}}s\right.\\
&\left.+|I_k^*(u(t_k))-I_k^*(v(t_k))|\right)|H_{ik}|\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))-f(\tau,v(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right.\\
&\left.+|I_k(u(t_k))-I_k(v(t_k))|\right)|\alpha_i|(t_{i+1}-t_i)\\
&+\frac{T}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))-f(\tau,v(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right.\\
&\left.+|I_k(u(t_k))-I_k(v(t_k))|\right)\\
&+\frac{T}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))-f(s,v(s))|d_{q_{k-1}}s\right.\\
&\left.+|I_k^*(u(t_k))-I_k^*(v(t_k))|\right)(T-t_k)\\
&+\frac{T}{|\Lambda|}\int_{t_m}^{T}\int_{t_m}^{s}|f(\tau,u(\tau))-f(\tau,v(\tau))|d_{q_m}\tau d_{q_m}s\\
&+\int_{t_m}^{T}\int_{t_m}^{s}|f(\tau,u(\tau))-f(\tau,v(\tau))|d_{q_m}\tau d_{q_m}s\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))-f(\tau,v(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right.\\
&\left.+|I_k(u(t_k))-I_k(v(t_k))|\right)\\
&+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))-f(s,v(s))|d_{q_{k-1}}s\right.\\
&\left.+|I_k^*(u(t_k))-I_k^*(v(t_k))|\right)(T-t_k)\\
\le&\ \frac{T}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\int_{t_i}^{\tau}L_1d_{q_i}rd_{q_i}\tau d_{q_i}s\|u-v\|\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}L_1d_{q_{k-1}}s+L_3\right)|H_{ik}|\|u-v\|
\end{aligned}
$$

$$+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}L_1 d_{q_{k-1}}\tau d_{q_{k-1}}s\right.$$
$$\left.+L_2\right)|\alpha_i|(t_{i+1}-t_i)\|u-v\|$$
$$+\frac{T}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}L_1 d_{q_{k-1}}\tau d_{q_{k-1}}s+L_2\right)\|u-v\|$$
$$+\frac{T}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}L_1 d_{q_{k-1}}s+L_3\right)(T-t_k)\|u-v\|$$
$$+\frac{T}{|\Lambda|}\int_{t_m}^{T}\int_{t_m}^{s}L_1 d_{q_m}\tau d_{q_m}s\|u-v\|$$
$$+\int_{t_m}^{T}\int_{t_m}^{s}L_1 d_{q_m}\tau d_{q_m}s\|u-v\|$$
$$+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}L_1 d_{q_{k-1}}\tau d_{q_{k-1}}s+L_2\right)\|u-v\|$$
$$+\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}L_1 d_{q_{k-1}}s+L_3\right)(T-t_k)\|u-v\|$$
$$=\Phi_1\|u-v\|.$$

Therefore,

$$\|\mathscr{A}u-\mathscr{A}v\|\leq\Phi_1\|u-v\|.$$

From (17), $\mathscr{A}$ is a contraction. As a consequence of the Banach fixed point theorem, we conclude that $\mathscr{A}$ has a fixed point which is the unique solution of the problem (1). □

Our second main result is an existence result based on Leray-Schauder's nonlinear alternative.

Lemma 3.2. (Nonlinear alternative for single valued maps)[19]. Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0\in U$. Suppose that $F:\overline{U}\to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either

(i) F has a fixed point in $\overline{U}$, or
(ii) there is a $u\in\partial U$ (the boundary of U in C) and $\theta\in(0,1)$ with $u=\theta F(u)$.

Theorem 3.3. Assume that:

(H_4) There exist a continuous nondecreasing function $\psi:[0,\infty)\to(0,\infty)$ and a continuous function $p:J\to\mathbb{R}^+$ such that

$$|f(t,u)|\leq p(t)\psi(|u|)\ \text{ for each }\ (t,u)\in J\times\mathbb{R}.$$

(H_5) There exist continuous nondecreasing functions $\varphi_1,\varphi_2:[0,\infty)\to(0,\infty)$ such that

$$|I_k(u)|\leq\varphi_1(|u|)\ \text{ and }\ |I_k^*(u)|\leq\varphi_2(|u|)$$

for all $u\in\mathbb{R}$, $k=1,2,\ldots,m$.

(H_6) There exists a constant $M^*>0$ such that

$$\frac{M^*}{p_0\psi(M^*)Q_0+\varphi_1(M^*)Q_1+\varphi_2(M^*)Q_2}>1,$$

where $p_0=\max\{p(t):t\in J\}$ and

$$Q_0=\frac{T}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\frac{(t_{i+1}-t_i)^3}{1+q_i+q_i^2}+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}(t_k-t_{k-1})|H_{ik}|$$
$$+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}|\alpha_i|\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}(t_{i+1}-t_i)$$
$$+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m+1}\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}$$
$$+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}(t_k-t_{k-1})(T-t_k),$$
$$Q_1=\frac{T}{|\Lambda|}\sum_{i=1}^{m}i|\alpha_i|(t_{i+1}-t_i)+\frac{m(T+|\Lambda|)}{|\Lambda|},$$
$$Q_2=\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}|H_{ik}|+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}(T-t_k).$$

Then the impulsive boundary value problem (1) has at least one solution on J.

Proof. Firstly, we shall show that $\mathscr{A}$ *maps bounded sets (balls) into bounded sets in* $PC(J,\mathbb{R})$. For a positive number $\overline{\rho}$, let $B_{\overline{\rho}}=\{u\in PC(J,\mathbb{R}):\|u\|\leq\overline{\rho}\}$ be a bounded ball in $PC(J,\mathbb{R})$. Then, for $t\in J$, we have

$$|(\mathscr{A}u)(t)|$$
$$\leq\frac{t}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\int_{t_i}^{\tau}|f(r,u(r))|d_{q_i}rd_{q_i}\tau d_{q_i}s$$
$$+\frac{t}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s+|I_k^*(u(t_k))|\right)|H_{ik}|$$
$$+\frac{t}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right.$$
$$\left.+|I_k(u(t_k))|\right)|\alpha_i|(t_{i+1}-t_i)$$
$$+\frac{t}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right.$$
$$\left.+|I_k(u(t_k))|\right)$$
$$+\frac{t}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s+|I_k^*(u(t_k))|\right)(T-t_k)$$
$$+\frac{t}{|\Lambda|}\int_{t_m}^{T}\int_{t_m}^{s}|f(\tau,u(\tau))|d_{q_m}\tau d_{q_m}s$$
$$+\int_{t_k}^{t}\int_{t_k}^{s}|f(\tau,u(\tau))|d_{q_k}\tau d_{q_k}s$$
$$+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right.$$
$$\left.+|I_k(u(t_k))|\right)$$
$$+\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s+|I_k^*(u(t_k))|\right)(t-t_k)$$

$$
\begin{aligned}
\leq\ & \frac{T}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\int_{t_i}^{\tau}|f(r,u(r))|d_{q_i}rd_{q_i}\tau d_{q_i}s \\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s+|I_k^*(u(t_k))|\right)|H_{ik}| \\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right. \\
&\left.+|I_k(u(t_k))|\right)|\alpha_i|(t_{i+1}-t_i) \\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right. \\
&\left.+|I_k(u(t_k))|\right) \\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s\right. \\
&\left.+|I_k^*(u(t_k))|\right)(T-t_k) \\
&+\frac{T+|\Lambda|}{|\Lambda|}\int_{t_m}^{T}\int_{t_m}^{s}|f(\tau,u(\tau))|d_{q_m}\tau d_{q_m}s \\
\leq\ & \frac{T}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\int_{t_i}^{\tau}p_0\psi(\|u\|)d_{q_i}rd_{q_i}\tau d_{q_i}s \\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}p_0\psi(\|u\|)d_{q_{k-1}}s+\varphi_2(\|u\|)\right)|H_{ik}| \\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}p_0\psi(\|u\|)d_{q_{k-1}}\tau d_{q_{k-1}}s\right. \\
&\left.+\varphi_1(\|u\|)\right)|\alpha_i|(t_{i+1}-t_i) \\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}p_0\psi(\|u\|)d_{q_{k-1}}\tau d_{q_{k-1}}s\right. \\
&\left.+\varphi_1(\|u\|)\right) \\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}p_0\psi(\|u\|)d_{q_{k-1}}s\right. \\
&\left.+\varphi_2(\|u\|)\right)(T-t_k) \\
&+\frac{T+|\Lambda|}{|\Lambda|}\int_{t_m}^{T}\int_{t_m}^{s}p_0\psi(\|u\|)d_{q_m}\tau d_{q_m}s \\
\leq\ & \frac{p_0\psi(\overline{\rho})T}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\frac{(t_{i+1}-t_i)^3}{1+q_i+q_i^2} \\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(p_0\psi(\overline{\rho})(t_k-t_{k-1})+\varphi_2(\overline{\rho})\right)|H_{ik}| \\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(p_0\psi(\overline{\rho})\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}\right. \\
&\left.+\varphi_1(\overline{\rho})\right)|\alpha_i|(t_{i+1}-t_i) \\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(p_0\psi(\overline{\rho})\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\varphi_1(\overline{\rho})\right) \\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(p_0\psi(\overline{\rho})(t_k-t_{k-1})+\varphi_2(\overline{\rho})\right)(T-t_k) \\
&+\frac{T+|\Lambda|}{|\Lambda|}p_0\psi(\overline{\rho})\frac{(T-t_m)^2}{1+q_m} \\
:=\ & K.
\end{aligned}
$$

Therefore, we conclude that $\|\mathscr{A}u\|\leq K$.

Next we show that $\mathscr{A}$ *maps bounded sets into equicontinuous sets of* $PC(J,\mathbb{R})$. Let $\tau_1,\tau_2\in J_n$ for some $n\in\{0,1,2,\ldots,m\}$, $\tau_1<\tau_2$, $B_{\overline{\rho}}$ be a bounded set of $PC(J,\mathbb{R})$, and let $u\in B_{\overline{\rho}}$. Then we have:

$$
\begin{aligned}
&|(\mathscr{A}u)(\tau_2)-(\mathscr{A}u)(\tau_1)| \\
\leq\ & \frac{|\tau_2-\tau_1|}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\int_{t_i}^{\tau}|f(r,u(r))|d_{q_i}rd_{q_i}\tau d_{q_i}s \\
&+\frac{|\tau_2-\tau_1|}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s\right. \\
&\left.+|I_k^*(u(t_k))|\right)|H_{ik}| \\
&+\frac{|\tau_2-\tau_1|}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right. \\
&\left.+|I_k(u(t_k))|\right)|\alpha_i|(t_{i+1}-t_i) \\
&+\frac{|\tau_2-\tau_1|}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}|f(\tau,u(\tau))|d_{q_{k-1}}\tau d_{q_{k-1}}s\right. \\
&\left.+|I_k(u(t_k))|\right) \\
&+\frac{|\tau_2-\tau_1|}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s\right. \\
&\left.+|I_k^*(u(t_k))|\right)(T-t_k) \\
&+\frac{|\tau_2-\tau_1|}{|\Lambda|}\int_{t_m}^{T}\int_{t_m}^{s}|f(\tau,u(\tau))|d_{q_m}\tau d_{q_m}s \\
&+|\tau_2-\tau_1|\sum_{k=1}^{n}\left(\int_{t_{k-1}}^{t_k}|f(s,u(s))|d_{q_{k-1}}s+|I_k^*(u(t_k))|\right) \\
&+\left|\int_{t_n}^{\tau_2}\int_{t_n}^{s}f(\tau,u(\tau))d_{q_n}\tau d_{q_n}s-\int_{t_n}^{\tau_1}\int_{t_n}^{s}f(\tau,u(\tau))d_{q_n}\tau d_{q_n}s\right| \\
\leq\ & \frac{|\tau_2-\tau_1|}{|\Lambda|}p_0\psi(\overline{\rho})\sum_{i=0}^{m}|\alpha_i|\frac{(t_{i+1}-t_i)^3}{1+q_i+q_i^2} \\
&+\frac{|\tau_2-\tau_1|}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(p_0\psi(\overline{\rho})(t_k-t_{k-1})+\varphi_2(\overline{\rho})\right)|H_{ik}| \\
&+\frac{|\tau_2-\tau_1|}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(p_0\psi(\overline{\rho})\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}\right. \\
&\left.+\varphi_1(\overline{\rho})\right)|\alpha_i|(t_{i+1}-t_i) \\
&+\frac{|\tau_2-\tau_1|}{|\Lambda|}\sum_{k=1}^{m}\left(p_0\psi(\overline{\rho})\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\varphi_1(\overline{\rho})\right) \\
&+\frac{|\tau_2-\tau_1|}{|\Lambda|}\sum_{k=1}^{m}\left(p_0\psi(\overline{\rho})(t_k-t_{k-1})+\varphi_2(\overline{\rho})\right)(T-t_k) \\
&+\frac{|\tau_2-\tau_1|}{|\Lambda|}p_0\psi(\overline{\rho})\frac{(T-t_m)^2}{1+q_m} \\
&+|\tau_2-\tau_1|\sum_{k=1}^{n}\left(p_0\psi(\overline{\rho})(t_k-t_{k-1})+\varphi_2(\overline{\rho})\right)
\end{aligned}
$$

$$+|\tau_2-\tau_1|p_0\psi(\overline{\rho})\frac{[\tau_2+\tau_1+2t_n]}{1+q_n}.$$

The right-hand side of the above inequality is independent of u and tends to zero as $\tau_1\to\tau_2$. As a consequence of the previous results, together with the Arzelá-Ascoli theorem, we conclude that $\mathscr{A}:PC(J,\mathbb{R})\to PC(J,\mathbb{R})$ is completely continuous.

Our result will follows from the Leray-Schauder nonlinear alternative (Lemma 3) if we prove the boundendness of the set of all solutions to equations $u(t)=\lambda(\mathscr{A}u)(t)$ for some $0<\lambda<1$.

Let x be a solution. Thus, for each $t\in J$, we have

$$\begin{aligned}
&\lambda(\mathscr{A}u)(t)\\
=&\frac{\lambda t}{\Lambda}\sum_{i=0}^{m}\alpha_i\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\int_{t_i}^{\tau}f(r,u(r))d_{q_i}rd_{q_i}\tau d_{q_i}s\\
&+\frac{\lambda t}{\Lambda}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}f(s,u(s))d_{q_{k-1}}s+I_k^*(u(t_k))\right)H_{ik}\\
&+\frac{\lambda t}{\Lambda}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}f(\tau,u(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s\right.\\
&\left.+I_k(u(t_k))\right)\alpha_i(t_{i+1}-t_i)\\
&-\frac{\lambda t}{\Lambda}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}f(\tau,u(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k(u(t_k))\right)\\
&-\frac{\lambda t}{\Lambda}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}f(s,u(s))d_{q_{k-1}}s+I_k^*(u(t_k))\right)(T-t_k)\\
&-\frac{\lambda t}{\Lambda}\int_{t_m}^{T}\int_{t_m}^{s}f(\tau,u(\tau))d_{q_m}\tau d_{q_m}s\\
&+\lambda\int_{t_k}^{t}\int_{t_k}^{s}f(\tau,u(\tau))d_{q_k}\tau d_{q_k}s\\
&+\lambda\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}f(\tau,u(\tau))d_{q_{k-1}}\tau d_{q_{k-1}}s+I_k(u(t_k))\right)\\
&+\lambda\sum_{0<t_k<t}\left(\int_{t_{k-1}}^{t_k}f(s,u(s))d_{q_{k-1}}s+I_k^*(u(t_k))\right)(t-t_k).
\end{aligned}$$

This implies by (H_4) and (H_5) that for each $t\in J$, we have

$$\begin{aligned}
&|\lambda(\mathscr{A}u)(t)|\\
\le&\frac{T}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\int_{t_i}^{\tau}p_0\psi(\|u\|)d_{q_i}rd_{q_i}\tau d_{q_i}s\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}p_0\psi(\|u\|)d_{q_{k-1}}s+\varphi_2(\|u\|)\right)|H_{ik}|\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}p_0\psi(\|u\|)d_{q_{k-1}}\tau d_{q_{k-1}}s\right.\\
&\left.+\varphi_1(\|u\|)\right)|\alpha_i|(t_{i+1}-t_i)\\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{s}p_0\psi(\|u\|)d_{q_{k-1}}\tau d_{q_{k-1}}s\right.\\
&\left.+\varphi_1(\|u\|)\right)\\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(\int_{t_{k-1}}^{t_k}p_0\psi(\|u\|)d_{q_{k-1}}s\right.\\
&\left.+\varphi_2(\|u\|)\right)(T-t_k)\\
&+\frac{T+|\Lambda|}{|\Lambda|}\int_{t_m}^{T}\int_{t_m}^{s}p_0\psi(\|u\|)d_{q_m}\tau d_{q_m}s\\
\le&\frac{T}{|\Lambda|}\sum_{i=0}^{m}|\alpha_i|p_0\psi(\|u\|)\frac{(t_{i+1}-t_i)^3}{1+q_i+q_i^2}\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}(p_0\psi(\|u\|)(t_k-t_{k-1})+\varphi_2(\|u\|))|H_{ik}|\\
&+\frac{T}{|\Lambda|}\sum_{i=1}^{m}\sum_{k=1}^{i}\left(p_0\psi(\|u\|)\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}\right.\\
&\left.+\varphi_1(\|u\|)\right)|\alpha_i|(t_{i+1}-t_i)\\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}\left(p_0\psi(\|u\|)\frac{(t_k-t_{k-1})^2}{1+q_{k-1}}+\varphi_1(\|u\|)\right)\\
&+\frac{T+|\Lambda|}{|\Lambda|}\sum_{k=1}^{m}(p_0\psi(\|u\|)(t_k-t_{k-1})+\varphi_2(\|u\|))(T-t_k)\\
&+\frac{T+|\Lambda|}{|\Lambda|}p_0\psi(\|u\|)\frac{(T-t_m)^2}{1+q_m}\\
=&p_0\psi(\|u\|)Q_0+\varphi_1(\|u\|)Q_1+\varphi_2(\|u\|)Q_2.
\end{aligned}$$

Consequently, we have

$$\frac{\|x\|}{p_0\psi(\|u\|)Q_0+\varphi_1(\|u\|)Q_1+\varphi_2(\|u\|)Q_2}\le 1.$$

In view of (H_6), there exists M^* such that $\|u\|\neq M^*$. Let us set

$$U=\{u\in PC(J,\mathbb{R}):\|u\|<M^*\}.$$

Note that the operator $\mathscr{A}:\overline{U}\to PC(J,\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $u\in\partial U$ such that $u=\lambda\mathscr{A}u$ for some $\lambda\in(0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3), we deduce that $\mathscr{A}$ has a fixed point $u\in\overline{U}$ which is a solution of the problem (1). This completes the proof. □

4 Examples

In this section, we will give some examples to illustrate our main results.

Example 4.1. Consider the following integral boundary value problem of nonlinear second-order impulsive

q_k-difference equation

$$\begin{cases} D^2_{\left(\frac{k^3-k+2}{k^4+2k+3}\right)^{\frac{4}{3}}} u(t) = \dfrac{2t|u(t)|}{(e^{4t}+2)^2(|u(t)|+4)} + e^t \sin \pi t, \\ \qquad t \in J = [0,1],\, t \neq t_k, \\ \Delta u(t_k) = \dfrac{|u(t_k)|}{5(e^{k-1}+2)+|u(t_k)|}, \quad t_k = \dfrac{k}{5},\ k = 1,2,3,4, \\ D_{\left(\frac{k^3-k+2}{k^4+2k+3}\right)^{\frac{4}{3}}} u(t_k^+) - D_{\left(\frac{(k-1)^3-k+3}{(k-1)^4+2k+1}\right)^{\frac{4}{3}}} u(t_k) = \dfrac{2|u(t_k)|}{4(3k+5)+|u(t_k)|}, \\ \qquad t_k = \frac{k}{5}, k = 1,2,3,4, \\ u(0) = 0, \quad u(1) = \displaystyle\sum_{i=0}^{4} (|i-2|+4) \int_{t_i}^{t_{i+1}} u(s) d_{\left(\frac{i^3-i+2}{i^4+2i+3}\right)^{\frac{4}{3}}} s. \end{cases} \tag{18}$$

Here $q_k = \left((k^3-k+2)/(k^4+2k+3)\right)^{4/3}$ for $k = 0,1,2,3,4$, $m = 4$, $T = 1$, $\alpha_i = |i-2|+4$, $f(t,u) = (2t|u(t)|)/((e^{4t}+2)^2(|u(t)|+4)) + e^t \sin \pi t$, $I_k(u) = |u|/(5(e^{k-1}+2)+|u|)$ and $I_k^*(u) = 2|u|/(4(3k+5)+|u|)$. Since

$$|f(t,u) - f(t,v)| \leq (1/18)|u-v|,$$

$$|I_k(u) - I_k(v)| \leq (1/15)|u-v|$$

and

$$|I_k^*(u) - I_k^*(v)| \leq (1/16)|u-v|,$$

then (H_1) and (H_2) are satisfied with $L_1 = (1/18)$, $L_2 = (1/15)$, $L_3 = (1/16)$. We can show that

$\Phi_1 \approx 0.8846645855 < 1.$

Hence, by Theorem 3.1, the impulsive q_k-difference boundary value problem (18) has a unique solution on $[0,1]$.

Example 4.2. Consider the following integral boundary value problem of nonlinear second-order impulsive q_k-difference equation

$$\begin{cases} D^2_{\left(\frac{k^2+2k+1}{k^2+3k+2}\right)^{\frac{3}{4}}} u(t) = \dfrac{(-t^2+2t+3e^{-t})u(t)}{\sin^2 u(t) + 36\pi(e^t+4)^2} \\ \qquad + \dfrac{1}{12\pi(e^t+4)^2}, \quad t \in J = [0,1],\, t \neq t_k, \\ \Delta u(t_k) = \dfrac{\sin u(t_k)+1}{3\pi(k+2)}, \\ \qquad t_k = \dfrac{k}{5},\ k = 1,2,3,4, \\ D_{\left(\frac{k^2+2k+1}{k^2+3k+2}\right)^{\frac{3}{4}}} u(t_k^+) - D_{\left(\frac{(k-1)^2+2k-1}{(k-1)^2+3k-1}\right)^{\frac{3}{4}}} u(t_k) = \dfrac{4u(t_k)+6}{7\pi(k+1)}, \\ \qquad t_k = \dfrac{k}{5},\ k = 1,2,3,4, \\ u(0) = 0, \\ u(1) = \sum_{i=0}^{4} \left(\frac{1}{|i-2|+4}\right) \int_{t_i}^{t_{i+1}} u(s) d_{\left(\frac{i^2+2i+1}{i^2+3i+2}\right)^{\frac{3}{4}}} s. \end{cases} \tag{19}$$

Set $q_k = \left((k^2+2k+1)/(k^2+3k+2)\right)^{3/4}$ for $k = 0,1,2,3,4$, $m = 4$, $T = 1$, $\alpha_i = 1/(|i-2|+4)$, $f(t,u) = ((-t^2+2t+3e^{-t})u)/(\sin^2 u + 36\pi(e^t+4)^2) + 1/(12\pi(e^t+4)^2)$, $I_k(u) = (\sin u + 1)/(3\pi(k+2))$ and $I_k^*(u) = (4u+6)/(7\pi(k+1))$. Clearly,

$$|f(t,u)| = \left| \frac{(-t^2+2t+3e^{-t})u(t)}{\sin^2 u(t) + 36\pi(e^t+4)^2} + \frac{1}{12\pi(e^t+4)^2} \right| \leq \left(\frac{2t+3}{3(e^t+4)^2} \right) \frac{|u+3|}{12\pi},$$

$$|I_k(u)| = \left| \frac{\sin u(t_k)+1}{3\pi(k+2)} \right| \leq \frac{|u|+1}{9\pi},$$

and

$$|I_k^*(u)| = \left| \frac{4u(t_k)+6}{7\pi(k+1)} \right| \leq \frac{2|u|+3}{7\pi}.$$

Choosing $p(t) = (2t+3)(3(e^t+4))^2$, $p_0 = 1/15$, $\psi(|u|) = (|u|+3)/(12\pi)$, $\varphi_1(|u|) = (|u|+1)/(9\pi)$ and $\varphi_2(|u|) = (2|u|+3)/(7\pi)$, we obtain

$$\frac{M^*}{0.7167460235M^* + 0.9199843824} > 1,$$

which implies that $M^* > 3.247913388$. Hence, by Theorem 3.3, the boundary value problem (19) has at least one solution on $[0,1]$.

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