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Some Classes of Generating Relations Associated with a Family of the Generalized Gauss Type Hypergeometric Functions

Shy-Der Lin[1], *H. M. Srivastava*[2,*] *and Jen-Chih Yao*[3,4]

[1] Department of Applied Mathematics, Chung Yuan Christian University, Chung-Li 32023, Taiwan, Republic of China
[2] Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia V8W 3R4, Canada
[3] Center for Fundamental Science, Kaohsiung Medical University, Kaohsiung 80708, Taiwan, Republic of China
[4] Department of Mathematics, King Abdulaziz University, P. O. Box 80203, Jeddah 21589, Saudi Arabia

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Abstract: In recent years, several interesting families of generating functions for various classes of hypergeometric and generalized hypergeometric functions in one, two and more variables were investigated systematically. Here, in this sequel, we aim at establishing several (presumably) new generating relations for the generalized Gauss type hypergeometric functions which are introduced by means of some generalizations of the classical Beta function $B(\alpha,\beta)$. Special cases of the main results are also presented.

Keywords: Generating functions; Gamma and Beta functions; Generalized Beta functions; Generalized Gauss type hypergeometric functions; Linear, bilinear and bilateral (or mixed multilateral) generating functions.

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1 Introduction and Definitions

Various families of generating functions, especially for sequences involving (for example) the generalized hypergeometric function ${}_rF_s$ with r numerator and s denominator parameters, play an important rôle in the investigation of many useful properties of the sequences which they generate. They are used in finding several interesting properties, characteristics and formulas for the generated numbers and polynomials in a wide variety of research subjects in (for example) analytic number theory and modern combinatorics (see, for details, [6,8,12,18,19]; see also [13,14,20,22]). For a systematic introduction to, and for several interesting (and useful) applications of the various methods of obtaining linear, bilinear and bilateral (or mixed multilateral) generating functions for a fairly wide variety of sequences of special functions (and polynomials) in one, two and more variables, among much abundant available literature, we refer the interested reader to the extensive work presented in the monograph on this subject by Srivastava and Manocha [23].

We begin by recalling the fact that, in the widely-scattered literature on the subject of this paper, one can find several interesting generalizations of the familiar (Euler's) Gamma function $\Gamma(z)$ as well as the corresponding generalizations and extensions of the Beta function $B(\alpha,\beta)$, the hypergeometric functions ${}_1F_1$ and ${}_2F_1$, and the generalized hypergeometric functions ${}_rF_s$ with r numerator and s denominator parameters. For example, for an appropriately bounded sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$ of essentially arbitrary (real or complex) numbers, Srivastava *et al.* [25, p. 243 *et seq.*] recently considered the function $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)$ given by

$$\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right):=\begin{cases}\displaystyle\sum_{\ell=0}^{\infty}\kappa_\ell\,\frac{z^\ell}{\ell!} & (|z|<R;\ R>0;\ \kappa_0:=1)\\[2ex] \mathfrak{M}_0\,z^\omega\,\exp(z)\left[1+O\left(\dfrac{1}{|z|}\right)\right] & (|z|\to\infty;\ \mathfrak{M}_0>0;\ \omega\in\mathbb{C})\end{cases} \tag{1}$$

* Corresponding author e-mail: shyder@cycu.edu.tw, harimsri@math.uvic.ca, yaojc@kmu.edu.tw

for some suitable constants $\mathfrak{M}_0$ and ω depending essentially upon the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$. Then, in terms of the function $\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};z\right)$ defined by (1), Srivastava *et al.* [25] introduced some remarkably deep generalizations of the extended Gamma function $\Gamma_\mathfrak{p}(z)$, the extended Beta function $B_\mathfrak{p}(\alpha,\beta)$ and the extended hypergeometric function $F_\mathfrak{p}(a,b;c;z)$ (see, for details, [4] *and* [5]) by

$$\Gamma_\mathfrak{p}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(z) := \int_0^\infty t^{z-1}\,\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-t-\frac{\mathfrak{p}}{t}\right)dt \quad (2)$$

$$\left(\Re(z)>0;\ \Re(\mathfrak{p})\geqq 0\right),$$

$$\begin{aligned}\mathfrak{B}_\mathfrak{p}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta) &= \mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta;\mathfrak{p})\\ &:= \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\\ &\quad\cdot\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-\frac{\mathfrak{p}}{t(1-t)}\right)dt \qquad (3)\end{aligned}$$

$$\left(\min\{\Re(\alpha),\Re(\beta)\}>0;\ \Re(\mathfrak{p})\geqq 0\right)$$

and

$$\begin{aligned}&{}_2\mathfrak{F}_1^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\mathfrak{p}\right)}(a,b;c;z)\\ &:= \frac{1}{B(b,c-b)}\sum_{n=0}^\infty (a)_n\,\mathfrak{B}_\mathfrak{p}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(b+n,c-b)\,\frac{z^n}{n!} \qquad (4)\end{aligned}$$

$$\left(|z|<1;\ \Re(c)>\Re(b)>0;\ \Re(\mathfrak{p})\geqq 0\right),$$

provided that the defining integrals in (2), (3) and (4) exist. Here, and in what follows, $(\lambda)_\nu$ $(\lambda,\nu\in\mathbb{C})$ denotes the Pochhammer symbol (or the shifted factorial) defined (in general) by

$$\begin{aligned}(\lambda)_\nu &:= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)}\\ &= \begin{cases}1 & (\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\})\\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}),\end{cases} \qquad (5)\end{aligned}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [23, p. 21 *et seq.*]), $\mathbb{N}$ being (as usual) the set of positive integers.

In the same paper published in 2012, Srivastava *et al.* [25, pp. 256–257, Section 6] *further* extended the definitions (3), (4) *and* many other related definitions by introducing one *additional* parameter $\mathfrak{q}$ $\left(\textit{with}\ \Re(\mathfrak{q})\geqq 0\right)$. Thus, in terms of the Θ-function given by (1), we have the following *two-parameter* extensions of the definitions in (3) and (4) (see, for details, [25, p. 256, Eqs. (6.1) and (6.2)]):

$$\begin{aligned}\mathfrak{B}_{\mathfrak{p},\mathfrak{q}}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta) &= \mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta;\mathfrak{p},\mathfrak{q})\\ &:= \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\\ &\quad\cdot\Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};-\frac{\mathfrak{p}}{t}-\frac{\mathfrak{q}}{1-t}\right)dt \qquad (6)\end{aligned}$$

$$\left(\min\{\Re(\alpha),\Re(\beta)\}>0;\ \min\{\Re(\mathfrak{p}),\Re(\mathfrak{q})\}\geqq 0\right)$$

and

$$\begin{aligned}&{}_2\mathfrak{F}_1^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\mathfrak{p},\mathfrak{q}\right)}(a,b;c;z)\\ &:= \frac{1}{B(b,c-b)}\sum_{n=0}^\infty (a)_n\,\mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(b+n,c-b;\mathfrak{p},\mathfrak{q})\,\frac{z^n}{n!} \qquad (7)\end{aligned}$$

$$\left(|z|<1;\ \Re(c)>\Re(b)>0;\ \min\{\Re(\mathfrak{p}),\Re(\mathfrak{q})\}\geqq 0\right),$$

provided that the defining integrals in (6) and (7) exist. The generalized Beta function

$$\mathfrak{B}_{\mathfrak{p},\mathfrak{q}}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta)$$

defined by (6) was used recently by Luo and Raina [7] with a view to extending the family of generalized hypergeometric functions (see also [5]). Moreover, the special case of the Gamma function

$$\Gamma_\mathfrak{p}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(z)$$

given by (2) when

$$\kappa_\ell\equiv 1 \qquad (\ell\in\mathbb{N}_0) \qquad (8)$$

was applied recently in another paper by Srivastava *et al.* [16] (see also [29] and the references cited therein to many other related recent works on this subject). Obviously, when $\mathfrak{q}=\mathfrak{p}$, the definitions in (6) and (7) would reduce immediately to those in (3) and (4), respectively.

We now recall that some fundamental properties and characteristics of the following generalized Beta type function

$$B_p^{(\alpha,\beta;m)}(a,c)$$

were considered very recently by Parmar (see, for details, [11, p. 37, Eq. (19)]):

$$\begin{aligned}&B_\mathfrak{p}^{(\gamma,\delta;\mathfrak{m})}(\alpha,\beta)\\ &:= \int_0^1 t^{\alpha-1}\,(1-t)^{\beta-1}\,{}_1F_1\left(\gamma;\delta;-\frac{\mathfrak{p}}{t^\mathfrak{m}\,(1-t)^\mathfrak{m}}\right)dt \qquad (9)\end{aligned}$$

$$\left(\Re(\mathfrak{p}) \geqq 0;\ \min\{\Re(\alpha), \Re(\beta), \Re(\mathfrak{m})\} > 0;\right.$$
$$\left.\gamma \in \mathbb{C};\ \delta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right),$$

where

$$\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} \qquad \left(\mathbb{Z}^- := \{-1,-2,-3,\cdots\}\right).$$

Clearly, the case $\mathfrak{m} = 1$ of the definition (9), which (for $\mathfrak{m} = 1$ *and* $\gamma = \delta$) can be found in the earlier works of Chaudhry *et al.* (see [4, p. 20, Eq. (1.7)] and [5, p. 591, Eq. (1.7)]), corresponds to the special case of the generalized Beta type function defined by (3) when

$$\kappa_\ell = \frac{(\gamma)_n}{(\delta)_n} \qquad \left(n, \ell \in \mathbb{N}_0;\ \gamma \in \mathbb{C};\ \delta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right). \tag{10}$$

In our present investigation, we shall make use of a substantially more general family of generalized Beta type functions defined by

$$\mathfrak{B}_{\mathfrak{p};\mu,\nu}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta) = \mathfrak{B}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha,\beta;\mathfrak{p};\mu,\nu)$$
$$:= \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}$$
$$\cdot\, \Theta\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}; -\frac{\mathfrak{p}}{t^\mu (1-t)^\nu}\right) dt \tag{11}$$

$$\left(\min\{\Re(\alpha), \Re(\beta), \Re(\mu), \Re(\nu)\} > 0;\ \Re(\mathfrak{p}) \geqq 0\right),$$

which, in the special case when $\mu = \nu = 1$, yields the definition (3). On the other hand, upon setting $\mu = \nu = \mathfrak{m}$ and choosing the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$ as in (10), the definition (11) coincides with that in (9).

Motivated by the definitions of the Gauss type functions in (4) and (7), and also by their various special cases studied in earlier works (see, for example, [5, p. 591, Eqs. (2.1) and (2.2)]; see also [9, p. 39, Chapter 4], [10, p. 4606, Section 3] and [11, p. 44])), we introduce here a family of generalized Gauss type hypergeometric functions as follows:

$${}_{r+q}\mathfrak{F}_{s+q}^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\mathfrak{p};\mu,\nu\right)}\left[\begin{matrix} a_1,\cdots,a_r,\alpha_1,\cdots,\alpha_q; \\ c_1,\cdots,c_s,\gamma_1,\cdots,\gamma_q; \end{matrix}\ z\right]$$
$$:= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{r}(a_j)_n}{\prod_{j=1}^{s}(c_j)_n} \prod_{j=1}^{q} \left\{ \frac{B^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha_j+n, \gamma_j-\alpha_j;\mathfrak{p};\mu,\nu)}{B^{\left(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}\right)}(\alpha_j, \gamma_j-\alpha_j;\mathfrak{p};\mu,\nu)} \right\} \frac{z^n}{n!} \tag{12}$$

$$\left(q,r,s \in \mathbb{N}_0;\ \Re(\gamma_j) > \Re(\alpha_j) > 0\ \ (j = 1,\cdots,q);\right.$$
$$\left.\min\{\Re(\mu), \Re(\nu)\} > 0;\ \Re(\mathfrak{p}) \geqq 0\right),$$

where, as usual, an empty product is interpreted as 1 and the involved parameters and the argument z are *tacitly* assumed to be so constrained that the series on the right-hand side is absolutely convergent. The special case of the definition (12) when

$$\mu = \nu = 1 \qquad \text{and} \qquad q = r = s = 1 \tag{13}$$

$$(a_1 = 1;\ \alpha_1 = b;\ \gamma_1 = c)$$

coincides precisely with the definition (4). Also, for

$$\mu = \nu = \mathfrak{m} \qquad \text{and} \qquad q = r = s = 1 \tag{14}$$

$$(a_1 = 1;\ \alpha_1 = b;\ \gamma_1 = c),$$

and with the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$ specialized as in (10), the definition (12) would obviously correspond to the Gauss type hypergeometric function

$$F_{\mathfrak{p}}^{(\gamma,\delta;\mathfrak{m})}(a,b;c;z)$$

defined by (see [11, p. 44])

$$F_{\mathfrak{p}}^{(\gamma,\delta;\mathfrak{m})}(a,b;c;z) := \sum_{n=0}^{\infty} (a)_n \frac{B_p^{(\gamma,\delta;\mathfrak{m})}(b+n, c-b)}{B(b,c-b)} \frac{z^n}{n!} \tag{15}$$

$$\left(|z| < 1;\ \Re(\mathfrak{p}) \geqq 0;\ \min\{\Re(\alpha), \Re(\beta), \Re(\mathfrak{m})\} > 0;\right.$$
$$\left.\Re(c) > \Re(b) > 0;\ \gamma \in \mathbb{C};\ \delta \in \mathbb{C} \setminus \mathbb{Z}_0^-\right),$$

which, in the special case when $\mathfrak{m} = 1$, was introduced and studied by Chaudhry *et al.* [5, p. 591, Eqs. (2.1) and (2.2)]. We remark in passing that a special case of the generalized Gauss type hypergeometric function in (12) when

$$\mu = \nu = 1 \qquad \text{and} \qquad q - 1 = r = s = 0 \tag{16}$$

$$(\alpha_1 = b;\ \gamma_1 = c),$$

and with the sequence $\{\kappa_\ell\}_{\ell\in\mathbb{N}_0}$ specialized as in (8), was also investigated earlier by Chaudhry *et al.* [5, p. 592, Eq. (2.3)]. For various other investigations involving generalizations of the hypergeometric function ${}_rF_s$ of r numerator and s denominator parameters, the interested reader may be referred to the recent works [15, 17, 21, 24, 26, 27, 28, 29, 30].

This paper is motivated by such recent investigations as (for example) [1, 2, 3, 23]. We propose here to derive several classes of generating relations involving the generalized Gauss type hypergeometric function which we have defined by (12). We also consider some interesting (and potentially useful) special cases of our main results.

2 A Set of Main Results

First of all, a generalized binomial coefficient $\binom{\lambda}{\mu}$ may be defined (for real or complex parameters λ and μ) by

$$\binom{\lambda}{\mu} := \frac{\Gamma(\lambda+1)}{\Gamma(\mu+1)\Gamma(\lambda-\mu+1)} =: \binom{\lambda}{\lambda-\mu} \qquad (\lambda, \mu \in \mathbb{C}),$$

so that, in the special case when $\mu = n\ \ (n \in \mathbb{N}_0)$, we have

$$\binom{\lambda}{n} = \frac{\lambda(\lambda-1)\cdots(\lambda-n+1)}{n!} = \frac{(-1)^n(-\lambda)_n}{n!} \qquad (n \in \mathbb{N}_0),$$

where $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ denotes the general Pochhammer symbol given, as above, by (5).

We now state the following lemma (due to R. Srivastava [27, Lemma]), which will be required in our proof of Theorem 1 below.

Lemma (R. Srivastava [27, p. 133, Lemma]). *Let $\{\Xi_n\}_{n\in\mathbb{N}_0}$ be a suitably bounded sequence of essentially arbitrary real or complex parameters. Suppose also that*

$$\lambda \in \mathbb{C} \qquad \text{and} \qquad \rho, \sigma \geqq 0 \quad (\rho+\sigma>0).$$

Then

$$\sum_{n=0}^{\infty}\binom{\lambda+n-1}{n}\left(\sum_{k=0}^{\infty}\frac{(\lambda+n)_{\rho k}}{(1-\lambda-n)_{\sigma k}}\,\Xi_k\,\frac{z^k}{k!}\right)t^n = (1-t)^{-\lambda}\sum_{k=0}^{\infty}\frac{(\lambda)_{\rho k}}{(1-\lambda)_{\sigma k}}\left({}_2F_1\left[\begin{matrix}-\rho k, \sigma k;\\ \lambda;\end{matrix}\; t\right]\right)\Xi_k\,\frac{[z(1-t)^{\sigma-\rho}]^k}{k!} \tag{17}$$

$$\left(|t|<1;\ \lambda\in\mathbb{C};\ \rho,\sigma\geqq 0\ (\rho+\sigma>0)\right),$$

provided that each member of the assertion (17) *exists.*

Throughout this investigation, exceptional values of the complex parameter λ which would render either side of such generating functions as (for example) the assertion (17) of the Lemma undefined or invalid are tacitly excluded. Also, since the generalized hypergeometric series for ${}_rF_s$ would reduce to its first term 1 whenever one or more of its r numerator parameters take on the value 0, we are led easily to the following special cases of the Lemma when $\sigma \to 0+$ or when $\rho \to 0+$ (see, for details, [27]):

$$\sum_{n=0}^{\infty}\binom{\lambda+n-1}{n}\left(\sum_{k=0}^{\infty}(\lambda+n)_{\rho k}\,\Xi_k\,\frac{z^k}{k!}\right)t^n = (1-t)^{-\lambda}\sum_{k=0}^{\infty}(\lambda)_{\rho k}\,\Xi_k\,\frac{[z(1-t)^{-\rho}]^k}{k!}$$

$$(|t|<1;\ \lambda\in\mathbb{C};\ \rho>0) \tag{18}$$

and

$$\sum_{n=0}^{\infty}\binom{\lambda+n-1}{n}\left(\sum_{k=0}^{\infty}\frac{z^k}{k!\,(1-\lambda-n)_{\sigma k}}\,\Xi_k\right)t^n = (1-t)^{-\lambda}\sum_{k=0}^{\infty}\frac{\Xi_k}{(1-\lambda)_{\sigma k}}\,\frac{[z(1-t)^{\sigma}]^k}{k!}$$

$$(|t|<1;\ \lambda\in\mathbb{C};\ \sigma>0), \tag{19}$$

provided that both members of each of the generating functions (18) and (19) exist.

Theorem 1. *Let the array of N parameters*

$$\frac{\lambda}{N}, \frac{\lambda+1}{N}, \cdots, \frac{\lambda+N-1}{N} \qquad (\lambda\in\mathbb{C};\ N\in\mathbb{N})$$

be abbreviated by $\Delta(N;\lambda)$, the array being empty when $N=0$. Then each of the following generating relations holds true for the generalized Gauss type hypergeometric function

$${}_{r+q}\mathfrak{F}_{s+q}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\mathfrak{p};\mu,\nu)}$$

defined by (12):

$$\sum_{n=0}^{\infty}\binom{\lambda+n-1}{n}\,{}_{N+r+q}\mathfrak{F}_{s+q}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\mathfrak{p};\mu,\nu)}\left[\begin{matrix}\Delta(N;\lambda+n), a_1,\cdots,a_r,\alpha_1,\cdots,\alpha_q;\\ c_1,\cdots,b_s,\gamma_1,\cdots,\gamma_q;\end{matrix}\; z\right]t^n$$

$$= (1-t)^{-\lambda}\,{}_{N+r+q}\mathfrak{F}_{s+q}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\mathfrak{p};\mu,\nu)}\left[\begin{matrix}\Delta(N;\lambda), a_1,\cdots,a_r,\alpha_1,\cdots,\alpha_q;\\ c_1,\cdots,c_s,\gamma_1,\cdots,\gamma_q;\end{matrix}\; \frac{z}{(1-t)^N}\right] \tag{20}$$

$$(|t|<1;\ \lambda\in\mathbb{C};\ N\in\mathbb{N})$$

and

$$\sum_{n=0}^{\infty}\binom{\lambda+n-1}{n}\,{}_{r+q}\mathfrak{F}_{N+s+q}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\mathfrak{p};\mu,\nu)}\left[\begin{matrix}a_1,\cdots,a_r,\alpha_1,\cdots,\alpha_q;\\ \Delta(N;1-\lambda-n), c_1,\cdots,c_s,\gamma_1,\cdots,\gamma_q;\end{matrix}\; z\right]t^n$$

$$= (1-t)^{-\lambda}\,{}_{r+q}\mathfrak{F}_{N+s+q}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\mathfrak{p};\mu,\nu)}\left[\begin{matrix}a_1,\cdots,a_r,\alpha_1,\cdots,\alpha_q;\\ \Delta(N;1-\lambda), c_1,\cdots,c_s,\gamma_1,\cdots,\gamma_q;\end{matrix}\; z(1-t)^N\right] \tag{21}$$

$$(|t|<1;\ \lambda\in\mathbb{C};\ N\in\mathbb{N}),$$

provided that both members of each of the generating functions (20) *and* (21) *exist.*

Proof. In the generating function (18), we set

$$\rho = N \qquad \text{and} \qquad \Xi_k = \frac{\prod_{j=1}^{r}(a_j)_k}{\prod_{j=1}^{s}(c_j)_k}\prod_{j=1}^{q}\left\{\frac{B^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\alpha_j+k,\gamma_j-\alpha_j;\mathfrak{p};\mu,\nu)}{B^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0})}(\alpha_j,\gamma_j-\alpha_j;\mathfrak{p};\mu,\nu)}\right\} \tag{22}$$

$$(N\in\mathbb{N}_0;\ k,q,r,s\in\mathbb{N}_0),$$

make use of the following familiar identity:

$$(\lambda)_{Nn} = n^{Nn}\left(\frac{\lambda}{N}\right)_n\left(\frac{\lambda+1}{N}\right)_n\cdots\left(\frac{\lambda+N-1}{N}\right)_n \qquad (n\in\mathbb{N}_0;\ N\in\mathbb{N};\ \lambda\in\mathbb{C}),$$

and replace z by

$$\frac{z}{N^N} \qquad (N\in\mathbb{N}).$$

Then, upon interpreting each of the resulting k-sums by means of the definition (12), we obtain the assertion (20) of Theorem 1.

Our demonstration of the assertion (21) of Theorem 1 is much akin to that of the assertion (20), which has already been given above.

Remark 1. In their special cases when

$$\kappa_\ell \equiv 1 \quad (\ell\in\mathbb{N}_0) \qquad \text{and} \qquad \mathfrak{p}=0, \tag{23}$$

the generalized Gauss type hypergeometric functions involved in the generating functions (20) and (21) were investigated earlier by (among others) Chen and Srivastava [3, p. 171 *et seq.*].

We next introduce the sequences

$$\{\omega_n^{(\lambda,N)}(z)\}_{n\in\mathbb{N}_0} \qquad \text{and} \qquad \{\zeta_n^{(\lambda,N)}(z)\}_{n\in\mathbb{N}_0}$$

of generalized Gauss type hypergeometric functions defined by

$$\begin{aligned}\omega_n^{(\lambda,N)}(z) &= \omega_n^{(\lambda,N)}[a_1,\cdots,a_r,\alpha_1,\cdots,\alpha_q;c_1,\cdots,c_s,\gamma_1,\cdots,\gamma_q:z]\\ &:= {}_{N+r+q}\mathfrak{F}_{s+q}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\mathfrak{p};\mu,\nu)}\left[\begin{matrix}\Delta(N;\lambda+n),a_1,\cdots,a_r,\alpha_1,\cdots,\alpha_q;\\ c_1,\cdots,c_s,\gamma_1,\cdots,\gamma_q;\end{matrix}\ z\right]\end{aligned} \tag{24}$$

and

$$\begin{aligned}\zeta_n^{(\lambda,N)}(z) &= \zeta_n^{(\lambda,N)}[a_1,\cdots,a_r,\alpha_1,\cdots,\alpha_q;c_1,\cdots,c_s,\gamma_1,\cdots,\gamma_q:z]\\ &:= {}_{r+q}\mathfrak{F}_{N+s+q}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\mathfrak{p};\mu,\nu)}\left[\begin{matrix}a_1,\cdots,a_r,\alpha_1,\cdots,\alpha_k;\\ \Delta(N;1-\lambda-n),c_1,\cdots,c_s,\gamma_1,\cdots,\gamma_q;\end{matrix}\ z\right].\end{aligned} \tag{25}$$

By applying the generating functions (20) and (21) *with* $\lambda \mapsto \lambda+m \quad (m\in\mathbb{N}_0)$, we are led to the following interesting Consequence of Theorem 1.

Theorem 2. *The following generating relationships hold true for the sequences*

$$\{\omega_n^{(\lambda,N)}(z)\}_{n\in\mathbb{N}_0} \qquad \textit{and} \qquad \{\zeta_n^{(\lambda,N)}(z)\}_{n\in\mathbb{N}_0}$$

of generalized and extended hypergeometric functions defined by (23) *and* (24), *respectively*:

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{\lambda+m+n-1}{n}\omega_{m+n}^{(\lambda,N)}(z)t^n &= (1-t)^{-\lambda-m}\,\omega_m^{(\lambda,N)}\left(\frac{z}{(1-t)^N}\right)\end{aligned} \tag{26}$$

$$(|t|<1;\ m\in\mathbb{N}_0;\ \lambda\in\mathbb{C})$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{\lambda+m+n-1}{n}\zeta_{m+n}^{(\lambda,N)}(z)t^n &= (1-t)^{-\lambda-m}\,\zeta_m^{(\lambda,N)}\left(z(1-t)^N\right)\end{aligned} \tag{27}$$

$$(|t|<1;\ m\in\mathbb{N}_0;\ \lambda\in\mathbb{C}).$$

Remark 2. The specialized cases of each of the generating functions (26) and (27) under the conditions given by (23) were derived earlier by Chen and Srivastava [3, p. 171, Eqs. (5.15) and (5.16)], who also considered various interesting consequences and further extensions involving multiple sequences.

3 Applications of Theorem 2

In this section, we apply Theorem 2 in order to derive various interesting classes of linear, bilinear and bilateral (or mixed multilateral) generating functions for the sequences

$$\{\omega_n^{(\lambda,N)}(z)\}_{n\in\mathbb{N}_0} \qquad \text{and} \qquad \{\zeta_n^{(\lambda,N)}(z)\}_{n\in\mathbb{N}_0}$$

defined by (23) and (24), respectively, in terms of the generalized Gauss type hypergeometric function

$${}_{r+q}\mathfrak{F}_{s+q}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\mathfrak{p};\mu,\nu)}$$

defined by (12).

Theorem 3. *Corresponding to a non-vanishing function* $\varPi_\rho(\xi_1,\cdots,\xi_u)$ *of u complex variables* $\xi_1,\cdots,\xi_u \quad (u\in\mathbb{N})$ *and involving a complex parameter* ρ, *called the order*, *let*

$$\begin{aligned}\Lambda_{m,p,q}^{(1)}[z;\xi_1,\cdots,\xi_u;t] &:= \sum_{n=0}^{\infty}\mathfrak{A}_n\,\omega_{m+nq}^{(\lambda+\sigma nq,N)}(z)\cdot\varPi_{\mu+np}(\xi_1,\cdots,\xi_u)t^n\end{aligned} \tag{28}$$

$$\left(\mathfrak{A}_n\neq 0;\ m\in\mathbb{N}_0;\ N,p,q\in\mathbb{N};\ \sigma\in\mathbb{C}\right)$$

and

$$\begin{aligned}\Lambda_{m,p,q}^{(2)}[z;\xi_1,\cdots,\xi_u;t] &:= \sum_{n=0}^{\infty}\mathfrak{B}_n\,\zeta_{m+nq}^{(\lambda+\sigma nq,N)}(z)\cdot\varPi_{\rho+np}(\xi_1,\cdots,\xi_u)t^n\end{aligned} \tag{29}$$

$$\left(\mathfrak{a}_n\neq 0;\ m\in\mathbb{N}_0;\ N,p,q\in\mathbb{N};\ \sigma\in\mathbb{C}\right).$$

Suppose also that

$$\begin{aligned}\varTheta_{m,n,q}^{\lambda,p,\rho,\sigma}(z;\xi_1,\cdots,\xi_u;\eta) &:= \sum_{\ell=0}^{[n/q]}\binom{\lambda+m+\sigma q\ell+n-1}{n-q\ell}\mathfrak{A}_\ell\,\omega_{m+n}^{(\lambda+\sigma q\ell,N)}(z)\\ &\quad\cdot\varPi_{\rho+p\ell}(\xi_1,\cdots,\xi_u)\eta^\ell\end{aligned} \tag{30}$$

and

$$\begin{aligned}\varPhi_{m,n,q}^{\lambda,p,\mu,\sigma}(z;\xi_1,\cdots,\xi_u;\eta) &:= \sum_{\ell=0}^{[n/q]}\binom{\lambda+m+\sigma q\ell+n-1}{n-q\ell}\mathfrak{B}_\ell\,\zeta_{m+n}^{(\lambda+\sigma q\ell,N)}(z)\\ &\quad\cdot\varPi_{\mu+p\ell}(\xi_1,\cdots,\xi_s)\eta^\ell,\end{aligned} \tag{31}$$

where the sequences

$$\{\omega_k^{(\lambda,N)}(z)\}_{n\in\mathbb{N}_0} \qquad \textit{and} \qquad \{\zeta_k^{(\lambda,N)}(z)\}_{n\in\mathbb{N}_0}$$

are defined by (24) *and* (25), *respectively. Then*

$$\sum_{n=0}^{\infty} \Theta_{m,n,q}^{\lambda,p,\rho,\sigma}(z;\xi_1,\cdots,\xi_u;\eta)t^n = (1-t)^{-\lambda-m} \cdot \Lambda_{m,p,q}^{(1)}\left[\frac{z}{(1-t)^N};\xi_1,\cdots,\xi_u;\frac{\eta t^q}{(1-t)^{(\sigma+1)q}}\right] \qquad (|t|<1) \tag{32}$$

and

$$\sum_{n=0}^{\infty} \Phi_{m,n,q}^{\lambda,p,\rho,\sigma}(z;\xi_1,\cdots,\xi_u;\eta)t^n = (1-t)^{-\lambda-m} \cdot \Lambda_{m,p,q}^{(2)}\left[z(1-t)^N;\xi_1,\cdots,\xi_u;\frac{\eta t^q}{(1-t)^{(\sigma+1)q}}\right] \qquad (|t|<1), \tag{33}$$

provided that each member of the assertions (32) *and* (33) *exists.*

Proof. For convenience, let the first member of the assertion (32) of Theorem 3 be denoted by $\mathscr{S}$. Then, upon substituting for the polynomials

$$\Theta_{m,n,q}^{\lambda,p,\mu,\sigma}(z;\xi_1,\cdots,\xi_s;\eta)$$

from (30) into the left-hand side of (32), we obtain

$$\begin{aligned}\mathscr{S} &:= \sum_{n=0}^{\infty} t^n \sum_{\ell=0}^{[n/q]} \binom{\lambda+m+\sigma q\ell+n-1}{n-q\ell} \mathfrak{A}_\ell\, \omega_{m+n}^{(\lambda+\sigma q\ell,N)}(z) \cdot \Pi_{\rho+p\ell}(\xi_1,\cdots,\xi_u)\eta^\ell \\ &= \sum_{\ell=0}^{\infty} \mathfrak{A}_\ell\, \Pi_{\rho+p\ell}(\xi_1,\cdots,\xi_s)(\eta t^q)^\ell \cdot \sum_{n=0}^{\infty} \binom{\lambda+m+(\sigma+1)q\ell+n-1}{n} \omega_{m+n+q\ell}^{(\lambda+\sigma q\ell,N)}(z)t^n \\ &\qquad (\Pi_{\rho+p\ell}(\xi_1,\cdots,\xi_s)\neq 0),\end{aligned} \tag{34}$$

where we have inverted the order of the double summation involved.

The inner n-series in (34) can be summed by appealing to the generating function (26) with m and λ replaced by $m+q\ell$ and $\lambda+\sigma q\ell$, respectively ($\ell \in \mathbb{N}_0$; $q \in \mathbb{N}$; $\sigma \in \mathbb{C}$). We thus find from (34) that

$$\mathscr{S} = (1-t)^{-\lambda-m} \sum_{\ell=0}^{\infty} \mathfrak{A}_\ell\, \omega_{m+q\ell}^{(\lambda+\sigma q\ell,N)}\left(\frac{z}{(1-t)^N}\right) \cdot \Pi_{\mu+p\ell}(\xi_1,\cdots,\xi_s)\left(\frac{\eta t^q}{(1-t)^{(\sigma+1)q}}\right)^\ell \qquad (|t|<1), \tag{35}$$

which, in view of the definition (28), is precisely the second member of the assertion (32) of Theorem 3.

We have thus completed the proof of the assertion (32) of Theorem 3 under the assumption that the double series involved in the first two steps of our proof are absolutely (and uniformly) convergent. Thus, in general, the assertion (32) of Theorem 3 holds true (at least as a relation between *formal power series*) for those values of the various parameters and variables involved for which each member of the assertion (32) exists.

The method of proof of the first assertion (32) of Theorem 3 can be applied *mutatis mutandis* in order to derive the secon assertion (33) of Theorem 3 below, which would evidently yield bilateral or mixed multilateral generating relations for the sequence

$$\{\zeta_n^{(\lambda,N)}(z)\}_{n\in\mathbb{N}_0}$$

defined by (25) in terms of the generalized Gauus type hypergeometric function

$${}_{r+q}\mathfrak{F}_{s+q}^{(\{\kappa_\ell\}_{\ell\in\mathbb{N}_0};\mathfrak{p};\mu,\nu)}$$

defined by (12).

Remark 3. A special case of Theorem 3 when the conditions in (23) are satisfied would correspond to a known result due to Chen and Srivastava [3, p. 180, Theorem 1]. Furthermore, the multivariable extensions of the special cases of Theorem 3 under the conditions in (23) can also be found in the work of Chen and Srivastava [3, pp. 182–183, Theorems 2 and 3]. As a matter of fact, all such classes of bilateral (or mixed multilateral) generating functions for various sequences of functions in one or more variables can be derived whenever one can find generating functions for these sequences, which fit easily in the following general pattern (see, for example, [23, p. 437, Equation 8.5(1)]):

$$\begin{aligned}\sum_{n=0}^{\infty} \mathfrak{C}_{\mu,n}\, \Psi_{\mu+n}(z_1,\cdots,z_\nu)t^n &= \vartheta(z_1,\cdots,z_\nu;t) \cdot \{\varphi(z_1,\cdots,z_\nu;t)\}^{-\mu} \\ &\quad \cdot \Psi_\mu(\psi_1(z_1,\cdots,z_\nu;t),\cdots,\psi_\nu(z_1,\cdots,z_\nu;t)) \\ &\qquad (\mu\in\mathbb{C}),\end{aligned} \tag{36}$$

where the coefficients $\mathfrak{C}_{\mu,n}$ $(n\in\mathbb{N}_0)$ are independent of $z_1,\cdots,z_\nu$ and t, and ϑ, φ and $\psi_1,\cdots,\psi_\nu$ are suitable functions of $z_1,\cdots,z_\nu$ and t. Obviously, each of the generating functions (26) and (27), which we have applied in proving Theorem 3, does indeed belong to the family given by (36) when $\nu=1$ and $\mu=m$ $(m\in\mathbb{N}_0)$.

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References

[1] P. Agarwal and C. L. Koul, On generating functions, *J. Rajasthan Acad. Phys. Sci.* **2** (2003), 173–180.
[2] P. Agarwal, M. Chand and S. D. Purohit, A note on generating functions involving generalized Gauss hypergeometric functions, *Nat. Acad. Sci. Lett.* **37** (2014), 457–459.
[3] M.-P. Chen and H. M. Srivastava, Orthogonality relations and generating functions for Jacobi polynomials and related hypergeometric functions, *Appl. Math. Comput.* **68** (1995), 153–188.
[4] M. A. Chaudhry, A. Qadir, M. Rafique and S. M. Zubair, Extension of Euler's beta function, *J. Comput. Appl. Math.* **78** (1997), 19–32.
[5] M. A. Chaudhry, A. Qadir, H. M. Srivastava and R. B. Paris, Extended hypergeometric and confluent hypergeometric functions, *Appl. Math. Comput.* **159** (2004), 589–602.
[6] A. Erdélyi, W. Mangus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. **III**, McGraw-Hill Book Company, New York, Toronto and London, 1955.
[7] M.-J. Luo and R. K. Raina, Extended generalized hypergeometric functions and their applications, *Bull. Math. Anal. Appl.* **5** (4) (2013), 65–77.
[8] E. B. McBride, *Obtaining Generating Functions*, Springer Tracts in Natural Philosophy, Vol. **21**, Springer-Verlag, New York, Heidelberg and Berlin, 1971.
[9] E. Özergin, Some Properties of Hypergeometric Functions, Ph.D. Thesis, Eastern Mediterranean University, North Cyprus, Turkey, 2011.
[10] E. Özergin, M. A. Özarslan and A. Altın, Extension of gamma, beta and hypergeometric functions, *J. Comput. Appl. Math.* **235** (2011), 4601–4610.
[11] R. K. Parmar, A new generalization of Gamma, Beta, hypergeometric and confluent hypergeometric functions, *Matematiche* (*Catania*) **69** (2013), 33–52.
[12] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960; Reprinted by Chelsea Publishing Company, Bronx, New York, 1971.
[13] H. M. Srivastava, Certain generating functions of several variables, *Z. Angew. Math. Mech.* **57** (1977), 339–340.
[14] H. M. Srivastava, Generating relations and other results associated with some families of the extended Hurwitz-Lerch Zeta functions, *SpringerPlus* **2** (2013), Article ID 2: 67, 1–14.
[15] H. M. Srivastava, P. Agarwal and S. Jain, Generating functions for the generalized Gauss hypergeometric functions, *Appl. Math. Comput.* **247** (2014), 348–352.
[16] H. M. Srivastava, A. Çetinkaya and İ. O. Kıymaz, A certain generalized Pochhammer symbol and its applications to hypergeometric functions, *Appl. Math. Comput.* **226** (2014), 484–491.
[17] H. M. Srivastava, M. A. Chaudhry and R. P. Agarwal, The incomplete Pochhammer symbols and their applications to hypergeometric and related functions, *Integral Transforms Spec. Funct.* **23** (2012), 659–683.
[18] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, Dordrecht, Boston and London, 2001.
[19] H. M. Srivastava and J. Choi, *Zeta and q-Zeta Functions and Associated Series and Integrals*, Elsevier Science Publishers, Amsterdam, London and New York, 2012.
[20] H. M. Srivastava, S. Gaboury and B.-J. Fugère), Further results involving a class of generalized Hurwitz-Lerch Zeta functions, *Russian J. Math. Phys.* **21** (2014), 521–537.
[21] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.
[22] H. M. Srivastava, M.-J. Luo and R. K. Raina, New results involving a class of generalized Hurwitz-Lerch zeta functions and their applications, *Turkish J. Anal. Number Theory* **1** (2013), 26–35.
[23] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.
[24] H. M. Srivastava, M. A. Özarslan and C. Kaanoğlu, Some families of generating functions for a certain class of three-variable polynomials, *Integral Transforms Spec. Funct.* **21** (2010), 885–896.
[25] H. M. Srivastava, R. K. Parmar and P. Chopra, A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions, *Axioms* **1** (2012), 238–258.
[26] R. Srivastava, Some properties of a family of incomplete hypergeometric functions, *Russian J. Math. Phys.* **20** (2013), 121–128.
[27] R. Srivastava, Some classes of generating functions associated with a certain family of extended and generalized hypergeometric functions, *Appl. Math. Comput.* **243** (2014), 132–137.
[28] R. Srivastava and N. E. Cho, Generating functions for a certain class of incomplete hypergeometric polynomials, *Appl. Math. Comput.* **219** (2012), 3219–3225.
[29] R. Srivastava and N. E. Cho, Some extended Pochhammer symbols and their applications involving generalized hypergeometric polynomials, *Appl. Math. Comput.* **234** (2014), 277–285.
[30] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*: *An Introduction to the General Theory of Infinite Processes and of Analytic Functions*; *with an Account of the Principal Transcendental Functions*, Fourth Edition (Reprinted), Cambridge University Press, Cambridge, London and New York, 1973.

Shy-Der Lin is a Full Professor in the Department of Applied Mathematics *and* the Department of Business Administration at Chung Yuan Christian University in Chung-Li in Taiwan (Republic of China). He holds a Ph.D. degree in Technology Management from the National Taiwan University of Science and Technology in Taipei City in Taiwan (Republic of China). His interests include Inventory Management, Financial Engineering, Financial Mathematics, Fractional Calculus, Special Functions and Differential Equations. His work has been published in journals such as Journal of Fractional Calculus, Hiroshima Mathematical Journal, Indian Journal of Pure and Applied Mathematics, International Journal of Quality and Reliability Management, Journal of Operations Research Society, Computer and Industrial Engineering, Journal of Information and Optimization Sciences, Journal of Statistics and Management Systems, Applied Mathematics and Computation, Computers and Mathematics with Applications, Journal of Mathematical Analysis and Applications, Applied Mathematics Letters, Applied Mathematical Modelling, Integral Transforms and Special Functions, Acta Applicandae Mathematicae, Revista de la Academia Canaria de Ciencias, Rendiconti del Seminario Matematico dellUniversità e Politecnico di Torino, Russian Journal of Mathematical Physics, Taiwanese Journal of Mathematics, Applied Mathematics, Czechoslovak Mathematical Journal, African Journal of Business Management, Boundary Value Problems, Filomat, Applied Mathematics and Information Sciences, and other international scientific research journals.

H. M. Srivastava For the author's biographical and other professional details (including the lists of his most recent publications such as Journal Articles, Books, Monographs and Edited Volumes, Book Chapters, Encyclopedia Chapters, Papers in Conference Proceedings, Forewords to Books and Journals, *et cetera*), the interested reader should look into the following Web Site:
http://www.math.uvic.ca/faculty/harimsri

Jen-Chih Yao is a Chair Professor in the Center for Fundamental Science at Kaohsiung Medical University in Kaohsiung in Taiwan (Republic of China). He received his Ph.D. degree in Operations Research from Stanford University in U.S.A. in the year 1990. His interests include Nonlinear Analysis, Optimization, Variational Analysis and Optimal Control. His work has been published in journals such as Journal of Optimization Theory and Applications, Journal of Approximation Theory, Set-Valued and Variational Analysis, Applied Mathematics and Computation, Computers and Mathematics with Applications, Journal of Mathematical Analysis and Applications, Applied Mathematics Letters, Applied Mathematical Modelling, SIAM Journal on Optimization, SIAM Journal on Control and Optimization, Mathematical Programming, Acta Applicandae Mathematicae, Fixed Point Theory and Applications, Journal of Nonlinear and Convex Analysis, Taiwanese Journal of Mathematics, Applied Mathematics and Information Sciences, and other international scientific research journals.