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Common Fixed Point Theorems in Intuitionistic Fuzzy Metric Spaces through Conditions of Integral Type

Shaban Sedghi[1]*, Nabi Shobe[2], and A. Aliouche[3]

[1]Department of Mathematics, Islamic Azad University-Ghaemshahr Branch,
Ghaemshahr P. O. Box 163, Iran
Email Address: sedghi_gh@yahoo.com

[2]Department of Mathematics, Islamic Azad University-Babol Branch, Iran
Email Address: nabi_shobe@yahoo.com

[3]Department of Mathematics University of Larbi Ben M'Hidi,
Oum-El-Bouaghi, 04000, Algeria
Email Address: alioumath@yahoo.fr

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We establish common fixed point theorems in intuitioistic fuzzy metric spaces for weakly compatible mappings satisfying property (E.A) introduced by [1] or common property (E.A) introduced by Liu et al [18] and common fixed point theorems for weakly compatible mappings using contractive conditions of integral type. Our theorems generalize theorems 2.3, 2.4 and 2.6 of [25].

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1 Introduction and Preliminaries

Motivated by the potential applicability of fuzzy topology to quantum physics, particularly in connection with both string and $E-$infinity theory developed by El Naschie [9–11, 28]. One of the most important problems in fuzzy topology is to obtain an appropriate concept of an intuitionistic fuzzy metric space and an intuitionistic fuzzy normed space. This problems have been investigated by Park [19] and Saadati and Park [22] respectively and they introduced and studied a notion of an intuitionistic fuzzy metric (normed) space.

*The corresponding author.

Intuitionistic fuzzy metric notation is useful in modelling some phenomena where it is necessary to study the relationship between two probability functions as will observe in [15]; for instance, it has a direct physic motivation in the context of the two-slit experiment as the foundation of E-infinity of high energy physics, recently studied by El Naschie in [12, 13].

Since the intuitionistic fuzzy metric space has extra conditions, see [15], [25] modified the idea of intuitionistic fuzzy metric spaces and presented the new notion of intuitionistic fuzzy metric spaces with the help of the notion of continuous $t-$representable. The authors [3, 5, 8, 21, 30] proved fixed point theorems using contractive conditions of integral type.

Lemma 1.1. ([7]) *Consider the set L^* and operation $\leq_{L^*}$ defined by:*

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \textit{ and } x_1 + x_2 \leq 1\},$$

$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Longleftrightarrow x_1 \leq y_1$ *and* $x_2 \geq y_2$, *for every* $(x_1, x_2), (y_1, y_2) \in L^*$. *Then* $(L^*, \leq_{L^*})$ *is a complete lattice.*

Definition 1.1. ([4]) An intuitionistic fuzzy set $\mathcal{A}_{\zeta,\eta}$ in a universe U is an object $\mathcal{A}_{\zeta,\eta} = \{(\zeta_{\mathcal{A}}(u), \eta_{\mathcal{A}}(u)) | u \in U\}$, where for all $u \in U$, $\zeta_{\mathcal{A}}(u) \in [0,1]$ and $\eta_{\mathcal{A}}(u) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of u in $\mathcal{A}_{\zeta,\eta}$, and furthermore, they satisfy $\zeta_{\mathcal{A}}(u) + \eta_{\mathcal{A}}(u) \leq 1$.

For every $z_i = (x_i, y_i) \in L^*$, if $c_i \in [0,1]$ such that $\sum_{j=1}^{n} c_j = 1$, then it is easy to see that

$$c_1(x_1, y_1) + \cdots + c_n(x_n, y_n) = \sum_{j=1}^{n} c_j(x_j, y_j) = (\sum_{j=1}^{n} c_j x_j, \sum_{j=1}^{n} c_j y_j) \in L^*. \tag{1.1}$$

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a triangular norm $T = *$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \longrightarrow [0,1]$ satisfying $T(1, x) = 1 * x = x$, for all $x \in [0,1]$. A triangular conorm $S = \diamond$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \longrightarrow [0,1]$ satisfying $S(0, x) = 0 \diamond x = x$ for all $x \in [0,1]$. Using the lattice $(L^*, \leq_{L^*})$, these definitions can be straightforwardly extended.

Definition 1.2. ([6]) A triangular norm ($t-$norm) on L^* is a mapping $\mathcal{T} : (L^*)^2 \longrightarrow L^*$ satisfying the following conditions:

1) $\forall x \in L^*$, $\mathcal{T}(x, 1_{L^*}) = x)$, (boundary condition)

2) $\forall (x, y) \in (L^*)^2$, $(\mathcal{T}(x, y) = \mathcal{T}(y, x))$, (commutativity)

3) $\forall (x, y, z) \in (L^*)^3$, $(\mathcal{T}(x, \mathcal{T}(y, z)) = \mathcal{T}(\mathcal{T}(x, y), z))$, (associativity)

4) $\forall (x, x', y, y') \in (L^*)^4$, $x \leq_{L^*} x'$ and $y \leq_{L^*} y' \Longrightarrow \mathcal{T}(x, y) \leq_{L^*} \mathcal{T}(x', y'))$, (monotonicity).

Definition 1.3. ([6,7]) A continuous $t-$norm $\mathcal{T}$ on L^* is called continuous *t–representable* if and only if there exist a continuous $t-$norm $*$ and a continuous $t-$conorm $\diamond$ on $[0,1]$ such that for all $x=(x_1,x_2), y=(y_1,y_2)\in L^*$,

$$\mathcal{T}(x,y)=(x_1 * y_1, x_2 \diamond y_2).$$

Now define a sequence $\mathcal{T}^n$ recursively by $\mathcal{T}^1=\mathcal{T}$ and

$$\mathcal{T}^n(x^{(1)},\cdots,x^{(n+1)})=\mathcal{T}(\mathcal{T}^{n-1}(x^{(1)},\cdots,x^{(n)}),x^{(n+1)})$$

for $n\geq 2$ and $x^{(i)}\in L^*$.

Definition 1.4. ([6, 7]) A negator on L^* is any decreasing mapping $\mathcal{N}: L^* \longrightarrow L^*$ satisfying $\mathcal{N}(0_{L^*})=1_{L^*}$ and $\mathcal{N}(1_{L^*})=0_{L^*}$. If $\mathcal{N}(\mathcal{N}(x))=x$, for all $x\in L^*$, then $\mathcal{N}$ is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N:[0,1]\longrightarrow [0,1]$ satisfying $N(0)=1$ and $N(1)=0$. N_s denotes the standard negator on $[0,1]$ defined by for all $x\in[0,1]$, $N_s(x)=1-x$.

Definition 1.5. Let M and N be fuzzy sets from $X^2\times(0,+\infty)$ into $[0,1]$ such that $M(x,y,t)+N(x,y,t)\leq 1$ for all $x,y\in X$ and $t>0$. The 3-tuple $(X,\mathcal{M}_{M,N},\mathcal{T})$ is said to be an *intuitionistic fuzzy metric space* if X is an arbitrary (non-empty) set, $\mathcal{T}$ is a continuous $t-$representable and $\mathcal{M}_{M,N}$ is a mapping $X^2\times(0,+\infty)\rightarrow L^*$ (an intuitionistic fuzzy set, see Definition 1.2) satisfying the following conditions for every $x,y\in X$ and $t,s>0$:

(a) $\mathcal{M}_{M,N}(x,y,t)>_{L^*} 0_{L^*}$;
(b) $\mathcal{M}_{M,N}(x,y,t)=1_{L^*}$ if and only if $x=y$;
(c) $\mathcal{M}_{M,N}(x,y,t)=\mathcal{M}_{M,N}(y,x,t)$;
(d) $\mathcal{M}_{M,N}(x,y,t+s)\geq_{L^*}\mathcal{T}(\mathcal{M}_{M,N}(x,z,t),\mathcal{M}_{M,N}(z,y,s))$;
(e) $\mathcal{M}_{M,N}(x,y,\cdot):(0,\infty)\longrightarrow L^*$ is continuous.

In this case $\mathcal{M}_{M,N}$ is called an *intuitionistic fuzzy metric space*. Here

$$\mathcal{M}_{M,N}(x,y,t)=(M(x,y,t),N(x,y,t)).$$

Example 1.6. Let (X,d) be a metric space. Denote $\mathcal{T}(a,b)=(a_1b_1,\min(a_2+b_2,1))$ for all $a=(a_1,a_2)$ and $b=(b_1,b_2)\in L^*$ and let M and N be fuzzy sets on $X^2\times(0,\infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x,y,t)=(M(x,y,t),N(x,y,t))=(\frac{ht^n}{ht^n+md(x,y)},\frac{md(x,y)}{ht^n+md(x,y)}),$$

for all $t,h,m,n\in\mathbb{R}_+$. Then, $(X,\mathcal{M}_{M,N},\mathcal{T})$ is an intuitionistic fuzzy metric space.

Example 1.7. Let $X = \mathbf{N}$. Define $\mathcal{T}(a,b) = (\max(0, a_1 + b_1 - 1), a_2 + b_2 - a_2 b_2)$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$ and let M and N be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:

$$\mathcal{M}_{M,N}(x,y,t) = (M(x,y,t), N(x,y,t)) = \begin{cases} (\frac{x}{y}, \frac{y-x}{y}) & if \quad x \leq y \\ (\frac{y}{x}, \frac{x-y}{x}) & if \quad y \leq x. \end{cases}$$

for all $x, y \in X$ and $t > 0$. Then, $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is an intuitionistic fuzzy metric space.

Definition 1.8. 1) A sequence $\{x_n\}$ is said to be *convergent* to $x \in X$ in the intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ and denoted by $x_n \stackrel{\mathcal{M}_{M,N}}{\longrightarrow} x$ if $\mathcal{M}_{M,N}(x_n, x, t) \longrightarrow 1_{L^*}$ as $n \longrightarrow \infty$ for every $t > 0$

2) A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ is called a *Cauchy sequence* if for each $0 < \varepsilon < 1$ and $t > 0$, there exists $n_0 \in \mathbf{N}$ such that

$$\mathcal{M}_{M,N}(x_n, y_m, t) >_{L^*} (N_s(\varepsilon), \varepsilon),$$

and for each $n, m \geq n_0$; here N_s is the standard negator.

3) An intuitionistic fuzzy metric space is said to be *complete* if and only if every Cauchy sequence in this space is convergent.

Lemma 1.2. ([22]) *Let $\mathcal{M}_{M,N}$ be an intuitionistic fuzzy metric. Then, for any $t > 0$, $\mathcal{M}_{M,N}(x,y,t)$ is nondecreasing with respect to t in $(L^*, \leq_{L^*})$ for all x, y in X.*

Definition 1.9. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic fuzzy metric space. For $t > 0$, we define the *open ball* $B(x,r,t)$ with center $x \in X$ and radius $0 < r < 1$ by

$$B(x,r,t) = \{y \in X : \mathcal{M}_{M,N}(x,y,t) >_{L^*} (N_s(r), r)\}.$$

A subset $A \subset X$ is called *open* if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x,r,t) \subset A$. Let $\tau_{\mathcal{M}_{M,N}}$ denote the family of all open subset of X. $\tau_{\mathcal{M}_{M,N}}$ is called the *topology induced by the intuitionistic fuzzy metric space*.

Note that this topology is Hausdorff (see [19]).

Definition 1.10. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitionistic fuzzy metric space. A subset A of X is said to be IF*-bounded* if there exist $t > 0$ and $0 < r < 1$ such that $\mathcal{M}_{M,N}(x,y,t) >_{L^*} (N_s(r), r)$ for each $x, y \in A$.

Definition 1.11. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitioistic fuzzy metric space. $\mathcal{M}$ is said to be continuous on $X \times X \times]0, \infty[$ if

$$\lim_{n \to \infty} \mathcal{M}_{M,N}(x_n, y_n, t_n) = \mathcal{M}_{M,N}(x,y,t)$$

whenever $\{(x_n, y_n, t_n)\}$ is a sequence in $X \times X \times]0, \infty[$ which converges to a point $(x,y,t) \in X \times X \times]0, \infty[$; i.e., $\lim_n \mathcal{M}_{M,N}(x_n, x, t) = \lim_n \mathcal{M}_{M,N}(y_n, y, t) = 1_{\mathcal{L}^*}$ and $\lim_n \mathcal{M}_{M,N}(x, y, t_n) = \mathcal{M}_{M,N}(x,y,t)$.

Lemma 1.3. ([25]) *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitioistic fuzzy metric space. Then $\mathcal{M}$ is a continuous function on $X \times X \times]0, \infty[$.*

In the sequel, A and S are self-mappings of an intuitioistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ and $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = u \in X.$$

Definition 1.12. A and S are said to be

1) weakly commuting [2] if for all $x \in X$ and $t > 0$

$$\mathcal{M}_{M,N}(SAx, ASx, t) \leq \mathcal{M}_{M,N}(Ax, Sx, t)$$

2) compatible [2] if

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(ASx_n, SAx_n, t) = 1_{\mathcal{L}^*}, \text{ for all } t > 0,$$

3) compatible of type (α) [2] if

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(SAx_n, A^2x_n, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(ASx_n, S^2x_n, t) = 1_{\mathcal{L}^*} \text{ for all } t > 0,$$

4) compatible of type (β) [2] if

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(S^2x_n, A^2x_n, t) = 1_{\mathcal{L}^*} \text{ for all } t > 0,$$

5) semi-compatible if

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(ASx_n, Su, t) = 1_{\mathcal{L}^*} \text{ for all } t > 0,$$

6) weakly compatible [16] if they commute at their coincidence points; i.e., $Ax = Sx$ for some $x \in X$ implies that $ASx = SAx$,

7) $R-$weakly commuting [29] if there exists $R > 0$ such that for all $x \in X$ and $t > 0$

$$\mathcal{M}_{M,N}(SAx, ASx, Rt) \leq \mathcal{M}_{M,N}(Ax, Sx, t) \tag{1.2}$$

If $R = 1$ in (1.2) we obtain the definition of weakly commuting.

8) pointwise $R-$weakly commuting [20] if for all $x \in X$, there exists an $R > 0$ such that (1.2) holds.

Remark 1.13. (A, S) is $R-$weakly commuting implies that (A, S) is compatible, but the converse is not true in general, see [27].

Remark 1.14. ([27]) The semi-compatibility of the pair (A, S) does not imply the semi-compatibility of (S, A).

Remark 1.15. It is proved in [20] that $R-$weak commutativity is equivalent to commutativity at coincidence points; i.e., A and S are pointwise $R-$weakly commuting if and only if they are weakly compatible.

Proposition 1.1. ([2, 26]) *If A and S are $R-$weakly commuting, or compatible, or compatible of type (α), or compatible of type (β), or semi-compatible, then they are weakly compatible.*

The converse is not true in general.

Example 1.16. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be an intuitioistic fuzzy metric space, where $X = [0, 10]$ and

$$\mathcal{M}_{M,N}(x, y, t) = (\frac{t}{t + |x - y|}, \frac{|x - y|}{t + |x - y|}) \quad \text{for all } t > 0 \text{ and } x, y \in X.$$

Denote $\mathcal{T}(a, b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$. Define S and A by:

$$Sx = \begin{cases} 3 & \text{if} \quad x \in (0, 2], \\ 0 & \text{if} \quad x \in \{0\} \cup (2, 10] \end{cases}, \quad Ax = \begin{cases} 0 & \text{if} \quad x = 0, \\ x + 8 & \text{if} \quad x \in (0, 2], \\ x - 2 & \text{if} \quad x \in (2, 10] \end{cases}.$$

We have $Ax = Sx$ iff $x = 0$. $SA(0) = AS(0) = 0$. Then, (A, S) is weakly compatible.

Let $\{x_n\}$ be a sequence in X defined by: $x_n = 2 + 1/n$, $n \geq 1$.
$Sx_n = S(2 + \frac{1}{n}) = 0$, $Ax_n = A(2 + \frac{1}{n}) = \frac{1}{n}$.
$Ax_n, Sx_n \to u = 0$ as $n \to \infty$, $SAx_n = S(\frac{1}{n}) = 3$, $ASx_n = A(0) = 0$,
$S^2 x_n = S(0) = 0$, $A^2 x_n = A(\frac{1}{n}) = 8 + \frac{1}{n}$. Since for all $t > 0$

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{M}_{M,N}(SAx_n, ASx_n, t) =& \mathcal{M}_{M,N}(3, 0, t) = (\frac{t}{t+3}, \frac{3}{t+3}), \\
\lim_{n\to\infty} \mathcal{M}_{M,N}(SAx_n, A^2 x_n, t) =& \mathcal{M}_{M,N}(3, 8, t) = (\frac{t}{t+5}, \frac{5}{t+5}), \\
\lim_{n\to\infty} \mathcal{M}_{M,N}(S^2 x_n, A^2 x_n, t) =& \mathcal{M}_{M,N}(0, 8, t) = (\frac{t}{t+8}, \frac{8}{t+8}), \\
\lim_{n\to\infty} \mathcal{M}_{M,N}(ASx_n, Su, t) =& \mathcal{M}_{M,N}(0, 0, t) = 1_{L^*},
\end{aligned}$$

(A, S) is not compatible, nor compatible of type (α), nor compatible of type (β), but (A, S) is semi- compatible.

Example 1.17. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ as in the above example. Define A and S by:

$$Ax = \begin{cases} 2 - x & \text{if} \quad x \in [0, 1), \\ 2 & \text{if} \quad x \in [1, 2] \end{cases}, \quad Sx = \begin{cases} x & \text{if} \quad x \in [0, 1), \\ 2 & \text{if} \quad x \in [1, 2]. \end{cases}$$

We have $Sx = Ax$ iff $x \in [1, 2]$. $SA(x) = AS(x) = 2$ for all $x \in [1, 2]$. Then, (A, S) is weakly compatible. Let $\{x_n\}$ be a sequence in X defined by: $x_n = 1 - 1/n$, $n \geq 1$.

$Sx_n = x_n,\ Ax_n = 2 - x_n,\ Ax_n,\ Sx_n \to 1 = u$ as $n \to \infty$
$SAx_n = 2,\ ASx_n = 2 - x_n$. As for all $t > 0$

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(ASx_n, Su, t) = \mathcal{M}_{M,N}(1,2,t) = (\frac{t}{t+1}, \frac{1}{t+1}),$$
$$\lim_{n\to\infty} \mathcal{M}_{M,N}(SAx_n, Au, t) = \mathcal{M}_{M,N}(2,2,t) = 1_{L^*},$$

therefore (A, S) is not semi-compatible, but (S, A) is semi-compatible,

Proposition 1.2. ([2,27]) 1) *Assume that S is continuous. Then, (A, S) is semi-compatible if and only if (A, S) is compatible.*

2) *Assume that A and S are continuous. Then, compatibility, compatibility of type (α) and compatibility of type (β) are equivalent*

Definition 1.18. The pair (A, S) satisfies the property (E.A) [1] if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(Ax_n, u, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(Sx_n, u, t) = 1_{L^*},$$

for some $u \in X$ and all $t > 0$.

Example 1.19. Let $X = \mathbb{R}$ and

$$\mathcal{M}_{M,N}(x, y, t) = (\frac{t}{t+|x-y|}, \frac{|x-y|}{t+|x-y|}),$$

for every $x, y \in X$ and $t > 0$. Let A and S defined by

$$Ax = 2x + 1, \quad Sx = x + 2.$$

Consider the sequence $x_n = 1/n + 1,\ n = 1, 2, \cdots$. We have

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(Ax_n, 3, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(Sx_n, 3, t) = 1_{L^*},$$

for every $t > 0$. Then the pair (A, S) satisfies the property (E.A).

In the next example, we show that there are some mappings which do not satisfy property (E.A).

Example 1.20. Let $X = \mathbb{R}$ and

$$\mathcal{M}_{M,N}(x, y, t) = (\frac{t}{t+|x-y|}, \frac{|x-y|}{t+|x-y|}),$$

for every $x, y \in X$ and $t > 0$. Let $Ax = x + 1$ and $Sx = x + 2$. If there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{M}_{M,N}.(Ax_n, u, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(Bx_n, u, t) = 1_{L^*},$$

for some $u \in X$, we conclude that $x_n \to u - 1$ and $x_n \to u - 2$ which is a contradiction.

Hence the pair (A, S) do not satisfy the property (E.A).

Definition 1.21. The pairs (A,S) and (B,T) of a an intuitioistic fuzzy metric space $(X,\mathcal{M}_{M,N},\mathcal{T})$ satisfy a common property (E.A) [18], if there exists two sequences $\{x_n\}$ and $\{y_n\}$ such that for some $u\in X$ and for all $t>0$

$$\begin{aligned}\lim_{n\to\infty}\mathcal{M}_{M,N}(Ax_n,u,t) &= \lim_{n\to\infty}\mathcal{M}_{M,N}(Sx_n,u,t)=\lim_{n\to\infty}\mathcal{M}_{M,N}(By_n,u,t)\\ &= \lim_{n\to\infty}M(Ty_n,u,t)=1_{L^*}. \ (1.3)\end{aligned}$$

If $B=A$ and $T=S$ in (1.3), we obtain the definition of property (E.A).

Example 1.22. Let $X=[1,\infty)$ and

$$\mathcal{M}_{M,N}(x,y,t)=(\frac{t}{t+|x-y|},\frac{|x-y|}{t+|x-y|}),$$

Define $A,B,S,T:X\to X$ by

$$Ax=2+\frac{x}{3},Bx=2+\frac{x}{2},Sx=1+\frac{2}{3}x,Tx=1+x.$$

Define sequences $\{x_n\}$ and $\{y_n\}$ by $x_n=3+1/n$, $y_n=2+1/n$, $n=1,2,\ldots$.

Since for all $t>0$

$$\begin{aligned}&\lim_{n\to\infty}\mathcal{M}_{M,N}(Ax_n,3,t)=\lim_{n\to\infty}\mathcal{M}_{M,N}(By_n,3,t)=\lim_{n\to\infty}\mathcal{M}_{M,N}(Sx_n,3,t)\\ &=\mathcal{M}_{M,N}(Ty_n,3,t)=1_{L^*},\end{aligned}$$

therefore the pairs (A,S) and (B,T) satisfy a common property (E.A)

Lemma 1.4. ([2,23,24]) *Let $(X,\mathcal{M}_{M,N},\mathcal{T})$ be an intuitioistic fuzzy metric space. Define $E_{\lambda,\mathcal{M}}:X^2\longrightarrow\mathbf{R}^+\cup\{0\}$ by*

$$E_{\lambda,\mathcal{M}}(x,y)=\inf\{t>0:\mathcal{M}_{M,N}(x,y,t)>_{L^*}(N_s(\lambda),\lambda)$$

for each $0<\lambda<1$ and $x,y\in X$. Then we have
(i) *For any $0<\mu<1$ there exists $0<\lambda<1$ such that*

$$E_{\mu,\mathcal{M}}(x_1,x_n)\le E_{\lambda,\mathcal{M}}(x_1,x_2)+E_{\lambda,\mathcal{M}}(x_2,x_3)+\cdots+E_{\lambda,\mathcal{M}}(x_{n-1},x_n)$$

for any $x_1,\ldots,x_n\in X$;
(ii) *The sequence $\{x_n\}_{n\in\mathbf{N}}$ is convergent in the intuitioistic fuzzy metric $(X,\mathcal{M}_{M,N},\mathcal{T})$ if and only if $E_{\lambda,\mathcal{M}}(x_n,x)\overset{\mathcal{M}_{M,N}}{\longrightarrow}0$. Also the sequence $\{x_n\}_{n\in\mathbf{N}}$ is Cauchy sequence if and only if it is Cauchy with $E_{\lambda,\mathcal{M}}$.*

Lemma 1.5. ([25]) *Let $(X,\mathcal{M}_{M,N},\mathcal{T})$ be an intuitioistic fuzzy metric space. If*

$$\mathcal{M}_{M,N}(x_n,x_{n+1},t)\ge_{L^*}\mathcal{M}_{M,N}(x_0,x_1,k^nt)$$

for some $k>1$ and $n\in\mathbb{N}$. Then $\{x_n\}$ is a Cauchy sequence.

Definition 1.23. ([14]) We say that the intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ has property (C), if it satisfies the following condition:

$$\mathcal{M}_{M,N}(x,y,t) = C \text{ for all } t > 0 \text{ implies } C = 1_{L^*}.$$

It is our purpose in this paper to prove common fixed point theorems in intuitioistic fuzzy metric spaces for weakly compatible mappings satisfying property (E.A) introduced by [1] or common property (E.A) introduced by Liu et al [18] and common fixed point theorems for weakly compatible mappings using contractive conditions of integral type. Our theorems generalize theorems 2.3, 2.4 and 2.6 of [25].

2 Main Results

Let Φ be the set of all continuous functions $\phi : L^* \longrightarrow L^*$, such that $\phi(t) >_{L^*} t$ for all $t \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$.

Example 2.1. Let $\phi : L^* \longrightarrow L^*$ defined by $\phi(t_1, t_2) = (\sqrt{t_1}, 0)$ for every $t = (t_1, t_2) \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$.

Theorem 2.1. *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a complete intuitionistic fuzzy metric space and A, B, S and T be self-mappings of X satisfying the following conditions:*

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X), \tag{2.1}$$

$$\int_0^{\mathcal{M}_{M,N}(Ax,By,t)} \varphi(s)ds \geq_{L^*} \phi\Big(\int_0^{\mathcal{L}_{M,N}(x,y,t)} \varphi(s)ds\Big), \tag{2.2}$$

for all $x, y \in X$, where $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a Lesbegue integrable mapping which is summable satisfying for each $0 < \epsilon < 1$,

$$0 < \int_0^{\epsilon} \varphi(s)ds < 1, \quad \int_0^1 \varphi(s)ds = 1, \tag{2.3}$$

and

$$\begin{aligned}\mathcal{L}_{M,N}(x,y,t) = \min\{&\mathcal{M}_{M,N}(Sx,Ty,t), \mathcal{M}_{M,N}(Ax,Sx,t), \mathcal{M}_{M,N}(By,Ty,t),\\ &\mathcal{M}_{M,N}(Sx,By,t), \mathcal{M}_{M,N}(Ax,Ty,t)\}.\end{aligned}$$

Suppose that the pair (A, S) or (B, T) satisfies the property (E.A), one of $A(X)$ or $B(X)$ or $S(X)$ or $T(X)$ is a closed subset of X and the pairs (A, S) and (B, T) are weakly compatible. Then, A, B, S and T have a unique common fixed point in X.

Proof. Suppose that the pair (B, T) satisfies the property (E.A). Therefore, there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} \mathcal{M}_{M,N}(Bx_n, z, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(Tx_n, z, t) = 1_{L^*}$$

for some $z \in X$ and all $t > 0$. As $B(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in X such that $Bx_n = Sy_n$, hence $\lim\limits_{n\to\infty} \mathcal{M}_{M,N}(Sy_n, z, t) = 1_{L^*}$. We prove that $\lim_{n\to\infty} \mathcal{M}_{M,N}(Ay_n, z, t) = 1_{L^*}$. Suppose that $\lim_{n\to\infty} \mathcal{M}_{M,N}(Ay_n, z, t) = l < 1_{L^*}$.

Using (2.2) we have

$$\int_0^{\mathcal{M}_{M,N}(Ay_n,Bx_n,t)} \varphi(s)ds \geq_{L^*} \phi(\int_0^{\mathcal{L}_{M,N}(y_n,x_n,t)} \varphi(s)ds), \tag{2.4}$$

where

$$\begin{aligned} L_{\mathcal{M}_{M,N}}(y_n, x_n, t) = \min\{&\mathcal{M}_{M,N}(Sy_n, Tx_n, t), \mathcal{M}_{M,N}(Ay_n, Sy_n, t), \\ &\mathcal{M}_{M,N}(Bx_n, Tx_n, t), \mathcal{M}_{M,N}(Ay_n, Tx_n, t), \mathcal{M}_{M,N}(Sy_n, Bx_n, t)\}, \end{aligned}$$

Then

$$\lim_{n\to\infty} L_{\mathcal{M}_{M,N}}(y_n, x_n, t) = l.$$

Taking the limit as $n \to \infty$ in (2.4) we get

$$\begin{aligned} \int_0^l \varphi(s)ds &\geq_{L^*} \phi(\int_0^l \varphi(s)ds) \\ &>_{L^*} \int_0^l \varphi(s)ds, \end{aligned}$$

which is a contradiction. Then $\lim_{n\to\infty} \mathcal{M}_{M,N}(Ay_n, z, t) = 1_{L^*}$.

Assume that $S(X)$ is a closed subset of X. Then, there exists $u \in X$ such that $Su = z$. If $Au \neq z$, using (2.2) we get

$$\int_0^{\mathcal{M}_{M,N}(Au,Bx_n,t)} \varphi(s)ds \geq_{L^*} \phi(\int_0^{\mathcal{L}_{M,N}(u,x_n,t)} \varphi(s)ds), \tag{2.5}$$

where

$$\begin{aligned} \mathcal{L}_{M,N}(u, x_n, t) = \min\{&\mathcal{M}_{M,N}(Su, Tx_n, t), \mathcal{M}_{M,N}(Au, Su, t), \mathcal{M}_{M,N}(Bx_n, Tx_n, t), \\ &\mathcal{M}_{M,N}(Au, Tx_n, t), \mathcal{M}_{M,N}(Su, Bx_n, t)\}. \end{aligned}$$

Hence

$$\lim_{n\to\infty} \mathcal{L}_{M,N}(u, x_n, t) = \mathcal{M}_{M,N}(Au, z, t)$$

Letting $n \to \infty$ in (2.5), we obtain

$$\begin{aligned} \int_0^{\mathcal{M}_{M,N}(Au,z,t)} \varphi(s)ds &\geq {}_{L^*}\phi(\int_0^{\mathcal{M}_{M,N}(Au,z,t)} \varphi(s)ds) \\ &> {}_{L^*} \int_0^{\mathcal{M}_{M,N}(Au,z,t)} \varphi(s)ds \end{aligned}$$

Therefore, $\mathcal{M}_{M,N}(Au,z,t) = 1_{L^*}$; i.e., $Au = Su = z$. Since $A(X) \subset T(X)$, there exists $v \in X$ such that $Tv = z$. If $z \neq Bv$ using (2.2) we get

$$\int_0^{\mathcal{M}_{M,N}(Au,Bv,t)} \varphi(s)ds \geq_{L^*} \phi(\int_0^{\mathcal{L}_{M,N}(u,v,t)} \varphi(s)ds),$$

where

$$\begin{aligned}
\mathcal{L}_{M,N}(u,v,t) &= \min\{\mathcal{M}_{M,N}(Su,Tv,t), \mathcal{M}_{M,N}(Au,Su,t), \mathcal{M}_{M,N}(Bv,Tv,t), \\
&\quad \mathcal{M}_{M,N}(Su,Bv,t), \mathcal{M}_{M,N}(Au,Tv,t)\} \\
&= \mathcal{M}_{M,N}(z,Bv,t).
\end{aligned}$$

Hence

$$\begin{aligned}
\int_0^{\mathcal{M}_{M,N}(Au,Bv,t)} \varphi(s)ds &\geq_{L^*} \phi(\int_0^{\mathcal{M}_{M,N}(z,Bv,t)} \varphi(s)ds) \\
&>_{L^*} \int_0^{\mathcal{M}_{M,N}(z,Bv,t)} \varphi(s)ds,
\end{aligned}$$

which is a contradiction. Then, $z = Bv = Tv$. Since the pairs (A,S) and (B,T) are weakly compatible we have $ASu = SAu$ and $TBv = BTv$; i.e., $Az = Sz$ and $Bz = Tz$.

If $Az \neq z$ using (2.2), we get

$$\begin{aligned}
\int_0^{\mathcal{M}_{M,N}(Az,z,t)} \varphi(s)ds &= \int_0^{\mathcal{M}_{M,N}(Az,Bv,t)} \varphi(s)ds \\
&\geq_{L^*} \phi(\int_0^{\mathcal{L}_{M,N}(z,v,t)} \varphi(s)ds) \\
&= \phi(\int_0^{\mathcal{M}_{M,N}(Az,Bv,t)} \varphi(s)ds) \\
&>_{L^*} \int_0^{\mathcal{M}_{M,N}(Az,z,t)} \varphi(s)ds
\end{aligned}$$

which is a contradiction. Therefore, $Az = z$. Similarly, we can prove that $z = Bz = Tz$. Then, z is a common fixed point of A, B, S and T. The uniqueness of z follows from (2.2). □

Now we give an example to support our theorem 2.2.

Example 2.2. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a intuitionistic fuzzy metric space, where $X = [0,1]$. Denote $\mathcal{T}(a,b) = (a_1 b_1, \min(a_2 + b_2, 1))$ for all $a = (a_1, a_2)$ and $b = (b_1, b_2) \in L^*$. For each $t \in (0, \infty)$, define

$$\mathcal{M}_{M,N}(x,y,t) = (\frac{t}{t + |x-y|}, \frac{|x-y|}{t+|x-y|}) \text{ for all } x, y \in X,$$

$A, B, S, T : X \to X$ by

$$Ax = Bx = 1,$$

$$Sx = \begin{cases} 1 & \text{if} \quad x \text{ is rational}, \\ 0 & \text{if} \quad x \text{ is irrational}, \end{cases}, \qquad Tx = \begin{cases} 1 & \text{if} \quad x \text{ is rational}, \\ \frac{1}{3} & \text{if} \quad x \text{ is irrational}, \end{cases},$$

$$\phi(t_1, t_2) = (\sqrt{t_1}, 0) \quad \text{for } t = (t_1, t_2) \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$$

and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by

$$\varphi(s) = \max\{s^{1/s-2}(1 - \ln s), 0\} \quad \text{for } s > 0 \quad \text{and} \quad \varphi(0) = 0.$$

Then, it is clear that for all $\epsilon > 0$, $\int\limits_0^{\epsilon} \varphi(s)ds = \epsilon^{1/\epsilon} > 0$ and for all $x, y \in X$ and $t > 0$

$$\begin{aligned} \int\limits_0^{\mathcal{M}_{M,N}(Ax,By,t)} \varphi(s)ds &= \int\limits_0^{1} \varphi(s)ds = 1 \\ &\geq_{L^*} \phi(\int_0^{\mathcal{L}_{M,N}(x,y,t)} \varphi(s)ds). \end{aligned}$$

It is easy to see that the other conditions of theorem 2.2 are satisfied, consequently, 1 is the unique common fixed point of A, B, S and T.

If $\varphi(t) = 1$ in theorem 2.2 we obtain a generalization of theorem 2.3 of [25].

Theorem 2.2. *Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ be a complete intuitionistic fuzzy metric space and A, B, S and T be self-mappings of X satisfying* (2.2). *Suppose that the pairs (A, S) and (B, T) satisfy a common property (E.A), $S(X)$ and $T(X)$ are closed subsets of X and the pairs (A, S) and (B, T) are weakly compatible Then, A, B, S and T have a unique common fixed point in X.*

Proof. Suppose that (A, S) and (B, T) satisfy a common property (E.A). Then, there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that for some $z \in X$ and for all $t > 0$.

$$\begin{aligned} \lim_{n\to\infty} \mathcal{M}_{M,N}(Ax_n, z, t) &= \lim_{n\to\infty} \mathcal{M}_{M,N}(Sx_n, z, t) = \lim_{n\to\infty} \mathcal{M}_{M,N}(By_n, z, t) \\ &= \lim_{n\to\infty} \mathcal{M}_{M,N}(Ty_n, z, t) = 1_{L^*}. \end{aligned}$$

Assume that $S(X)$ and $T(X)$ are closed subsets of X. Then, $z = Su = Tv$ for some $u, v \in X$.

If $Au \neq z$, using (2.2) we obtain

$$\int_0^{\mathcal{M}_{M,N}((Au,By_n,t)} \varphi(s)ds \geq_{L^*} \phi(\int_0^{\mathcal{L}_{M,N}(u,y_n,t)} \varphi(s)ds), \tag{2.6}$$

where

$$\begin{aligned}L(u,y_n,t) =& \min\{\mathcal{M}_{M,N}(Su,Ty_n,t),\mathcal{M}_{M,N}(Au,Su,t),\mathcal{M}_{M,N}(By_n,Ty_n,t),\\ &\mathcal{M}_{M,N}(Au,Ty_n,t),\mathcal{M}_{M,N}(Su,By_n,t)\}\\ =& \min\{\mathcal{M}_{M,N}(z,Ty_n,t),\mathcal{M}_{M,N}(Au,z,t),\mathcal{M}_{M,N}(By_n,Ty_n,t),\\ &\mathcal{M}_{M,N}(Au,Ty_n,t),\mathcal{M}_{M,N}(z,By_n,t)\}.\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\mathcal{L}_{M,N}(u,y_n,t)=\mathcal{M}_{M,N}(Au,z,t).$$

Letting $n\to\infty$ in (2.6) we get

$$\begin{aligned}\int_0^{\mathcal{M}_{M,N}(Au,z,t)}\varphi(s)ds \quad &\geq_{L^*} \quad \phi(\int_0^{\mathcal{M}_{M,N}(Au,z,t)}\varphi(s)ds))\\ &>_{L^*} \quad \int_0^{\mathcal{M}_{M,N}(Au,z,t)}\varphi(s)ds.\end{aligned}$$

which is a contradiction. Hence, $\mathcal{M}_{M,N}(Au,z,t)=1_{L^*}$; i.e., $Au=Su=Tv=z$. The rest of the proof follows as in theorem 2.2. □

Theorem 2.3. *Let A,B,S and T be self-mappings of a complete intuitionistic fuzzy metric space $(X,\mathcal{M}_{M,N},\mathcal{T})$ which has the property (C), satisfying (2.1) and there exists $k>1$ such that*

$$\int_0^{\mathcal{M}_{M,N}(Ax,By,t)}\varphi(s)ds \geq_{L^*} \phi(\min\begin{pmatrix}\int_0^{\mathcal{M}_{M,N}(Sx,Ty,kt)}\varphi(s)ds, & \int_0^{\mathcal{M}_{M,N}(Ax,Sx,kt)}\varphi(s)ds,\\ \int_0^{\mathcal{M}_{M,N}(By,Ty,kt)}\varphi(s)ds & \end{pmatrix}), \tag{2.7}$$

for every $x,y\in X$ and all $t>0$, where $\varphi:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ is a Lesbegue integrable mapping which is summable and satisfying (2.3). Suppose that one of $S(X)$ and $T(X)$ is a closed subset of X and the pairs (A,S) and (B,T) are weakly compatible Then, A,B,S and T have a unique common fixed point in X.

Proof. Let $x_0\in X$ be an arbitrary point in X. We can define inductively a sequence $\{y_n\}$ in X such that

$$y_{2n}=Ax_{2n}=Tx_{2n+1}, y_{2n+1}=Bx_{2n+1}=Sx_{2n+2}, \text{ for } n=0,1,2,\cdots. \tag{2.8}$$

First, we prove that $\{y_n\}$ is a Cauchy sequence in X. Set $d_n(t)=\mathcal{M}_{M,N}(y_n,y_{n+1},t)$, $t>0$.

Using (2.7) we have

$$\int_0^{d_{2n}(t)}\varphi(s)ds = \int_0^{\mathcal{M}_{M,N}(y_{2n},y_{2n+1},t)}\varphi(s)ds$$

$$
\begin{aligned}
&= \int_0^{\mathcal{M}_{M,N}(Ax_{2n},Bx_{2n+1},t)} \varphi(s)ds \\
&\geq_{L^*} \phi(\min\begin{pmatrix} \int_0^{\mathcal{M}_{M,N}(y_{2n-1},y_{2n},kt)} \varphi(s)ds, & \int_0^{\mathcal{M}_{M,N}(y_{2n},y_{2n-1},kt)} \varphi(s)ds, \\ \int_0^{\mathcal{M}_{M,N}(y_{2n},y_{2n+1},kt)} \varphi(s)ds & \end{pmatrix}) \\
&= \phi(\min\begin{pmatrix} \int_0^{d_{2n-1}(kt)} \varphi(s)ds, & \int_0^{d_{2n-1}(kt)} \varphi(s)ds, \\ \int_0^{d_{2n}(kt)} \varphi(s)ds & \end{pmatrix})
\end{aligned}
$$

If

$$\int_0^{d_{2n}(kt)} \varphi(s)ds <_{L^*} \int_0^{d_{2n-1}(kt)} \varphi(s)ds$$

for some $n \in \mathbb{N}$ in the above inequality we get

$$
\begin{aligned}
\int_0^{d_{2n}(t)} \varphi(s)ds \quad &\geq_{L^*} \quad \phi(\int_0^{d_{2n}(kt)} \varphi(s)ds) \\
&>_{L^*} \quad \int_0^{d_{2n}(t)} \varphi(s)ds
\end{aligned}
$$

which is a contradiction. Hence

$$\int_0^{d_{2n}(t)} \varphi(s)ds \geq_{L^*} \int_0^{d_{2n-1}(kt)} \varphi(s)ds.$$

Similarly

$$\int_0^{d_{2n+1}(t)} \varphi(s)ds \geq_{L^*} \int_0^{d_{2n}(kt)} \varphi(s)ds.$$

Therefore

$$\int_0^{d_n(t)} \varphi(s)ds \geq_{L^*} \int_0^{d_{n-1}(kt)} \varphi(s)ds.$$

Then $d_n(t) \geq_{L^*} d_{n-1}(kt)$; i.e.,

$$\mathcal{M}(y_n, y_{n+1}, t) \geq_{L^*} \mathcal{M}(y_{n-1}, y_n, kt) \geq_{L^*} ... \geq_{L^*} \mathcal{M}(y_0, y_1, k^n t).$$

By Lemma 1.5, it follows that $\{y_n\}$ is a Cauchy sequence and the completeness of X implies that $\{y_n\}$ converges to z in X. So

$$\lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z.$$

Assume that $S(X)$ is closed. Then there exists $u \in X$ such that $Su = z$. If $z \neq Au$ using (2.7) we obtain

$$\int_0^{\mathcal{M}_{M,N}(Au,Bx_{2n+1},t)} \varphi(s)ds$$

$$\geq_{L^*} \phi(\min\begin{pmatrix} \int_0^{\mathcal{M}_{M,N}(Su,Tx_{2n+1},kt)} \varphi(s)ds, & \int_0^{\mathcal{M}_{M,N}(Au,Su,kt)} \varphi(s)ds, \\ \int_0^{\mathcal{M}_{M,N}(Bx_{2n+1},Tx_{2n+1},kt)} \varphi(s)ds & \end{pmatrix}),$$

Letting $n \to \infty$ we get

$$\int_0^{\mathcal{M}_{M,N}(Au,z,t)} \varphi(s)ds >_{L^*} \int_0^{\mathcal{M}_{M,N}(Au,z,t)} \varphi(s)ds$$

which is a contradiction. Hence $Au = Su = z$. Since $A(X) \subset T(X)$, there exist $v \in X$, such that $Tv = z$.

If $z \neq Bv$ using (2.7) we have

$$\begin{aligned} \int_0^{\mathcal{M}_{M,N}(z,Bv,t)} \varphi(s)ds = &\int_0^{\mathcal{M}_{M,N}(Au,Bv,t)} \varphi(s)ds \\ &\geq_{L^*} \phi(\min\begin{pmatrix} \int_0^{\mathcal{M}_{M,N}(Su,Tv,kt)} \varphi(s)ds, & \int_0^{\mathcal{M}_{M,N}(Au,Su,kt)} \varphi(s)ds, \\ \int_0^{\mathcal{M}_{M,N}(Bv,Tv,kt)} \varphi(s)ds & \end{pmatrix}) \\ &>_{L^*} \int_0^{\mathcal{M}_{M,N}(z,Bv,t)} \varphi(s)ds \end{aligned}$$

which is a contradiction. Hence $Tv = Bv = Au = Su = z$. Since the pairs (A, S) and (B, T) are weakly compatible we have $ASu = SAu$ and $TBv = BTv$; i.e., $Az = Sz$ and $Bz = Tz$.

If $Az \neq z$ using (2.7), we get

$$\int_0^{\mathcal{M}_{M,N}(Az,z,t)} \varphi(s)ds = \int_0^{\mathcal{M}_{M,N}(Az,Bv,t)} \varphi(s)ds$$

$$\geq_{L^*} \phi(\min\begin{pmatrix} \int_0^{\mathcal{M}_{M,N}(Sz,Tv,kt)} \varphi(s)ds, & \int_0^{\mathcal{M}_{M,N}(Az,Sz,kt)} \varphi(s)ds, \\ \int_0^{\mathcal{M}_{M,N}(Bv,Tv,kt)} \varphi(s)ds & \end{pmatrix})$$

$$>_{L^*} \int_0^{\mathcal{M}_{M,N}(Az,z,t)} \varphi(s)ds,$$

which is a contradiction. Hence $Az = Sz = z$. Similarly we can prove that $z = Bz = Tz$. Therefore z is a common fixed point of A, B, S and T. The uniqueness of z follows from (2.7) and property (C). $\square$

If $\varphi(t) = 1$ in theorem 2.5 we get theorem 2.4 of [25].

Theorem 2.4. *Let A, B, S and T be self-mappings of a complete intuitionistic fuzzy metric space $(X, \mathcal{M}_{M,N}, \mathcal{T})$ which has the property (C), satisfying (2.1) and there exists $k > 1$ such that*

$$\int_0^{\mathcal{M}_{M,N}(Ax,By,t)} \varphi(s)ds \geq_{L^*} a(t) \int_0^{\mathcal{M}_{M,N}(Sx,Ty,kt)} \varphi(s)ds$$
$$+ b(t) \int_0^{\min\{\mathcal{M}_{M,N}(Ax,Sx,kt),\mathcal{M}_{M,N}(By,Ty,kt)\}} \varphi(s)ds$$
$$+ c(t) \int_0^{\max\{\mathcal{M}_{M,N}(Ax,Sx,kt),\mathcal{M}_{M,N}(By,Ty,kt)\}} \varphi(s)ds, (2.9)$$

for every $x, y \in X$, where $a, b, c : [0, \infty) \longmapsto [0, 1]$ are three functions such that

$$a(t) + b(t) + c(t) = 1 \ \textit{for all} \ t > 0,$$

$\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a Lesbegue integrable mapping which is summable and satisfying (2.3). Suppose that one of $S(X)$ and $T(X)$ is a closed subset of X and the pairs (A, S) and (B, T) are weakly compatible Then, A, B, S and T have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be an arbitrary point in X. We can define inductively a sequence $\{y_n\}$ in X defined by (2.8).

First, we prove that $\{y_n\}$ is a Cauchy sequence in X. Set $d_n(t) = \mathcal{M}_{M,N}(y_n, y_{n+1}, t)$, $t > 0$.

Using (2.9) we have

$$\int_0^{d_{2n+1}(t)} \varphi(s)ds = \int_0^{\mathcal{M}_{M,N}(y_{2n+2},y_{2n+1},t)} \varphi(s)ds$$

$$
\begin{aligned}
&= \int_0^{\mathcal{M}_{M,N}(Ax_{2n+2},Bx_{2n+1},t)} \varphi(s)ds \\
&\geq_{L^*} a(t) \int_0^{\mathcal{M}_{M,N}(Sx_{2n+2},Tx_{2n+1},kt)} \varphi(s)ds \\
&\quad + b(t) \int_0^{\min\{\mathcal{M}_{M,N}(Sx_{2n+2},Ax_{2n+2},kt),\mathcal{M}_{M,N}(Tx_{2n+1},Bx_{2n+1},kt)\}} \varphi(s)ds \\
&\quad + c(t) \int_0^{\max\{\mathcal{M}_{M,N}(Sx_{2n+2},Ax_{2n+2},kt),\mathcal{M}_{M,N}(Tx_{2n+1},Bx_{2n+1},kt)\}} \varphi(s)ds \\
&= a(t) \int_0^{d_{2n}(kt)} \varphi(s)ds + b(t) \int_0^{\min\{d_{2n+1}(kt),d_{2n}(kt)\}} \varphi(s)ds + \\
&\quad c(t) \int_0^{\max\{d_{2n+1}(kt),d_{2n}(kt)\}} \varphi(s)ds.
\end{aligned}
$$

If

$$
\int_0^{d_{2n}(kt)} \varphi(s)ds >_{L^*} \int_0^{d_{2n+1}(kt)} \varphi(s)ds
$$

for some $n \in \mathbb{N}$ in the above inequality we get

$$
\begin{aligned}
\int_0^{d_{2n+1}(t)} \varphi(s)ds \quad &>_{L^*} \int_0^{d_{2n+1}(kt)} \varphi(s)ds \\
&>_{L^*} \int_0^{d_{2n+1}(t)} \varphi(s)ds
\end{aligned}
$$

which is a contradiction. Hence

$$
\int_0^{d_{2n+1}(t)} \varphi(s)ds \geq_{L^*} \int_0^{d_{2n}(kt)} \varphi(s)ds.
$$

As in the proof of theorem 2.4, $\{y_n\}$ is a Cauchy sequence and the completeness of X implies that $\{y_n\}$ converges to z in X. Assume that $S(X)$ is closed. Then there exists $u \in X$ such that $Su = z$.

If $z \neq Au$ using (2.9) we have

$$\begin{aligned}\int_0^{\mathcal{M}_{M,N}(Au,Bx_{2n+1},t)} \varphi(s)ds \geq_{L^*} a(t) & \int_0^{\mathcal{M}_{M,N}(Su,Tx_{2n+1},kt)} \varphi(s)ds \\ &+ b(t) \int_0^{\min\{\mathcal{M}_{M,N}(Au,Su,kt),\mathcal{M}_{M,N}(Bx_{2n+1},Tx_{2n+1},kt)\}} \varphi(s)ds \\ &+ c(t) \int_0^{\max\{\mathcal{M}_{M,N}(Au,Su,kt),\mathcal{M}_{M,N}(Bx_{2n+1},Tx_{2n+1},kt)\}} \varphi(s)ds\end{aligned}$$

Letting $n \to \infty$ we get

$$\begin{aligned}\int_0^{\mathcal{M}_{M,N}(Au,z,t)} \varphi(s)ds \geq_{L^*} & \quad a(t) \quad + b(t) \int_0^{\mathcal{M}_{M,N}(Au,z,kt)} \varphi(s)ds + c(t) \\ & >_{L^*} \int_0^{\mathcal{M}_{M,N}(Au,z,t)} \varphi(s)ds\end{aligned}$$

which is a contradiction. Hence $Au = Su = z$. Since $A(X) \subseteq T(X)$, there exist $v \in X$, such that $Tv = z$.

If $z \neq Bv$ using (2.9) we obtain

$$\begin{aligned}\int_0^{\mathcal{M}_{M,N}(z,Bv,t)} \varphi(s)ds = & \int_0^{\mathcal{M}_{M,N}(Au,Bv,t)} \varphi(s)ds \\ \geq_{L^*} & a(t) \int_0^{\mathcal{M}_{M,N}(Su,Tv,kt)} \varphi(s)ds \\ & + b(t) \int_0^{\min\{\mathcal{M}_{M,N}(Au,Su,kt),\mathcal{M}_{M,N}(Bv,Tv,kt)\}} \varphi(s)ds \\ & + c(t) \int_0^{\max\{\mathcal{M}_{M,N}(Au,Su,kt),\mathcal{M}_{M,N}(Bv,Tv,kt)\}} \varphi(s)ds \\ = & a(t) + b(t) \int_0^{\mathcal{M}_{M,N}(Au,Bv,t)} \varphi(s)ds + c(t) >_{L^*} \int_0^{\mathcal{M}_{M,N}(Au,Bv,t)} \varphi(s)ds,\end{aligned}$$

which is a contradiction. Hence $Tv = Bv = Au = Su = z$. Since the pairs (A, S) and (B, T) are weakly compatible we have $ASu = SAu$ and $TBv = BTv$; i.e., $Az = Sz$ and $Bz = Tz$.

If $Az \neq z$ using (2.9), we have

$$\begin{aligned}
\int_0^{\mathcal{M}_{M,N}(Az,z,t)} \varphi(s)ds &= \int_0^{\mathcal{M}_{M,N}(Az,Bv,t)} \varphi(s)ds \\
&\geq_{L^*} a(t) \int_0^{\mathcal{M}_{M,N}(Sz,Tv,kt)} \varphi(s)ds \\
&\quad + b(t) \int_0^{\min\{\mathcal{M}_{M,N}(Az,Sz,kt),\mathcal{M}_{M,N}(Bv,Tv,kt)\}} \varphi(s)ds \\
&\quad + c(t) \int_0^{\max\{\mathcal{M}_{M,N}(Az,Sz,kt),\mathcal{M}_{M,N}(Bv,Tv,kt)\}} \varphi(s)ds \\
&= a(t) \int_0^{\mathcal{M}_{M,N}(Az,z,kt)} \varphi(s)ds + b(t) + c(t) \\
&>_{L^*} \int_0^{\mathcal{M}_{M,N}(Az,z,t)} \varphi(s)ds,
\end{aligned}$$

which is a contradiction. Hence $Az = Sz = z$. Similarly we can prove that $z = Bz = Tz$.. Therefore z is a common fixed point of A, B, S and T. The uniqueness of z follows from (2.9). □

If $\varphi(t) = 1$ in theorem 2.6 we obtain a generalization of theorem 2.6 of [25].

Example 2.3. Let $(X, \mathcal{M}_{M,N}, \mathcal{T})$ as in example 2.3. Define $A, B, S, T : X \to X$ by

$$Ax = Bx = 1,$$

$$Sx = \begin{cases} \frac{1}{3} & \text{if} \quad x \in [0,1), \\ 1 & \text{if} \quad x = 1, \end{cases} \qquad Tx = \begin{cases} \frac{1}{6} & \text{if} \quad x \in [0,1), \\ 1 & \text{if} \quad x = 1, \end{cases}$$

$$\phi(t_1, t_2) = (\sqrt{t_1}, 0) \quad \text{for } t = (t_1, t_2) \in L^* \setminus \{0_{L^*}, 1_{L^*}\},$$

$\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ by

$$\varphi(s) = \max\{s^{1/s-2}(1 - \ln s), 0\} \quad \text{for } s > 0 \text{ and } \varphi(0) = 0$$

and $a, b, c : [0, \infty) \longmapsto [0, 1]$ by

$$a(t) = \frac{t^2}{t^2 + t + 1}, \quad b(t) = \frac{t}{t^2 + t + 1}, c(t) = \frac{1}{t^2 + t + 1} \quad \text{for all } t > 0.$$

Then, it is clear that for all $\epsilon > 0$, $\int\limits_0^{\epsilon} \varphi(s)ds = \epsilon^{\frac{1}{\epsilon}} > 0$ and for all $x, y \in X$ and $t > 0$

$$\begin{aligned}
\int\limits_0^{\mathcal{M}_{M,N}(Ax,By,t)} \varphi(s)ds = \int\limits_0^1 \varphi(s)ds = 1 \\
&\geq_{L^*} a(t) \int\limits_0^{\mathcal{M}_{M,N}(Sx,Ty,kt)} \varphi(s)ds \\
&+ b(t) \int\limits_0^{\min\{\mathcal{M}_{M,N}(Ax,Sx,kt),\mathcal{M}_{M,N}(By,Ty,kt)\}} \varphi(s)ds \\
&+ c(t) \int\limits_0^{\max\{\mathcal{M}_{M,N}(Ax,Sx,kt),\mathcal{M}_{M,N}(By,Ty,kt)\}} \varphi(s)ds
\end{aligned}$$

It is easy to see that the other conditions of theorem 2.6 are satisfied, consequently, 1 is the unique common fixed point of A, B, S and T.

Moreover, for $\varphi(t) = 1$, theorem 2.6 of [25] is not applicable since S and T are not continuous.

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