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# On $\mathscr{I}-$Proximity Spaces

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**Abstract:** An *ideal* $\mathscr{I}$ on a nonempty set $X$ is a subfamily of $P(X)$ which is closed under finite unions and subsets. The purpose of this paper is to introduce $\delta_{\mathscr{I}}-$neighborhood in an $\mathscr{I}-$*proximity space* and provides an alternative description to the study of $\mathscr{I}-$proximity spaces. Moreover, a new topology $\tau^*_{\mathfrak{U}}$ via ideal and uniform space $(X,\mathfrak{U})$ is introduced. On comparing with an old topology, it is found that the present one is finer.

**Keywords:** $\mathscr{I}-$proximity space, uniform space, $*-$normal space, ideal

## 1 Introduction

The fundamental concept of *Efremovič proximity space* has been introduced by Efremovič [2]. In addition, Leader [13,14] and Lodato [15,16] have worked with weaker axioms than those of Efremovič proximity space enabling them to introduce an arbitrary topology on the underlying set. Furthermore, *proximity relations* are useful in solving problems based on human perception [20] that arise in areas such as image analysis [5] and face recognition [3]. Cyclic contraction and best proximity point are among the popular topics in the fixed point theory and many results have been obtained, for instance, [1,8,9,24]. For further results and applications of proximity relations. (See [12,11,21,22,19].)

The notion of *ideal topological spaces* was first studied by Kuratowski [7] and Vaidyanathaswamy [25]. *Compatibility* of the topology with an ideal $\mathscr{I}$ was first defined by Njastad [17]. In 1990, Jankovic and Hamlett [6] investigated further properties of ideal topological spaces. Zhan [27] introduced the uncertainties of ideal theory on hemirings.

This paper is an attempt to induce a new proximity relation via uniformity and ideal. In Section 2, all preliminaries and theorems of $\mathscr{I}-$proximity structures, uniformity and ideals which will be needed in the sequel are briefly mentioned. In Section 3, $\delta_{\mathscr{I}}-$neighborhood in an $\mathscr{I}-$proximity space and provides an alternative description to the study of $\mathscr{I}-$proximity spaces. In Section 4, the operator $*$ on $P(X)$ with respect to an ideal and uniformity $\mathfrak{U}$ on $X$ is introduced and various properties of it are investigated. Moreover, the new generated uniformity via ideal is presented which generates a topology $\tau^*_{\mathfrak{U}}$ finer than the old one. Furthermore, $\tau^*_{\mathfrak{U}} = \tau_{\delta_{\mathscr{I}}}$ is proved.

## 2 Preliminaries and basic definitions

Let $(X,\tau)$ be a topological space. For a subset $A$ of $X$, $\overline{A}$ and $A^o$ denote the closure and the interior of $A$ in $(X,\tau)$, respectively.

**Definition 2.1.** [6] A nonempty collection $\mathscr{I}$ of subsets of a set $X$ is called an *ideal* on $X$, if it satisfies the following assertions:

1. $A \in \mathscr{I}$ and $B \in \mathscr{I} \Rightarrow A \cup B \in \mathscr{I}$,
2. $A \in \mathscr{I}$ and $B \subseteq A \Rightarrow B \in \mathscr{I}$.

That is, $\mathscr{I}$ is closed under finite unions and subsets.

An *ideal topological space* is a topological space $(X,\tau)$ with an ideal $\mathscr{I}$ on $X$ and is denoted by $(X,\tau,\mathscr{I})$. For a subset $A \subseteq X$, $A^*(\mathscr{I},\tau) := \{x \in X : A \cap U \notin \mathscr{I}$ for every open set $U$ containing $x\}$ is called the *local function* of $A$ with respect to $\mathscr{I}$ and $\tau$. (See [6,7,23].) We simply write $A^*$ instead of $A^*(\mathscr{I},\tau)$ in case there is no chance for confusion.

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**Proposition 2.2.** [6] Let $(X,\tau)$ be a topological space and $\mathscr{I}$ be an ideal on $X$. Then the operator

$$Cl^*: P(X) \to P(X)$$

defined by:

$$Cl^*(A) = A \cup A^* \tag{2.1}$$

satisfies Kuratwski's axioms and induces a topology $\tau^*$ on $X$ given by

$$\tau^* = \{A \subseteq X : Cl^*(A^c) = A^c\}. \tag{2.2}$$

Where $A^c$ denotes the complement of $A$.

Indeed, for every ideal topological space $(X,\tau,\mathscr{I})$, there exists a topology $\tau^*$ finer than $\tau$. For a subset $A \subseteq X$, $Cl^*(A)$ and $Int^*(A)$ will denote the closure and the interior of $A$ with respect to $\tau^*$, respectively.

**Definition 2.3.** [10] Let $\mathscr{I}$ be an ideal on a nonempty set $X$. A binary relation $\delta_{\mathscr{I}}$ on $P(X)$ is called an *$\mathscr{I}-$proximity relation* on $X$ if $\delta_{\mathscr{I}}$ satisfies the following conditions:

$(IP_1)$ $A\delta_{\mathscr{I}}B \Rightarrow B\delta_{\mathscr{I}}A$,
$(IP_2)$ $A\delta_{\mathscr{I}}(B\cup C) \Leftrightarrow A\delta_{\mathscr{I}}B$ *or* $A\delta_{\mathscr{I}}C$,
$(IP_3)$ $A\not\delta_{\mathscr{I}}B\ \forall A \in \mathscr{I},\ B \in P(X)$,
$(IP_4)$ $A\cap B \notin \mathscr{I} \Rightarrow A\delta_{\mathscr{I}}B$,
$(IP_5)$ $A\not\delta_{\mathscr{I}}B \Rightarrow \exists\, C,D \subseteq X$ such that $A\not\delta_{\mathscr{I}}C^c$, $D^c\not\delta_{\mathscr{I}}B$ and $C\cap D \in \mathscr{I}$.

An *$\mathscr{I}-$proximity space* is a pair $(X,\delta_{\mathscr{I}})$ consisting of a set $X$ and an *$\mathscr{I}-$proximity relation* on $X$. We shall write $A\delta_{\mathscr{I}}B$ if the sets $A,B \subseteq X$ are $\delta_{\mathscr{I}}-$related, otherwise we shall write $A\not\delta_{\mathscr{I}}B$.

$\delta_{\mathscr{I}}$ is said to be *separated*, if it satisfies:

$(IP_6)$ $x\delta_{\mathscr{I}}y \Rightarrow x = y$.

**Lemma 2.4.** [10] If $A\delta_{\mathscr{I}}B$, $A \subseteq C$, and $B \subseteq D$, then $C\delta_{\mathscr{I}}D$.

**Theorem 2.5.** [10] Let $(X,\delta_{\mathscr{I}})$ be an $\mathscr{I}-$proximity space and $A^{\delta_{\mathscr{I}}} = \{x \in X : x\delta_{\mathscr{I}}A\}$. Then

$$A^{\delta_{\mathscr{I}}} - B^{\delta_{\mathscr{I}}} \subseteq (A-B)^{\delta_{\mathscr{I}}}.$$

**Theorem 2.6.** [10] Let $(X,\delta_{\mathscr{I}})$ be an $\mathscr{I}-$proximity space. Then the operator

$$Cl^{\delta_{\mathscr{I}}}: P(X) \to P(X)$$

defined by

$$Cl^{\delta_{\mathscr{I}}}(A) = A \cup A^{\delta_{\mathscr{I}}} \tag{2.3}$$

satisfies Kuratwski's axioms and induces a topology on $X$ called $\tau_{\delta_{\mathscr{I}}}$ given by:

$$\tau_{\delta_{\mathscr{I}}} = \{A \subseteq X : Cl^{\delta_{\mathscr{I}}}(A^c) = A^c\}. \tag{2.4}$$

**Theorem 2.7.** [10] Let $(X,\delta_{\mathscr{I}})$ be an $\mathscr{I}-$proximity space. Then the closure operator defined in (2.3) has the following property:

$$B\not\delta_{\mathscr{I}}A \Leftrightarrow B\not\delta_{\mathscr{I}}Cl^{\delta_{\mathscr{I}}}(A). \tag{2.5}$$

**Definition 2.8.** [10] A topological space $(X,\tau)$ is called *$*-$normal space* if $\forall F_1,F_2 \in \tau^{*^c}$ such that $F_1\cap F_2 \in \mathscr{I}$ then $\exists\, H,G \in \tau$ such that $F_1 \subseteq H$, $F_2 \subseteq G$ and $H\cap G \in \mathscr{I}$, where $\tau^{*^c}$ is the family of all $\tau^*-$closed sets.

**Theorem 2.9.** [10] Let $(X,\tau)$ be a $*-$normal space and $\delta_{\mathscr{I}}$ be a relation on $P(X)$ defined as:

$$A\delta_{\mathscr{I}}B \Leftrightarrow Cl^*(A)\cap Cl^*(B) \notin \mathscr{I}\ \forall A,B \subseteq X. \tag{2.6}$$

Then $\delta_{\mathscr{I}}$ is an $\mathscr{I}-$proximity relation on $X$.

**Definition 2.10.** [10] A topological space $(X,\tau)$ is called an *$\mathscr{I}-$proximizable space*, denoted by $\tau \approx \delta_{\mathscr{I}}$, if there exists $\mathscr{I}-$proximity relation $\delta_{\mathscr{I}}$ such that $\tau_{\delta_{\mathscr{I}}} = \tau^*$.

**Theorem 2.11.** [10] Let $\mathscr{I}$ be an ideal on a nonempty set $X$, $(X,\tau)$ be a $*-$normal $T_1$ space and $\delta_{\mathscr{I}}$ is the formula (2.6). Then $\tau \approx \delta_{\mathscr{I}}$.

**Definition 2.12.** [18] Let $X$ be a nonempty set, $A \subseteq X$ and $R \subseteq X\times X$. Then

$$R[A] = \{y \in X : (x,y) \in R \text{ for some } x \in A\}. \text{ For } x \in X,\quad R[x] = R[\{x\}].$$

**Definition 2.13.** [18] A *uniform structure* or (*uniformity*) $\mathfrak{U}$ on a set $X$ is a collection of subsets of $X\times X$ satisfying the following conditions:

1. Every $R \in \mathfrak{U}$ contains the diagonal $\Delta = \{(x,x) : x \in X\}$,
2. If $R_1 \in \mathfrak{U}$ and $R_2 \in \mathfrak{U}$, then $R_1\cap R_2 \in \mathfrak{U}$,
3. Given $R_1 \in \mathfrak{U}$, there exists a $R_2 \in \mathfrak{U}$ such that $R_2\circ R_2 \subseteq R_1$,
4. If $R_1 \in \mathfrak{U}$ and $R_1 \subseteq R_2$, then $R_2 \in \mathfrak{U}$,
5. If $R \in \mathfrak{U}$, then $R^{-1} \in \mathfrak{U}$.

The pair $(X,\mathfrak{U})$ is called an *uniform space*.

**Theorem 2.14.** [18] Let $(X,\mathfrak{U})$ be a uniform space. Then $\beta_x = \{R[x] : R \in \mathfrak{U}\}$ is a *neighborhood filter* and $\mathscr{N}_x = \{V \subseteq X : B \subseteq V \text{ for some } B \in \beta_x\}$ is a *neighborhood system* at $x$. Furthermore, $\tau_{\mathfrak{U}} = \{G \subseteq X :\ \forall x \in G\ \exists\, V \in \mathscr{N}_x \text{ such that} V \subseteq G\}$ is a topology on $X$.

**Theorem 2.15.** [18] Let $(X,\mathfrak{U})$ be a uniform space. Then the closure of $\tau_{\mathfrak{U}}$ defined as

$$Cl(A) = \cap_{R\in\mathfrak{U}} R[A]. \tag{2.7}$$

**Definition 2.16.** [4] Let $(X,\tau,\mathscr{I})$ be a topological space with an ideal $\mathscr{I}$ and $A \subseteq X$. Then $A$ is called *$\mathscr{I}-$compact* if for every open cover $\{H_j : j \in J\}$ of $A$, there exists a finite subcover $\{H_i : i = 1,\dots,n\}$ such that $(A - \cup_{i=1}^{i=n} H_i) \in \mathscr{I}$.

## 3 $\mathscr{I}-$ proximal neighborhood structures

**Definition 3.1.** A subset $B$ of an $\mathscr{I}-$proximity space $(X,\delta_{\mathscr{I}})$ is a *$\delta_{\mathscr{I}}-$neighborhood* of $A$ (in symbols $A \ll_{\mathscr{I}} B$) if and only if $A\not\delta_{\mathscr{I}}B^c$.

**Theorem 3.2.** Let $(X,\delta_{\mathscr{I}})$ be an $\mathscr{I}-$proximity space. Then

1. $A \ll_{\mathscr{I}} B$ implies $Cl^{\delta_{\mathscr{I}}}(A) \ll_{\mathscr{I}} B$,
2. $A \ll_{\mathscr{I}} B$ implies $A \ll_{\mathscr{I}} int^{\delta_{\mathscr{I}}}(B)$.

**Proof.**

1. Let $A \ll_{\mathscr{I}} B$. Then $A \not\delta_{\mathscr{I}} B^c$. Hence Theorem 2.7 implies $Cl^{\delta_{\mathscr{I}}}(A) \not\delta_{\mathscr{I}} B^c$; that is, $Cl^{\delta_{\mathscr{I}}}(A) \ll_{\mathscr{I}} B$.
2. $A \not\delta_{\mathscr{I}} B^c$ implies $A \not\delta_{\mathscr{I}} Cl^{\delta_{\mathscr{I}}}(B^c)$. Equivalently, $A \not\delta_{\mathscr{I}} (int^{\delta_{\mathscr{I}}}(B))^c$, i.e. $A \ll_{\mathscr{I}} int^{\delta_{\mathscr{I}}}(B)$.

**Theorem 3.3.** Let $(X, \delta_{\mathscr{I}})$ be an $\mathscr{I}-$proximity space. Then the relation $\ll_{\mathscr{I}}$ satisfies the following properties:

1. $X \ll_{\mathscr{I}} X$,
2. $A \ll_{\mathscr{I}} B$ implies $A \cap B^c \in \mathscr{I}$,
3. $A \subseteq B \ll_{\mathscr{I}} C \subseteq D$ implies $A \ll_{\mathscr{I}} D$,
4. $A \ll_{\mathscr{I}} B_i$ for $i = 1, 2, ..., n$ if and only if $A \ll_{\mathscr{I}} \cap_{i=1}^{n} B_i$,
5. $A \ll_{\mathscr{I}} B$ implies $B^c \ll_{\mathscr{I}} A^c$,
6. If $A \in \mathscr{I}$ or $B \in \mathscr{I}$, then $A \ll_{\mathscr{I}} B^c$,
7. $A \ll_{\mathscr{I}} B$ implies $\exists\ C, D \subseteq X$ such that $A \ll_{\mathscr{I}} C, D^c \ll_{\mathscr{I}} B$ and $C \cap D \in \mathscr{I}$,
8. If $\delta_{\mathscr{I}}$ is a separated $\mathscr{I}-$proximity, then $x \neq y \Rightarrow x \ll_{\mathscr{I}} \{y\}^c$.

**Proof.**

1. $(IP_3)$ and Definition 2.1 indicate $X \not\delta_{\mathscr{I}} \emptyset$. Hence $X \ll_{\mathscr{I}} X$.
2. If $A \ll_{\mathscr{I}} B$, then $(IP_4)$ implies $A \cap B^c \in \mathscr{I}$.
3. If $A \not\ll_{\mathscr{I}} D$, then $A \delta_{\mathscr{I}} D^c$. Lemma 2.4 implies that $B \delta_{\mathscr{I}} C^c$, i.e. $B \not\ll_{\mathscr{I}} C$, which is contradiction.
4. It suffices to consider $n = 2$. $A \ll_{\mathscr{I}} B_1$ and $A \ll_{\mathscr{I}} B_2 \Leftrightarrow A \not\delta_{\mathscr{I}} B_1^c$ and $A \not\delta_{\mathscr{I}} B_2^c \Leftrightarrow A \not\delta_{\mathscr{I}} (B_1^c \cup B_2^c) \Leftrightarrow A \not\delta_{\mathscr{I}} (B_1 \cap B_2)^c \Leftrightarrow A \ll_{\mathscr{I}} (B_1 \cap B_2)$.
5. $A \ll_{\mathscr{I}} B$ implies $A \not\delta_{\mathscr{I}} B^c$ and $(IP_1)$ implies $B^c \not\delta_{\mathscr{I}} A$, i.e. $B^c \ll_{\mathscr{I}} A^c$.
6. Let $A \in \mathscr{I}$. Hence $(IP_3)$ implies $A \not\delta_{\mathscr{I}} B$. It follows that $A \ll_{\mathscr{I}} B^c$. Similarly, if $B \in \mathscr{I}$.
7. $A \ll_{\mathscr{I}} B$ implies $A \not\delta_{\mathscr{I}} B^c$. $(IP_5)$ implies $\exists\ C, D$ such that $A \not\delta_{\mathscr{I}} C^c$, $B^c \not\delta_{\mathscr{I}} D^c$ and $C \cap D \in \mathscr{I}$; that is, $A \ll_{\mathscr{I}} C$, $D^c \ll_{\mathscr{I}} B$ and $C \cap D \in \mathscr{I}$.
8. $x \neq y \Rightarrow x \not\delta_{\mathscr{I}} y$ by $(IP_6) \Rightarrow x \ll_{\mathscr{I}} \{y\}^c$.

**Corollary 3.4.** $A_i \ll_{\mathscr{I}} B_i$ for $i = 1, 2, ..., n$ implies

$$\cap_{i=1}^{n} A_i \ll_{\mathscr{I}} \cap_{i=1}^{n} B_i \text{ and } \cup_{i=1}^{n} A_i \ll_{\mathscr{I}} \cup_{i=1}^{n} B_i.$$

**Theorem 3.5.** If $\ll_{\mathscr{I}}$ is a binary relation on $X$ satisfying $(1)-(7)$ in Theorem 3.3 and $\delta_{\mathscr{I}}$ is defined by

$$A \not\delta_{\mathscr{I}} B \Leftrightarrow A \ll_{\mathscr{I}} B^c, \tag{3.1}$$

then $\delta_{\mathscr{I}}$ is an $\mathscr{I}-$proximity relation on $X$. $B$ is a $\delta_{\mathscr{I}}-$neighborhood of $A$ if and only if $A \ll_{\mathscr{I}} B$. Moreover, if $\ll_{\mathscr{I}}$ also satisfies $(8)$ in Theorem 3.3, then $\delta_{\mathscr{I}}$ is separated.

**Proof.**

$(IP_1)$ $A \not\delta_{\mathscr{I}} B$ implies $A \ll_{\mathscr{I}} B^c$. By Theorem 3.3 (5), $B \ll_{\mathscr{I}} A^c$, and hence $B \not\delta_{\mathscr{I}} A$.

$(IP_2)$ $(A \cup B) \not\delta_{\mathscr{I}} C$ implies $(A \cup B) \ll_{\mathscr{I}} C^c$. Then by Theorem 3.3 (3), $A \ll_{\mathscr{I}} C^c$ and $B \ll_{\mathscr{I}} C^c$, i.e. $A \not\delta_{\mathscr{I}} C$ and $B \not\delta_{\mathscr{I}} C$. Conversely, if $A \not\delta_{\mathscr{I}} C$ and $B \not\delta_{\mathscr{I}} C$ then by part (1), $C \not\delta_{\mathscr{I}} A$ and $C \not\delta_{\mathscr{I}} B$; that is, $C \ll_{\mathscr{I}} A^c$ and $C \ll_{\mathscr{I}} B^c$. thus by Theorem 3.3 (4), $C \ll_{\mathscr{I}} (A^c \cap B^c)$, i.e. $C \ll_{\mathscr{I}} (A \cup B)^c$. Hence $C \not\delta_{\mathscr{I}} (A \cup B)$.

$(IP_3)$ If $A \in \mathscr{I}$. Then Theorem 3.3 (6) implies $A \ll_{\mathscr{I}} B^c$. Hence $A \not\delta_{\mathscr{I}} B$. Similarly if $B \in \mathscr{I}$.

$(IP_4)$ If $A \not\delta_{\mathscr{I}} B$, then $A \ll_{\mathscr{I}} B^c$. From Theorem 3.3 (2) we have $A \cap B \in \mathscr{I}$.

$(IP_5)$ Suppose $A \not\delta_{\mathscr{I}} B$, i.e. $A \ll_{\mathscr{I}} B^c$. By Theorem 3.3 (7), $\exists\ C, D \subseteq X$ such that $A \ll_{\mathscr{I}} C, B \ll_{\mathscr{I}} D$ and $C \cap D \in \mathscr{I}$. Thus $\exists\ C, D$ such that $A \not\delta_{\mathscr{I}} C^c$, $A \not\delta_{\mathscr{I}} D^c$ and $C \cap D \in \mathscr{I}$.

$(IP_6)$ Let $x \neq y$. Then Theorem 3.3 (8) implies $x \ll_{\mathscr{I}} \{y\}^c$. Thus $x \not\delta_{\mathscr{I}} y$. Hence $\delta_{\mathscr{I}}$ is separated.

**Theorem 3.6.** If $A \ll_{\mathscr{I}} B$ and $B \in \mathscr{I}$, then $A \in \mathscr{I}$.

**Proof.** $A \ll_{\mathscr{I}} B$ implies $A \not\delta_{\mathscr{I}} B^c$. $(IP_4)$, $B \in \mathscr{I}$ and Definition 2.1 imply $(A \cap B^c) \cup B \in \mathscr{I}$; that is, $A \cup B \in \mathscr{I}$. Then Definition 2.1 implies $A \in \mathscr{I}$.

**Theorem 3.7.** $A \ll_{\mathscr{I}} B\ \forall\ B \subseteq X$ if and only if $A \in \mathscr{I}$.

**Proof.** Let $A \ll_{\mathscr{I}} B\ \forall\ B \subseteq X$. It follows that $A \ll_{\mathscr{I}} \emptyset$, i.e. $A \not\delta_{\mathscr{I}} X$. Thus by $(IP_4)$, $A \in \mathscr{I}$. Conversely, if $A \in \mathscr{I}$ then $(IP_3)$ implies that $A \not\delta_{\mathscr{I}} B^c\ \forall\ B \subseteq X$. So, $A \ll_{\mathscr{I}} B\ \forall\ B \subseteq X$.

**Theorem 3.8.** Let $\mathscr{I}$ be an ideal on a nonempty set $X$, $\delta_{\mathscr{I}}$ be an $\mathscr{I}-$proximity on $X$ and $(X, \tau)$ be a $*-$normal $T_1$ space such that $\tau \approx \delta_{\mathscr{I}}$. If $A$ is $\mathscr{I}-$compact, $B$ is closed set in $\tau^*$ and $A \cap B \in \mathscr{I}$, then $A^{\delta_{\mathscr{I}}} \not\delta B^{\delta_{\mathscr{I}}}$.

**Proof.** For all $a \in A$, if $a \in B$. It follows that $\{a\} \in \mathscr{I}$, and hence $(IP_3)$ implies $a \not\delta_{\mathscr{I}} B$. Also, if $a \notin B$, $B$ is closed, hence $a \not\delta_{\mathscr{I}} B$. This result implies there exists $E \subseteq X$ such that $a \not\delta_{\mathscr{I}} E^c$ and $E \not\delta_{\mathscr{I}} B$; that is, $a \ll_{\mathscr{I}} E$ and $E \ll_{\mathscr{I}} B^c$. Theorem 3.2 part (2) implies $a \ll_{\mathscr{I}} int^{\delta_{\mathscr{I}}}(E) \subseteq E \ll_{\mathscr{I}} B^c$. Let $N_a = int^{\delta_{\mathscr{I}}}(E)$. Hence $N_a \not\delta_{\mathscr{I}} B$. Now $\{N_a : a \in A\}$ is an open cover of $A$. Since $A$ is $\mathscr{I}-$compact, there is a finite subcover $\{N_{a_i} : i = 1, 2, ..., n\}$ such that $(A - \cup_{i=1}^{i=n} N_{a_i}) \in \mathscr{I}$. Let $N = \cup_{i=1}^{i=n} N_{a_i}$. Then $(IP_2)$ implies $N \not\delta_{\mathscr{I}} B$. This result, combined with Theorem 2.7, implies $Cl^{\delta_{\mathscr{I}}}(N) \not\delta_{\mathscr{I}} B$. Since $(A - N) \in \mathscr{I}$, hence Theorem 2.5 and $(IP_3)$ imply $A^{\delta_{\mathscr{I}}} - N^{\delta_{\mathscr{I}}} \subseteq (A - N)^{\delta_{\mathscr{I}}} = \emptyset$. Therefore, $A^{\delta_{\mathscr{I}}} \subseteq N^{\delta_{\mathscr{I}}} \subseteq Cl^{\delta_{\mathscr{I}}}(N)$. This result, combined with Lemma 2.4 and $B$ is closed, implies $A^{\delta_{\mathscr{I}}} \not\delta B^{\delta_{\mathscr{I}}}$.

## 4 $\mathscr{I}-$proximity spaces induced by uniformity

**Theorem 4.1.** Let $(X, \mathfrak{U})$ be a uniform space. For all $R_1 \in \mathfrak{U}$ and $x, y \in X$ such that $y \in R_1[x]$. Then there exists a $R_2 \in \mathfrak{U}$ such that $R_1[y] \subseteq R_2[x]$.

**Proof.** Let $R_1 \in \mathfrak{U}$ and $x,y \in X$ such that $y \in R_1[x]$ and let $z \in R_1[y]$. It follows that $(x,y) \in R_1$ and $(y,z) \in R_1$, and hence $(x,z) \in R_1 \circ R_1$. This result, combined with $\mathfrak{U}$ as a uniformity, implies there exists a $R_2 \in \mathfrak{U}$ such that $R_1 \circ R_1 \subseteq R_2$. Consequently, $(x,z) \in R_2$; that is, $z \in R_2[x]$. Then the result.

**Theorem 4.2.** Let $(X,\mathfrak{U})$ be a uniform space and $\mathscr{I}$ be an ideal on $X$. Then the operator

$$*: P(X) \to P(X)$$

defined by

$$A^*_{\mathfrak{U}} = \{x \in X : R[x] \cap A \notin \mathscr{I} \ for\ all\ R \in \mathfrak{U}\} \tag{4.1}$$

satisfies the following:-

1. $\emptyset^*_{\mathfrak{U}} = \emptyset$,
2. If $A \subseteq B$, then $A^*_{\mathfrak{U}} \subseteq B^*_{\mathfrak{U}}$,
3. $(A \cup B)^*_{\mathfrak{U}} = A^*_{\mathfrak{U}} \cup B^*_{\mathfrak{U}}$,
4. $(A^*_{\mathfrak{U}})^* \subseteq A^*_{\mathfrak{U}}$,
5. If $\mathscr{I} \subseteq \mathscr{J}$, then $A^*_{\mathfrak{U}}(\mathscr{J}) \subseteq A^*_{\mathfrak{U}}(\mathscr{I})$.

**Proof.**

1. The formula (4.1) and $\emptyset \in \mathscr{I}$ imply $\emptyset^*_{\mathfrak{U}} = \emptyset$.
2. Let $x \in A^*_{\mathfrak{U}}$. Then $R[x] \cap A \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}$. Hence $A \subseteq B$, combined with Definition 2.1, implies $R[x] \cap B \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}$. So $x \in B^*_{\mathfrak{U}}$. Then the result.
3. $$\begin{aligned}(A \cup B)^*_{\mathfrak{U}} =& \{x \in X : R[x] \cap (A \cup B) \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}\}\\ =& \{x \in X : ((R[x] \cap A) \cup (R[x] \cap B)) \notin \mathscr{I}\\ & \forall\ R \in \mathfrak{U}\}\\ =& \{x \in X : R[x] \cap A \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}\}\ or\\ & \{x \in X : R[x] \cap B \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}\}\\ =& A^*_{\mathfrak{U}} \cup B^*_{\mathfrak{U}}.\end{aligned}$$
4. Let $x \in (A^*_{\mathfrak{U}})^*$. It follows that $R[x] \cap A^*_{\mathfrak{U}} \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}$, and hence $R[x] \cap A^*_{\mathfrak{U}} \neq \emptyset\ \forall\ R \in \mathfrak{U}$. Therefore, there exists $y \in R[x]$ and $y \in A^*_{\mathfrak{U}}$ such that $y \in R[x]$ and $R[y] \cap A \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}$. This result, combined with $R[y] \subseteq R[x]$ and Definition 2.1, implies $R[x] \cap A \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}$. Hence $x \in A^*_{\mathfrak{U}}$. Then the result.
5. Let $x \in A^*_{\mathfrak{U}}(\mathscr{J})$. Then $R[x] \cap A \notin \mathscr{J}\ \forall\ R \in \mathfrak{U}$. Since $\mathscr{I} \subseteq \mathscr{J}$, hence $R[x] \cap A \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}$. So, $x \in A^*_{\mathfrak{U}}(\mathscr{I})$. Then the result.

**Corollary 4.3.** Let $(X,\mathfrak{U})$ be an uniform space and $\mathscr{I}$ be an ideal on $X$. Then the operator

$$-: P(X) \to P(X)$$

defined by

$$\overline{A} = A \cup A^*_{\mathfrak{U}} \tag{4.2}$$

satisfies Kuratwski's axioms and induces a uniform topology on $X$ called $\tau^*(\mathfrak{U})$ given by

$$\tau^*(\mathfrak{U}) = \{A \subseteq X : \overline{A^c} = A^c\}. \tag{4.3}$$

**Proof.** The result is a direct consequence of Theorem 4.2.

**Theorem 4.4.** $\tau^*(\mathfrak{U})$ is finer than $\tau(\mathfrak{U})$.

**Proof.** To prove the theorem, it suffices to show that $\forall\ A \subseteq X$, $\overline{A} \subseteq Cl(A)$. Let $x \in \overline{A}$. It follows that $x \in A$ or $x \in A^*_{\mathfrak{U}}$. If $x \in A$, hence the result. Now, let $x \in A^*_{\mathfrak{U}}$. Then $R[x] \cap A \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}$. This indicates that $R[x] \cap A \neq \emptyset\ \forall\ R \in \mathfrak{U}$. Also $R^{-1}[x] \cap A \neq \emptyset\ \forall\ R \in \mathfrak{U}$. Hence there exists $y \in X$ such that $y \in R^{-1}[x]\ \forall\ R \in \mathfrak{U}$ and $y \in A$. Hence $(y,x) \in R\ \forall\ R \in \mathfrak{U}$ and $y \in A$. It follows that there exists $y \in A$ such that $x \in R[y]$; that is $x \in \cap_{R \in \mathfrak{U}} R[A]$. then the result.

The following two theorems indicate that every uniform space $(X,\mathfrak{U})$ has an associated $\mathscr{I}-$proximity relation.

**Theorem 4.5.** Let $\mathscr{I}$ be an ideal on a nonempty set $X$ and $\delta_{\mathscr{I}}$ be a binary relation on $P(X)$ defined as:

$$A\delta_{\mathscr{I}}B \Leftrightarrow R[A] \cap B \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}. \tag{4.4}$$

Then $\delta_{\mathscr{I}}$ is an $\mathscr{I}-$proximity relation on $X$.

**Proof.** First we prove that $\delta_{\mathscr{I}}$ satisfies $(IP_2)$, $\forall\ A,B,C \in P(X)$, $(A \cup B)\delta_{\mathscr{I}}C \Leftrightarrow R[(A \cup B)] \cap C \notin \mathscr{I}$ for all $R \in \mathfrak{U} \Leftrightarrow (R[A] \cup R[B]) \cap C \notin \mathscr{I}$ for all $R \in \mathfrak{U} \Leftrightarrow ((R[A] \cap C) \cup (R[B] \cap C)) \notin \mathscr{I}$ for all $R \in \mathfrak{U} \Leftrightarrow R[A] \cap C \notin \mathscr{I}$ for all $R \in \mathfrak{U}$ or $R[B] \cap C \notin \mathscr{I}$ for all $R \in \mathfrak{U} \Leftrightarrow A\delta_{\mathscr{I}}C$ or $B\delta_{\mathscr{I}}C$. Similarly, $A\delta_{\mathscr{I}}(B \cup C) \Leftrightarrow A\delta_{\mathscr{I}}B$ or $A\delta_{\mathscr{I}}C$. For $(IP_1)$, let $A\delta_{\mathscr{I}}B$. Hence $(IP_2)$ and $A \subseteq R^{-1}[A]$ imply $R^{-1}[A]\delta_{\mathscr{I}}B$. Hence $R[R^{-1}[A]] \cap B \notin \mathscr{I}$ for all $R \in \mathfrak{U}$. This result, combined with $R[R^{-1}[A]] \cap B \subseteq A \cap B$, implies $A \cap B \notin \mathscr{I}$. Hence $A \cap B \subseteq A \cap R[B]$ and Definition 2.1 imply $A \cap R[B] \notin \mathscr{I}$. Consequently, $B\delta_{\mathscr{I}}A$. For $(IP_3)$, let $A\delta_{\mathscr{I}}B$. Hence $(IP_2)$ and $A \subseteq R^{-1}[A]$ imply $R^{-1}[A]\delta_{\mathscr{I}}B$. It follows that $R[R^{-1}[A]] \cap B \notin \mathscr{I}$ for all $R \in \mathfrak{U}$. This result, combined with $R[R^{-1}[A]] \cap B \subseteq A \cap B$ and Definition 2.1, implies $A \cap B \notin \mathscr{I}$ for all $R \in \mathfrak{U}$. This shows that $A \notin \mathscr{I}$ and $B \notin \mathscr{I}$. For $(IP_4)$, let $A \cap B \notin \mathscr{I}$. Then Definition 2.1 implies $R[A] \cap B \notin \mathscr{I}$ for all $R \in \mathfrak{U}$; that is, $A\delta_{\mathscr{I}}B$. To check that $\delta_{\mathscr{I}}$ also satisfies condition $(IP_5)$, let $A\not\delta_{\mathscr{I}}B$. It follows that $R[A] \cap B \in \mathscr{I}$ for some $R \in \mathfrak{U}$. By taking $C = R[A]$ and $D = (R[A])^c$, we have the required properties.

**Theorem 4.6.** Let $\mathscr{I}$ be an ideal on a nonempty set $X$ and $\delta_{\mathscr{I}}$ be a binary relation on $P(X)$ defined as:

$$A\delta_{\mathscr{I}}B \Leftrightarrow R[A] \cap R[B] \notin \mathscr{I}\ for\ all\ R \in \mathfrak{U}. \tag{4.5}$$

Then $\delta_{\mathscr{I}}$ is an $\mathscr{I}-$proximity relation on $X$.

**Proof.** It is clear that $\delta_{\mathscr{I}}$ satisfies $(IP_1)$. For $(IP_2)$, $\forall\ A,B,C \in P(X)$ $(A \cup B)\delta_{\mathscr{I}}C \Leftrightarrow R[(A \cup B)] \cap R[C] \notin \mathscr{I}$ for all $R \in \mathfrak{U} \Leftrightarrow (R[A] \cup R[B]) \cap R[C] \notin \mathscr{I}$ for all $R \in \mathfrak{U} \Leftrightarrow ((R[A] \cap R[C]) \cup (R[B] \cap R[C])) \notin \mathscr{I}$ for all $R \in \mathfrak{U} \Leftrightarrow R[A] \cap R[C] \notin \mathscr{I}$ for all $R \in \mathfrak{U}$ or $R[B] \cap R[C] \notin \mathscr{I}$ for all $R \in \mathfrak{U} \Leftrightarrow A\delta_{\mathscr{I}}C$ or $B\delta_{\mathscr{I}}C$. For $(IP_3)$, let $A\delta_{\mathscr{I}}B$. Hence $(IP_1)$, $(IP_2)$, and $B \subseteq R^{-1}[B]$ imply that $A\delta_{\mathscr{I}}R^{-1}[B]$. Hence $R[A] \cap R(R^{-1}[B]) \notin \mathscr{I}$ for all $R \in \mathfrak{U}$. This result, combined with $R[A] \cap R(R^{-1}[B]) \subseteq R[A] \cap B$ and Definition 2.1, implies $R[A] \cap B \notin \mathscr{I}$ for all $R \in \mathfrak{U}$. This shows that $B \notin \mathscr{I}$. Similarly $A \notin \mathscr{I}$. For $(IP_4)$, let $A \cap B \notin \mathscr{I}$. Then Definition 2.1 implies $R[A] \cap R[B] \notin \mathscr{I}$ for all $R \in \mathfrak{U}$; that is, $A\delta_{\mathscr{I}}B$. To check that $\delta_{\mathscr{I}}$ also satisfies condition $(IP_5)$, let $A \not\delta_{\mathscr{I}} B$. It follows that $R[A] \cap R[B] \in \mathscr{I}$ for some $R \in \mathfrak{U}$. By taking $C = (R[B])^c$ and $D = R[B]$, we have the required properties.

**Theorem 4.7.** Let $(X,\mathfrak{U})$ be a uniform space, $\mathscr{I}$ be an ideal on $X$ and $A,B$ are subsets of $X$. Then the following two $\mathscr{I}-$proximity relations on $X$ are equivalent

1. $A\delta^1_{\mathscr{I}}B \Leftrightarrow R[A] \cap B \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}$,
2. $A\delta^2_{\mathscr{I}}B \Leftrightarrow R[A] \cap R[B] \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}$.

**Proof.** To prove the theorem, it suffices to show that $A\delta^1_{\mathscr{I}}B \Leftrightarrow A\delta^2_{\mathscr{I}}B$. Let $A\delta^1_{\mathscr{I}}B$. Then $R[A] \cap B \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}$ and hence Definition 2.1 implies $R[A] \cap R[B] \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}$; that is, $A\delta^2_{\mathscr{I}}B$. On the other hand, suppose $A\delta^2_{\mathscr{I}}B$. This result, combined with $B \subseteq R^{-1}[B]$ and Lemma 2.4 imply $A\delta^2_{\mathscr{I}}R^{-1}[B]$; that is, $R[A] \cap R[R^{-1}[B]] \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}$. Definition 2.1 and $R[R^{-1}[B]] \subseteq B$ imply $R[A] \cap B \notin \mathscr{I}\ \forall\ R \in \mathfrak{U}$. This shows that $A\delta^1_{\mathscr{I}}B$. Then the result.

The following theorem shows that the topology generated by the closure operator defined in (4.2) coincide with the topology generated by the closure operator defined in (2.3).

**Theorem 4.8** Let $(X,\mathfrak{U})$ be a uniform space and $\delta_{\mathscr{I}}$ is the formula (4.4). Then

$$\tau^*(\mathfrak{U}) = \tau_{\delta_{\mathscr{I}}}.$$

**Proof.** To prove the theorem, it suffices to show that $\forall\ A \subseteq X$, $\overline{A} = Cl^{\delta}(A)$. This follows from the fact that $x \in Cl^{\delta}(A) \Leftrightarrow x \in A$ or $x \in A^{\delta_{\mathscr{I}}} \Leftrightarrow x \in A$ or $x\delta_{\mathscr{I}}A \Leftrightarrow x \in A$ or $R[x] \cap A \notin \mathscr{I}\ \forall\ R \in \mathfrak{U} \Leftrightarrow x \in A$ or $x \in A^*_{\mathfrak{U}} \Leftrightarrow x \in \overline{A}$.

**Corollary 4.9.** Let $(X,\mathfrak{U})$ be a uniform space and $\delta_{\mathscr{I}}$ is the formula (4.5). Then

$$\tau^*(\mathfrak{U}) = \tau_{\delta_{\mathscr{I}}}.$$

**Proof.** The result is a direct consequence of Theorem 4.7 and Theorem 4.8.

## 5 Conclusion

A set $X$ with a *nearness relation* between its subsets is called a *proximity space* and every such structure induces a *topology* on $X$ defined via the closure operator: we say that a point $x$ lies in the closure of a subset $A$ if the subset $\{x\}$ is near $A$. It appears that the same topology on $X$ may correspond in this way to different proximities. Moreover many topological results may be inherited from statements concerning proximity spaces. It has to be recalled that in the same manner the proximity structure is induced by the *uniform relation* introduced by A. Weil [26].

In this paper, a new approaches of *proximity relations*, *$\mathscr{I}-$proximity relation*, via *ideals* and *uniform relations* have been introduced. $\delta_{\mathscr{I}}-$*neighborhood* in an $\mathscr{I}-$proximity relation has been introduced. This provides an alternative description to the study of $\mathscr{I}-$proximity spaces. Furthermore, the operator $*$ on $P(X)$ with respect to an ideal and uniformity $\mathfrak{U}$ on $X$ has been presented and various properties of it are investigated. The new generated uniformity via ideal is mentioned which generated a topology $\tau^*(\mathfrak{U})$ finer than the old one. In addition, $\tau^*(\mathfrak{U}) = \tau_{\delta_{\mathscr{I}}}$ is proved.

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