

Solution of Fractional Volterra Integral Equation and Non-Homogeneous Time Fractional Heat Equation using Integral Transform of Pathway Type

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Abstract: The aim of the present paper is to obtain the solution of certain integral equations by using P_α -transform. The concept of P_α -transform is introduced by Kumar [11]. The P_α -transform is binomial type containing many classes of transforms including the Laplace transform. We have found the solution of fractional Volterra equation with Caputo fractional derivative using P_α -transform. Also the solution of non-homogeneous time fractional heat equation in spherical domain with Caputo derivative has been found. The results for the classical Laplace transform are retrieved by letting $\alpha \rightarrow 1$.

Keywords: Volterra equation, Abel's integral equation, Fractional Volterra Equation, P_α -transform, Riemann-Liouville fractional derivative, Laplace transform, Non-homogeneous time fractional heat equation.

1 Introduction

The subject of fractional calculus deals with the investigation of integrals and derivatives of any arbitrary real or complex order, which unify and extend the notations of fractional order derivative and n-fold integral. Fractional calculus is now considered as a partial technique in many branches of science including physics (Oldham and Spanier [22]). Recently Srivastava et al. [30] gave the model of under-actuated mechanical system with fractional order derivative and Sharma et al. [24] studied advanced generalized fractional kinetic equation in Astrophysics.

In an integral equation, an unknown function to be determined, appears under one and more integral signs. The integral equation has been a subject of interest of mathematicians as well as physicists and engineers also. The development of integral equation has led to the formation of many real world engineering and physical problems and also in mathematical physics models, such as scattering in quantum mechanism, diffraction problem, conformal mapping and water waves. A large number of initial and boundary value problems can be converted into Volterra integral equation. The Volterra's population growth model, biological species living together, the heat transformation and heat radiation are many areas which are described by integral equations. Many scientific problems give rise to integral equations often arises in low frequency electromagnetic problems, electrostatic, electromagnetic scattering problems and elastic waves and many more [6]. The fractional order integral equations has numerous applications in porous media, rheology, control, electro chemistry, viscoelasticity, electromagnetism fluid structure, coupling and particle mechanics (see e.g. [16], [22], [26], [27]).

The general form of integral equation (Wazwaz [6]) is given by

$$u(x) = f(x) + \lambda \int_{g(x)}^{h(x)} K(x,t)u(t)dt, \tag{1}$$

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where $g(x)$ and $h(x)$ are the limits of integration, λ is a constant parameter, and $K(x,t)$ is called the *kernel* or the *nucleus* of the integral equation. The function $u(x)$ to be determined appears under the integral sign. The kernel $K(x,t)$ and the function $f(x)$ in equation (1) are given and the limits of integration $g(x)$ and $h(x)$ may be both variables, constant or mixed.

The general form of Volterra integral equations (Rahman [23]) is

$$u(x) = f(x) + \lambda \int_a^x K(x,t)u(t)dt, \quad (2)$$

where the limits of integration are functions of x and the unknown function $u(x)$ appears linearly under the integral sign.

Abel's integral equation (Gorenflo and Vessela [25], Kilbas and Saigo [1]) is given by

$$f(t) = \frac{\lambda}{\Gamma(\mu)} \int_0^t \frac{u(\tau)}{(t-\tau)^\mu} d\tau, \quad 0 < \mu < 1 \quad (3)$$

The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by Kilbas *et. al.* in [3] as:

$$I_{a^+}^\alpha f(t) = {}_a D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(u)}{(t-u)^{1-\alpha}} du, \quad (4)$$

where $-\infty \leq a < t \leq \infty$.

The Riemann-Liouville fractional derivatives of order $\alpha > 0$ and where $-\infty \leq a < t \leq \infty$ is defined by Kilbas *et. al.* in [3] as:

$${}_a D_t^\alpha f(t) = \frac{d^m}{dt^m} I_{a^+}^{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_a^t (t-s)^{m-1-\alpha} f(s) ds. \quad (5)$$

For $n \in \mathbb{N}$, we denote by $AC^n[a,b]$, the space of real valued functions $f(t)$ which have continuous derivatives up to order $n-1$ on $[a,b]$ such that $f^{(n-1)}(t)$ belongs to the space of absolutely continuous functions $AC[a,b]$:

$$AC^n[a,b] = \left\{ f: [a,b] \rightarrow \mathbb{R} : \frac{d^{n-1}}{dx^{n-1}} f(x) \in AC[a,b] \right\}.$$

The Caputo derivative of order $\alpha > 0$, $m = \lceil \alpha \rceil$ and $f \in AC^m[a,b]$ is defined as [21]

$${}_a^C D_t^\alpha f(t) = I_{a^+}^{m-\alpha} \frac{d^m}{dt^m} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} \frac{d^m}{ds^m} f(s) ds. \quad (6)$$

The Laplace transform is a valuable tool to deal with problems involving integral- differential equations, difference equations, partial differential equations and fractional differential equation [14]. The classical Laplace transform [19] of a function $f(t)$ of a real variable t , which is denoted by $L[f(t);s]$, is a function $F(s)$ of a complex variable s defined as

$$L[f(t);s] = F(s) = \int_0^\infty e^{-st} f(t) dt, \quad \Re(s) > 0 \quad (7)$$

whenever it exists. The inverse Laplace transform $L^{-1}F(s)$ is given by

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds, \quad (8)$$

The Laplace Transform of the Riemann-Liouville fractional derivative in equation (5) is given by Samko *et. al.* in [29] as

$$L[{}_0 D_t^\alpha f(t) : s] = s^\alpha f(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \quad (9)$$

where $(n-1) < \alpha \leq n$.

After the development of the Laplace transform many transforms were derived by a number of authors which are applicable in many practical situations.

In this paper we solve integral equations by P_α -transform of pathway type which is introduced by Dilip Kumar [11] which is motivated by the non-extensive statistical mechanics introduced by Tsallis ([7], [8]) and the pathway model introduced by Mathai in 2005 ([4], [5]).

Pathway model is based on the principle of switching among three different families of functions, say generalized

extended type-1 beta family, type-2 beta family and gamma family. This type of switching property can be used in practical situations where one needs to fit a parametric family of distributions to experimental data or to switch among three different functional forms. When the pathway parameter is allowed to vary, we get three different forms. In real scaler case, the pathway model is defined as

$$\begin{aligned}
 f(x) &= c_1 |x|^\gamma [1 - a(1 - \alpha)|x|^\delta]^{-\frac{\eta}{1-\alpha}}, \quad 1 - a(1 - \alpha)|x|^\delta > 0, \alpha < 1 \\
 &= c_2 |x|^\gamma [1 + a(\alpha - 1)|x|^\delta]^{-\frac{\eta}{\alpha-1}}, \quad -\infty < x < \infty, \alpha > 1 \\
 &= c_3 |x|^\gamma e^{-a\eta|x|^\delta}, \quad -\infty < x < \infty, \alpha \rightarrow 1
 \end{aligned}
 \tag{10}$$

where $a > 0, \delta > 0, \gamma > 0, \eta > 0$. c_1, c_2 and c_3 are the normalizing constants if we consider each of them as statistical density. Three different functional forms are, generalized form extended type-1 beta, type-2 beta form respectively. The Tsallis statistics ([7], [8]) and superstatistics are covered by the pathway model. In recent years, the pathway model and Tsallis statistics have been applied in many areas like thermonuclear reaction rate theory in astrophysics ([13], [17], [18]) and in applied analysis ([2], [10], [12]) by Kumar and co-workers. In 2011, the Kumar introduced a fractional type integral transform called P - transform or pathway transform defined by

$$(P_v^{\rho, \beta, \alpha} f)(x) = \int_0^\infty D_{\rho, \beta}^{v, \alpha}(xt) f(t) dt, x > 0,
 \tag{11}$$

where $D_{\rho, \beta}^{v, \alpha}(x)$ denotes the function

$$D_{\rho, \beta}^{v, \alpha}(x) = \int_0^{[\frac{1}{a(1-\alpha)}]^\frac{1}{\rho}} y^{v-1} [1 - a(1 - \alpha)y^\rho]^{-\frac{1}{1-\alpha}} e^{-xy^{-\beta}} dy, x > 0
 \tag{12}$$

with $v \in C, \beta > 0, \rho > 0, a > 0, \alpha < 1$ or

$$D_{\rho, \beta}^{v, \alpha}(x) = \int_0^\infty y^{v-1} [1 + a(\alpha - 1)y^\rho]^{-\frac{1}{1-\alpha}} e^{-xy^{-\beta}} dy, x > 0
 \tag{13}$$

for $v \in C, \beta > 0, a > 0, \rho \in R, \alpha > 1$. When $D_{\rho, \beta}^{v, \alpha}(x)$ takes the form (12) or (13), the transform will be called as type-1 or type-2 P - transform, respectively, which are defined in the space $L_{v,r}(0, \infty)$ consisting of the Lebesgue measurable complex valued function f for which

$$\|f\|_{v,r} = \left\{ \int_0^\infty |t^v f(t)|^r \frac{dt}{t} \right\}^\frac{1}{r} < \infty
 \tag{14}$$

for $1 \leq r < \infty, v \in R$. The P -transform and the P_α -transform both are based on pathway idea but the P_α -transform deals the problem with much easy compared to P -transform.

The P_α - transform of a function $f(t)$ of a real variable t denoted by $P_\alpha[f(t); s]$ is a function $F(s)$ of a complex variable s , valid under certain conditions on $f(t)$, (given in Lemma 1) is defined by Kumar [11] as

$$P_\alpha[f(t); s] = F(s) = \int_0^\infty [1 + (\alpha - 1)s]^{-\frac{t}{\alpha-1}} f(t) dt, \alpha > 1
 \tag{15}$$

Here $\lim_{\alpha \rightarrow 1+} [1 + (\alpha - 1)s]^{-\frac{t}{\alpha-1}} = e^{-st}$ (15) defines a class of transforms. All these transforms are the paths going from the binomial form $[1 + (\alpha - 1)s]^{-\frac{t}{\alpha-1}}$ to the exponential form e^{-st} . In P_α - transform the variable t is shifted from the binomial factor $[1 + (\alpha - 1)s]^{-\frac{t}{\alpha-1}}$ to the exponent and hence this form is more suitable for obtaining translation, convolution etc. But, of course, when the pathway parameter α goes to 1, P_α - transform will go to the exponential form eventually leading to the Laplace transform i.e.,

$$\lim_{\alpha \rightarrow 1} P_\alpha[f(t); s] = L[f(t); s]
 \tag{16}$$

The convergence conditions for the P_α - transform of a function $f(t)$ to exist are given by the following results.

Lemma 1, [11] *If $f(t)$ is integrable over any finite interval (a, b) , $0 < a < t < b$, there exists a real number c such that,*

(i) *for any arbitrary $b > 0$, $\int_b^\rho e^{-ct} f(t) dt$ tends to a finite limit as $\rho \rightarrow \infty$*

(ii) *for any arbitrary $a > 0$, $\int_v^a |f(t) dt|$ tends to a finite limit as $v \rightarrow 0+$,*

then the P_α -transform $P_\alpha[f(t); s]$ exists for $\Re\left(\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right) > c$ for $s \in C$.

Theorem 1.[11] If

(i) $f(t)$ is integrable over a finite limit (a, b) , $0 < a < t < b$,

(ii) for arbitrary positive a , the integral $\int_v^a |f(t)| dt$ tends to a finite limit as $v \rightarrow 0+$

(iii) $f(t) = O(e^{ct})$, $c > 0$ as $t \rightarrow \infty$ where $O(\cdot)$ is the standard big O notation which means $f(t)$ is of order not exceeding e^{ct} ,

then the P_α -transform defined in (15) converges absolutely if $\Re\left(\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right) > c$, $\alpha > 1$.

If instead of condition (iii), we have the condition $f(t) = O(t^\gamma)$, $\Re(\gamma+1) > 0$ as $t \rightarrow \infty$, then the pathway-Laplace transform converges absolutely for $\Re\left(\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right) > 0$.

Corollary 1. If conditions of Theorem 1 are satisfied and $\alpha \rightarrow 1$, then the Laplace transform obtained as $\lim_{\alpha \rightarrow 1} P_\alpha[f(t); s] = L[f(t); s]$ defined in (15) converges absolutely if $\Re(s) > c$. Moreover instead of condition (iii) if $f(t) = O(t^\gamma)$, $\Re(\gamma+1) > 0$ as $t \rightarrow \infty$, then the Laplace transform obtained as $\lim_{\alpha \rightarrow 1} [f(t); s] = L[f(t); s]$ converges absolutely for $\Re(s) > 0$.

Theorem 2.[11] (Convolution Theorem for P_α -transform) If $F(s)$ and $G(s)$ are the P_α - transform of the functions $f(t)$ and $g(t)$, respectively, then the product $F(s)G(s)$ is the P_α - transform of the function $\int_0^t f(t-\tau)g(\tau)d\tau$. That is

$$F(s)G(s) = P_\alpha \left[\int_0^t f(t-\tau)g(\tau)d\tau; s \right] = P_\alpha[f(t); s]P_\alpha[g(t); s]. \quad (17)$$

Lemma 2.[11] For $v \in \mathbb{C}$, $\Re(v) > 0$ and for $\alpha > 1$, we have

$$P_\alpha [{}_0D_t^{-v} f(t); s] = \left\{ \frac{\alpha-1}{\ln[1+(\alpha-1)s]} \right\}^v P_\alpha[f(t); s] \quad (18)$$

where ${}_0D_t^{-v}$ is Riemann-Liouville fractional integral defined in equation (4).

Theorem 3.[11] If $f(t)$ and its derivatives up to order n are of exponential order and are P_α -transformable and if $f(t)$ and its derivatives up to $(n-1)$ th order are continuous with the exception of the origin and if n th derivative $f^{(n)}(t)$ is at least piecewise continuous and if $P_\alpha[f(t); s] = F(s)$ then

$$P_\alpha[f^{(n)}(t); s] = \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^n F(s) - \sum_{m=1}^n \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^{n-m} f^{(m-1)}(0+), \quad (19)$$

where $f(0+) = \lim_{\varepsilon \rightarrow 0} f(0+\varepsilon)$.

Motivated by the work of Kumar, in the present paper we find the P_α -transform of Caputo fractional derivatives and derive P_α -transform for Volterra and Abel integral equation. Further, in Section 3 we find the solution of fractional Volterra integral equation. We discuss its application for solving singular integral equation having Bessel function in its kernel. The solution of non homogeneous time fractional heat equation in a spherical domain has been discussed in Section 4.

2 Main Results

Theorem 4. If Caputo fractional derivatives of function $f(t)$ of order v exist and are P_α - transformable and if $P_\alpha[f(t); s] = F(s)$, then for $\alpha > 1$, we have

$$P_\alpha [{}_0^C D_t^v f(t); s] = \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^v F(s) - \sum_{k=0}^{n-1} \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^{v-k-1} f^{(k)}(0) \quad (20)$$

where $n-1 < v \leq n$.

Proof: Using the fact that ${}_0^C D_t^v f(t) = {}_0D_t^{v-n}(f^{(n)}(t)) = {}_0D_t^{-(n-v)}(f^{(n)}(t))$, Lemma (2) gives,

$$P_\alpha [{}_0^C D_t^v f(t); s] = \left\{ \frac{\alpha-1}{\ln[1+(\alpha-1)s]} \right\}^{n-v} P_\alpha(f^{(n)}(t)). \quad (21)$$

So

$$P_\alpha [{}^C_0D_t^\nu f(t);s] = \left\{ \frac{\ln[1 + (\alpha - 1)s]}{\alpha - 1} \right\}^{\nu-n} P_\alpha(f^{(n)}(t)) \tag{22}$$

Applying Theorem (3), we get

$$P_\alpha [{}^C_0D_t^\nu f(t);s] = \left\{ \frac{\ln[1 + (\alpha - 1)s]}{\alpha - 1} \right\}^{\nu-n} \left[\left\{ \frac{\ln[1 + (\alpha - 1)s]}{\alpha - 1} \right\}^n F(s) - \sum_{k=0}^{n-1} \left\{ \frac{\ln[1 + (\alpha - 1)s]}{\alpha - 1} \right\}^{n-k-1} f^{(k)}(0) \right]$$

Finally ,

$$P_\alpha [{}^C_0D_t^\nu f(t);s] = \left\{ \frac{\ln[1 + (\alpha - 1)s]}{\alpha - 1} \right\}^\nu F(s) - \sum_{k=0}^{n-1} \left\{ \frac{\ln[1 + (\alpha - 1)s]}{\alpha - 1} \right\}^{\nu-k-1} f^{(k)}(0). \tag{23}$$

Theorem 5. The solution of Volterra integral equation (2) using P_α -transform is given by $P_\alpha^{-1} \left\{ \frac{1}{1-\lambda P_\alpha K(x)} \right\} = \psi(x)$, where $P_\alpha K(x) \neq \frac{1}{\lambda}$, $\alpha > 1$

Proof: Apply P_α - transform on both side of (2) and using Theorem (2), we obtain

$$P_\alpha \{u(x)\} = P_\alpha \{f(x)\} + \lambda P_\alpha \{K(x)\} P_\alpha \{u(x)\} \tag{24}$$

Let the P_α - transform of $u(x)$ and $K(x-t)$ be $U(s)$ and $K(s)$, respectively, then by Theorem (2),

$$U(s) = F(s) + \lambda K(s)U(s) \tag{25}$$

Hence

$$U(s) = \frac{F(s)}{1 - \lambda(K(s))}; K(s) \neq \frac{1}{\lambda} \tag{26}$$

and inverse transform gives

$$u(x) = \int_0^x \psi(x-t)f(t)dt \tag{27}$$

where it is assumed that $P_\alpha^{-1} \left\{ \frac{1}{1-\lambda P_\alpha K(x)} \right\} = \psi(x)$.

The expression (27) is the solution of second kind Volterra integral equation of convolution type.

Theorem 6. For $\alpha > 1$ and $0 < \mu < 1$, then the solution of the Abel integral equation (3) is given by

$$u(t) = \frac{\sin \pi \mu}{\pi} \int_0^t (t - \tau)^{\mu-1} G(\tau) d\tau, \tag{28}$$

where $G(t) = P_\alpha^{-1} \left\{ F(s) \left(\frac{\alpha-1}{\ln[1+(\alpha-1)s]} \right) \right\}$.

Proof: The Abel integral equation is given by

$$f(t) = \int_0^t \frac{u(\tau)}{(t - \tau)^\mu} d\tau, t > 0 \tag{29}$$

Applying the P_α -transform on both side of equation (29) and using Theorem 4, we get

$$P_\alpha \{f(t)\} = P_\alpha \{u(t)\} P_\alpha \{t^{-\mu}\} \tag{30}$$

If we take $P_\alpha \{f(t)\} = F(s)$, $P_\alpha \{u(t)\} = U(s)$ and using formula of P_α -transform for power function given in Kumar [11, Eq. 32], we get

$$F(s) = U(s) \Gamma(1 - \mu) \left\{ \frac{\ln[1 + (\alpha - 1)s]}{\alpha - 1} \right\}^{\mu-1} \tag{31}$$

Which leads to

$$U(s) = \frac{F(s)}{\Gamma(1-\mu) \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^{\mu-1}} = \frac{\left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\} F(s)}{\Gamma(1-\mu) \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^{\mu}} \quad (32)$$

Using duplication formula for Gamma function (Rainville [15]) and [11, Eq. 32], we get

$$P_{\alpha}^{-1} \left\{ \frac{1}{\Gamma(1-\mu) \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^{\mu}; t} \right\} = \frac{t^{(\mu-1)}}{\Gamma(1-\mu)\Gamma(\mu)} = \frac{\sin\pi\mu}{\pi} t^{\mu-1} \quad (33)$$

Finally, Theorem 2 gives

$$u(t) = \frac{\sin\pi\mu}{\pi} \int_0^t (t-\tau)^{\mu-1} G(\tau) d\tau, \quad (34)$$

where $G(t) = P_{\alpha}^{-1} \left\{ F(s) \left(\frac{\alpha-1}{\ln[1+(\alpha-1)s]} \right) \right\}$.

3 Solution of fractional Volterra integral equation by using P_{α} - transform

Theorem 7. Consider fractional Volterra singular integral equation of the form

$${}_0^C D_t^{\nu} f(x) = g(x) + \lambda \int_x^{+\infty} K(x-t)f(t)dt, f(0) = 0, \quad (35)$$

in which $K(x,t) = K(x-t)$ is the kernel, $g(x)$ satisfies all conditions of Lemma (1) and $0 < \nu \leq 1$, then (35) has solution of the form

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \left\{ \frac{G(s)}{\lambda K(-s) - \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^{\nu}} [1 + (\alpha-1)s]^{\frac{t}{\alpha-1}} \right\} ds. \quad (36)$$

Proof: Apply P_{α} -transform on both sides of Eq. (35) denote $P_{\alpha}[f(x)] = F(s)$, $P_{\alpha}[g(x)] = G(s)$. Let $K(-s)$ be the P_{α} -transform of $K(x)$. Then by using Theorem 2, we obtain

$$\left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^{\nu} F(s) = G(s) + \lambda K(-s)F(s) \quad (37)$$

which gives,

$$F(s) = \frac{-G(s)}{\lambda K(-s) - \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^{\nu}}, \quad (38)$$

and consequently by Bromwich's integral we get the following relation,

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \left\{ \frac{G(s)}{\lambda K(-s) - \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^{\nu}} [1 + (\alpha-1)s]^{\frac{t}{\alpha-1}} \right\} ds \quad (39)$$

which can be solved by the use of Residue theorem (see Brown and Churchill [20]).

Here, we illustrate the application of the above theorem in finding solutions of some singular integral equations:

(i) Consider singular integral equation having Bessel function $J_0(2\sqrt{(x-t)})$ as its kernel

$${}_0^C D_t^{\nu} f(x) = e^{-ax} + \lambda \int_x^{+\infty} J_0(2\sqrt{(x-t)})f(t)dt, f(0) = 0, 0 < \nu \leq 1 \quad (40)$$

In view of (39), one can obtain solution of (40) as

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}x}}{\left(\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\} + a\right)\left(\lambda e^{\left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}} + \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}^{v+1}\right)} ds \quad (41)$$

By setting $\alpha \rightarrow 1$ in Eq. (41), we obtain the corresponding results for the classical Laplace transform as follows:

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{se^{sx}}{(s+a)(\lambda e^{\frac{1}{s}} + s^{v+1})} ds \quad (42)$$

(ii) Taking $v = 0.5$ in Eq. (41), we obtain an interesting result:
Solution of integral equation

$${}^C D_t^{0.5} f(x) = e^{-ax} + \lambda \int_x^{+\infty} J_0(2\sqrt{(x-t)})f(t)dt, \quad f(0) = 0, \quad 0 < v \leq 1 \quad (43)$$

is given by

$$f(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{Ax}}{\sqrt{A}(A+a)(\lambda e^{\frac{1}{A}} A^{-\frac{3}{2}} + 1)} ds \quad (44)$$

where $A = \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}$

Proof: We apply the P_α -transform of convolution of function and using the fact that

$$P_\alpha^{-1} \left\{ \frac{1}{\sqrt{A}(A+a)} \right\} = \int_0^x \frac{e^{a(\eta-x)}}{\sqrt{\pi x}} dx, \quad (45)$$

and also the following relationship

$$\begin{aligned} P_\alpha^{-1} \left\{ \frac{1}{1 + \lambda e^{\frac{1}{A}} A^{-\frac{3}{2}}} \right\} &= P_\alpha^{-1} \left\{ 1 - \left(\lambda e^{\frac{1}{A}} A^{-\frac{3}{2}}\right) + \left(\lambda e^{\frac{1}{A}} A^{-\frac{3}{2}}\right)^2 - \dots \right\} \\ &= P_\alpha^{-1} \left\{ 1 + \sum_{k=1}^{\infty} (-1)^k \lambda^k e^{\frac{k}{A}} A^{-\frac{3k}{2}} \right\} \\ &= \delta(x) + \sum_{k=1}^{\infty} (-1)^k \lambda^k \left(\frac{x}{k}\right)^{\frac{3k-2}{4}} I_{\frac{3k-2}{2}}(2\sqrt{kx}) \end{aligned} \quad (46)$$

where $A = \left\{\frac{\ln[1+(\alpha-1)s]}{\alpha-1}\right\}$. From equations (45) and (46), one gets the formal solution of equation (43) as follows:

$$f(x) = \left\{ \int_0^x \frac{e^{a(\eta-x)}}{\sqrt{\pi x}} d\eta \right\} * \left\{ \delta(x) + \sum_{k=1}^{\infty} (-1)^k \lambda^k \left(\frac{x}{k}\right)^{\frac{3k-2}{4}} I_{\frac{3k-2}{2}}(2\sqrt{kx}) \right\}.$$

(iii) The solution of the following system of fractional singular integral equations of the form,

$$\begin{aligned} {}^C D_t^\nu(x) &= g(x) - \lambda \int_x^{+\infty} k(x-t)\psi(t)dt \\ {}^C D_t^\nu(x) &= h(x) + \lambda \int_x^{+\infty} k(x-t)\phi(t)dt, \end{aligned} \quad (47)$$

with conditions $\phi(0) = 0, \psi(0) = 0$ and $0 \leq v \leq 1$, is given by

$$\begin{aligned} \Phi(x) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{A^\nu G(s) + \lambda K(-s)H(s)}{\lambda^2(K(-s))^2 + A^{2\nu}} e^{Ax} ds, \\ \Psi(x) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{A^\nu H(s) + \lambda K(-s)G(s)}{\lambda^2(K(-s))^2 + A^{2\nu}} e^{Ax} ds. \end{aligned} \quad (48)$$

where $A = \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}$

Proof: Multiplying second equation of (47) by i and adding to the first equation leads to

$${}^C_0D_t^\nu = (g + ih)(x) + i\lambda \int_x^\infty k(x-t)(\phi + i\psi)(t)dt. \quad (49)$$

Now let $(\phi + i\psi)(x) = \zeta(x)$, $(g + ih)(x) = f(x)$, $i\lambda = \xi$, then we can rewrite the above equation in the form

$${}^C_0D_t^\nu = f(x) + \xi \int_x^\infty k(x-t)\zeta(t)dt. \quad (50)$$

In view of (39), one can obtain solution of (50) as below:

Taking P_α transform of equation (50) leads to

$$\left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}^\nu \Phi(s) = F(s) + \xi K(-s)\Phi(s) \quad (51)$$

where $\Phi(s), F(s), K(s)$ are P_α -transform of the functions $\zeta(x), f(x), k(x)$, respectively.

Hence we get the following relationship

$$\Phi(s) = \frac{A^\nu G(s) + \lambda K(-s)H(s)}{\lambda^2(K(-s))^2 + A^{2\nu}} + i \frac{A^\nu H(s) + \lambda K(-s)G(s)}{\lambda^2(K(-s))^2 + A^{2\nu}} \quad (52)$$

$G(s), H(s)$ being P_α - transform, of $g(x), h(x)$, respectively. So we get

$$\tilde{\phi}(s) = \frac{A^\nu G(s) + \lambda K(-s)H(s)}{\lambda^2(K(-s))^2 + A^{2\nu}}, \tilde{\psi}(s) = \frac{A^\nu H(s) + \lambda K(-s)G(s)}{\lambda^2(K(-s))^2 + A^{2\nu}}$$

Finally, applying the complex inversion formula, the solution of (47) is obtained as

$$\begin{aligned} \Phi(x) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{A^\nu G(s) + \lambda K(-s)H(s)}{\lambda^2(K(-s))^2 + A^{2\nu}} e^{Ax} ds, \\ \Psi(x) &= \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{A^\nu H(s) + \lambda K(-s)G(s)}{\lambda^2(K(-s))^2 + A^{2\nu}} e^{Ax} ds \end{aligned} \quad (53)$$

where $A = \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}$.

(iv) Solution of the fractional Volterra singular integral equation of the form,

$${}^C_0D_t^\nu \phi(x) = f(x) + \lambda \int_0^x \ln(x-t)\phi(t)dt, \quad \phi(0) = 0, \quad 0 \leq \nu \leq 1, \quad (54)$$

is given by

$$\phi(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{AF(s)e^{Ax}}{A^{\nu+1} + \lambda(\xi + \ln A)} ds \quad (55)$$

where $A = \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}$.

Proof: After taking P_α - transform of above integral equation (54) and simplifying, one gets

$$P_\alpha[\Phi(x); s] = \frac{sF(s)}{s^{A+1} + \lambda(\xi + \ln A)} \quad (56)$$

in which $\xi \approx 0.577$ is Euler constant. Applying complex inversion formula to the above relation leads to

$$\phi(x) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{AF(s)e^{Ax}}{A^{\nu+1} + \lambda(\xi + \ln A)} ds \quad (57)$$

where $A = \left\{ \frac{\ln[1+(\alpha-1)s]}{\alpha-1} \right\}$.

4 Non-Homogeneous Time fractional Heat Equation in a Spherical Domain

Theorem 8. Let $f(t)$ be P_{α} - transformable function. For $0 \leq r < 1, t > 0, 0 < \alpha \leq 1$, the solution of the non-homogeneous time fractional heat equation

$${}_0^C D_t^{\alpha} u(r,t) = \frac{\partial^2 u(r,t)}{\partial r^2} + \frac{2}{r} \frac{\partial u(r,t)}{\partial r} - \lambda u(r,t) - f(t), \quad t > 0 \tag{58}$$

satisfying the boundary conditions $\lim_{r \rightarrow 0} |u(r,t)| < \infty, u_r(1,t) = 1$ and the initial conditions $u(r,0) = 0, f(0)=0$,

is given by

$$u(r,t) = \frac{1}{r} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \left(\frac{(\sinh r\sqrt{\lambda + A^{\nu}})}{A(\lambda + A \cosh(\sqrt{\lambda + A^{\nu}}) - \sinh(\sqrt{\lambda + A^{\nu}}))} - \frac{F(s)}{\lambda + A^{\nu}} \right) e^{st} ds. \tag{59}$$

Proof. Let us define $v(r,t) = ru(r,t)$. Then equation (58) becomes

$${}_0^C D_t^{\alpha} v(r,t) = \frac{\partial^2 v(r,t)}{\partial r^2} - \lambda v(r,t) - rf(t) \tag{60}$$

By taking the P_{α} - transform of equation (60) with respect to variable t and applying boundary conditions, we get

$$\left\{ \frac{\ln[1 + (\alpha - 1)s]}{\alpha - 1} \right\}^{\nu} V(r,s) = \frac{d^2 V(r,s)}{dr^2} - \lambda V(r,s) - rF(s), f(0) = 0 \tag{61}$$

where $V(r,s) = P_{\alpha}[v(r,t)]$.

or

$$\frac{d^2 V(r,s)}{dr^2} - \left(\lambda + \left\{ \frac{\ln[1 + (\alpha - 1)s]}{\alpha - 1} \right\}^{\nu} \right) V = rF(s) \tag{62}$$

with the boundary conditions

$$\lim_{r \rightarrow 0} |V(r,s)| = 0, \text{ and } V_r(1,s) - V(1,s) = \frac{1}{s}.$$

Equation (62) is second order ordinary differential equation. Its solution is given by

$$V(r,s) = \frac{(\sinh r\sqrt{\lambda + A^{\nu}})}{A(\lambda + A^{\nu} \cosh(\sqrt{\lambda + A^{\nu}}) - \sinh(\sqrt{\lambda + A^{\nu}}))} - \frac{F(s)}{\lambda + A^{\nu}} \tag{63}$$

where $A = \left\{ \frac{\ln[1 + (\alpha - 1)s]}{\alpha - 1} \right\}^{\nu}$. By using Bromwich's integral and taking inverse P_{α} - transform we get

$$v(r,t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \left(\frac{(\sinh r\sqrt{\lambda + A^{\nu}})}{s(\lambda + A \cosh(\sqrt{\lambda + A^{\nu}}) - \sinh(\sqrt{\lambda + A^{\nu}}))} - \frac{F(s)}{\lambda + A^{\nu}} \right) e^{st} ds, \tag{64}$$

and hence we obtain

$$u(r,t) = \frac{1}{r} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \left(\frac{(\sinh r\sqrt{\lambda + A^{\nu}})}{s(\lambda + A \cosh(\sqrt{\lambda + A^{\nu}}) - \sinh(\sqrt{\lambda + A^{\nu}}))} - \frac{F(s)}{\lambda + A^{\nu}} \right) e^{st} ds. \tag{65}$$

The P_{α} - transforms are useful when the boundary conditions are time dependent. Now consider the case when one of the boundary is moving. This type of problem arises in combustion problems where the boundary moves due to the burning of the fuel [9].

Example: Consider the following time dependent heat equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{66}$$

where $\beta t < x < \infty$, $t > 0$, $\beta \in \mathbb{R}$ and subject to the initial condition $u(x, 0) = 0$, $0 < x < \infty$ and boundary conditions $u(x, t)|_{x=\beta t} = f(t)$, $\lim_{x \rightarrow \infty} |u(x, t)| < \infty$, $t > 0$.

Then the solution of (66) is given by

$$u(x, t) = e^{-\frac{\beta(x-\beta t)}{2a^2}} \int_0^t f(t-\tau) \Phi(x-\beta\tau, \tau) d\tau \quad (67)$$

where $\Phi(x-\beta t, t)$ is given by

$$\frac{1}{2} \left[e^{-\frac{\beta(x-\beta t)}{2a^2}} \operatorname{erfc} \left(\frac{\eta}{2a\sqrt{t}} - \frac{\beta\sqrt{t}}{2a} \right) + e^{\frac{\beta(x-\beta t)}{2a^2}} \operatorname{erfc} \left(\frac{\eta}{2a\sqrt{t}} + \frac{\beta\sqrt{t}}{2a} \right) \right] \quad (68)$$

Proof: By introducing the new coordinate $\eta = x - \beta t$, the problem can be reformulated as

$$\frac{\partial u}{\partial t} - \beta \frac{\partial u}{\partial \eta} = a^2 \frac{\partial^2 u}{\partial \eta^2} \quad (69)$$

where $0 < \eta < \infty$, $t > 0$ and subject to the boundary conditions

$$u(0, t) = f(t), \quad \lim_{\eta \rightarrow \infty} |u(\eta, t)| < \infty, t > 0$$

and the initial condition $u(\eta, 0) = 0$, $0 < \eta < \infty$.

Taking the P_α -transform of the equation (69) with respect to t and denoting $P_\alpha[u(\eta, t)] = U(\eta, s)$ we obtain

$$\frac{d^2 U(\eta, s)}{d\eta^2} + \frac{\beta}{a^2} \frac{dU(\eta, s)}{d\eta} - \left\{ \frac{\ln[1 + (\alpha - 1)s]}{\alpha - 1} \right\} \frac{1}{a^2} U(\eta, s) = 0 \quad (70)$$

with

$$U(0, s) = F(s), \quad \lim_{\eta \rightarrow \infty} |U(\eta, s)| < \infty$$

The solution to the differential equation (70) is

$$U(\eta, s) = F(s) \exp \left(-\frac{\beta\eta}{2a^2} - \frac{\eta}{a} \sqrt{A + \frac{\beta^2}{4a^2}} \right) \quad (71)$$

where $A = \left\{ \frac{\ln[1 + (\alpha - 1)s]}{\alpha - 1} \right\}$.

Referring the result by Duffy [9, p.89, Eq. (2.274)], correspondingly for P_α -transform, we have

$$P_\alpha[\Phi(\eta, t)] = \exp \left(-\frac{\eta}{a} \sqrt{A + \frac{\beta^2}{4a^2}} \right) \quad (72)$$

where $\Phi(\eta, t)$ is given by

$$\frac{1}{2} \left[e^{-\frac{\beta\eta}{2a^2}} \operatorname{erfc} \left(\frac{\eta}{2a\sqrt{t}} - \frac{\beta\sqrt{t}}{2a} \right) + e^{\frac{\beta\eta}{2a^2}} \operatorname{erfc} \left(\frac{\eta}{2a\sqrt{t}} + \frac{\beta\sqrt{t}}{2a} \right) \right] \quad (73)$$

by taking inverse P_α -transform of (71) and applying the convolution theorem, we get

$$u(\eta, t) = e^{-\frac{\beta\eta}{2a^2}} \int_0^t f(t-\tau) \Phi(\eta, \tau) d\tau \quad (74)$$

and hence

$$u(x, t) = e^{-\frac{\beta(x-\beta t)}{2a^2}} \int_0^t f(t-\tau) \Phi(x-\beta\tau, \tau) d\tau. \quad (75)$$

5 Conclusion

This paper provides some new results in the areas of singular integral equations and fractional calculus. Furthermore, the implementation of the new integral transform (P_α -transform) for solving certain integral equation have been discussed.

The importance of using P_α -transform method is that we get a wider class of integrals varying from binomial to exponential function and it is very efficient technique for finding exact solution for certain singular integral equations. The method could lead to a promising approach for many applications in applied sciences.

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