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An Operator Method for Finding the Solution of Linear Fractional Order Fuzzy Differential Equations

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Abstract: In this paper, analytical investigations of linear fractional order fuzzy differential equations are obtained using a newfound operator method. Fuzzy fractional differential equations (FFDEs) subjected to initial conditions are dissected under the assumptions of generalized Hukuhara differentiability in conjunction with Caputo-type fuzzy fractional derivative. Consequently, all the prospects of fractional differentials of fuzzy-valued functions are deduced and discussed in detail under the notion of Caputo-type fuzzy fractional differentiability (CF_H-differentiability). Moreover, the novel method is illustrated on constructed systems of FFDEs and convex combination of r -level solutions for each system is measured, explicitly.

Keywords: Fuzzy-valued functions, fuzzy Riemann-Liouville fractional integral.

1 Introduction

Fractional calculus, being a significant area of study, has captivated the attention of many researchers. It owed useful applications in many physical and engineering processes [1,2,3,4]. Up till now, various forms of fractional operators have been introduced, such as, Riemann-Liouville, Riesz, Cresson, Jummarie, Caputo fractional derivatives, etc [5,6,7,8,9].

Correspondingly, fuzzy differential equations (FDEs) and fuzzy fractional differential equations (FFDEs), having wide applications in electronics and engineering, have been rapidly growing for last few decades. The primary introduction of fuzzy derivative was given by Chang et al. [10] which was then proceeded by Dubois et al. [11]. Later on, Puri et al. [12] generalized an embedding theorem and used it for fuzzy valued differential functions. Kandel [13] analyzed dynamical problems using FDE model. Comprehensive study of FDEs and fuzzy initial value problem (Cauchy problem) are found in papers of Kaleva [14,15] and Seikkala [16]. Agrawal et al. [17] elucidated the concept of FFDEs under Riemann-Liouville differentiability, following the concept of [17], Arshad et al. [18] further discussed existence and uniqueness of solutions of FFDEs.

Numerical and analytical approaches for solving FDEs as well as FFDEs are of identifiable attraction. For last few years, remarkable investigations have been executed by a number of authors for numerical solutions of FDEs and FFDEs. The extension principle approach and extremal solutions of deterministic initial value problem found in [16] have inspired several scholars for the solution of these equations. Significant contribution on these numerical methods is presented by Ma et al. [19] who pioneered the Euler method for FDEs, Khan et al. [20] extended the Sumudu transform to fuzzy Sumudu transform for analytical assessments of FDEs, Allahviranloo et al. [21] illustrated an operator method to obtain the integral forms of FDEs besides applied fuzzy Laplace transform to solve FFDEs considered under Caputo differentiability in [22], Ahmad et al. [23] proposed the fuzzification of the classical Euler method for FDE and also used Zadeh's extension principle in addition with an unconstrained optimization technique to solve FFDEs in [24], Salahshour et al. [25] studied FFDEs under the concept of Riemann-Liouville H-differentiability and used the method of [22] to obtained analytical solutions, Mazandarani et al. [26] explained modified fractional Euler method for the solutions of fuzzy fractional initial value problem. Shahriyar et al. [27] discussed three systems of fuzzy fractional differential equations and employed eigenvalue-eigenvector approach to obtain the solutions. Ghaemi et al. [28] modelled a fuzzy fractional kinetic equation, a model in chemical engineering for hemicelluloses hydrolysis reaction, and analyzed it using a spectral method to attain the concentration value of xylose in fuzzy environment. Very recently, Khan et al. [29] developed improved Euler's method

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for numerical simulation of linear and nonlinear fuzzy initial value problems of fractional order. In his paper, he worked out on the efficiency of the improved fractional Eulers method (IFEM) and exactness of results by its comparison with other methods.

Significantly, extending previously proposed operator method by Allahviranloo et al. [21], in this manuscript, under the assumptions of Caputo-type fuzzy fractional Hukuhara differentiability, analytical findings of linear fractional order fuzzy differential equations has been constructed. Initially, the concept of Hukuhara derivative of a fuzzy-valued function was generalized and extended in [12]. Furthermore, here, all possible Caputo-type fuzzy fractional Hukuhara derivatives of fuzzy-valued functions are established. Next, we particularize the algorithm for all systems of FFDEs formed by utilizing CF_H-differentiability. Subsequently, following the definition of fuzzy Riemann-Liouville fractional integration, integral solutions of lower and upper functions of some FFDEs are determined. For the ongoing applications of FFDEs in different fields of engineering, it has become an important task to innovate such methodologies that are rapidly convergent towards the accurate solutions. Thus, the proposed method sounds to be proficient and efficacious for FFDEs than the other familiar operator methods that already exist in literature.

2 Preliminaries

After the preliminary work on fuzzy set theory and fuzzy calculus [10,11,12], the basic definitions and properties of fuzzy numbers, fuzzy functions and its calculus have been mentioned repetitively by various authors in their research papers. For instance see Refs.[17-29]. Here, we briefly go through some descriptions and theorems of fuzzy fractional calculus that are essential for this paper. At this instant, let $\mathbf{E}_f$, $C^F[\alpha,\beta]$ and $L^F[\alpha,\beta]$ designate the set of all fuzzy numbers on real line, space of all continuous fuzzy-valued functions and space of all Lebesgue integrable fuzzy-valued functions on the interval $[\alpha,\beta]$, respectively.

2.1 Fuzzy Riemann-Liouville Fractional Integral

The fuzzy Riemann-Liouville fractional integral (FRLFI) of order $\upsilon \in \Re$, where $\Re$ is the set of real numbers, of a fuzzy-valued function $\tilde{\varpi}(x) \in C^F[\alpha,\beta] \cap L^F[\alpha,\beta]$ is described as:

$$[\Im^{\upsilon}\varpi(x)]^r = \left[\mathbf{I}^{\upsilon}\underline{\varpi}^r(x), \mathbf{I}^{\upsilon}\overline{\varpi}^r(x)\right], \quad 0 \le r \le 1, \tag{1}$$

where $\underline{\varpi}^r(x)$ and $\overline{\varpi}^r(x)$ are its r-level functions known as lower and upper functions, respectively. Also

$$\mathbf{I}^{\upsilon}\underline{\varpi}^r(x) = \frac{1}{\Gamma\upsilon}\int_0^x \frac{\underline{\varpi}^r(\eta)}{(x-\eta)^{1-\upsilon}}d\eta, \quad x>0, \quad \upsilon>0, \tag{2}$$

$$\mathbf{I}^{\upsilon}\overline{\varpi}^r(x) = \frac{1}{\Gamma\upsilon}\int_0^x \frac{\overline{\varpi}^r(\eta)}{(x-\eta)^{1-\upsilon}}d\eta, \quad x>0, \quad \upsilon>0. \tag{3}$$

An equivalent definition of fuzzy Riemann-Liouville fractional integral (FRLFI) is also found in Refs. [22,27].

2.2 Caputo-Type Fuzzy Fractional Hukuhara Differentiability

Let $\tilde{\varpi}(x) \in C^F[\alpha,\beta] \cap L^F[\alpha,\beta]$ then Caputo-type fuzzy fractional derivative in relation with fuzzy Riemann-Liouville fractional differential operator of order ν of $\tilde{\varpi}(x)$ is delineated as

$$^{c}\mathbf{D}^{\nu}\tilde{\varpi}(x) = {}^{RL}\mathbf{D}^{\nu}\left(\tilde{\varpi}(x) - \sum_{k=0}^{n-1}\frac{x^k}{k!}\tilde{\varpi}_0^{(k)}\right) \quad x>0, \ \nu \in (n-1,n), \ n \in \mathbb{N}, \tag{4}$$

where ${}^{RL}\mathbf{D}^{\nu}$ is the fuzzy Riemann-Liouville fractional differential operator (see e.g. refs. [22,27,29], which may be expressed as:

$$ {}^{c}\mathbf{D}^{\nu}\tilde{\varpi}\left(x\right)=\frac{1}{\Gamma\left(n-\nu\right)}\frac{d^{n}}{dx^{n}}\int_{0}^{x}\frac{\tilde{\varpi}\left(\xi\right)}{\left(x-\xi\right)^{1-n+\nu}}d\xi \quad \nu\in\left(n-1,n\right),\ \ n\in\mathbb{N}. \tag{5} $$

Next, for Caputo-type fuzzy fractional Hukuhara differentiability (CF_H-differentiability) of $\tilde{\varpi}\left(x\right)$ of order $0<\nu<1$, at $x_0\in\left(0,\beta\right)$ is described as follows:

Let $\varphi\left(x\right)=\frac{1}{\Gamma(1-\nu)}\int_{0}^{x}\frac{\tilde{\varpi}(\xi)-\sum_{k=0}^{n-1}\frac{\xi^{k}}{k!}\tilde{\varpi}_{0}^{(k)}}{(x-\xi)^{\nu}}d\xi$, then

$$ (a)\,{}^{c}\mathbf{D}^{\nu}\tilde{\varpi}\left(x_0\right)=\lim_{h\to0}\frac{\varphi\left(x_0+h\right)\ominus\varphi\left(x_0\right)}{h}=\lim_{h\to0}\frac{\varphi\left(x_0\right)\ominus\varphi\left(x_0-h\right)}{h}, $$

$$ (b)\,{}^{c}\mathbf{D}^{\nu}\tilde{\varpi}\left(x_0\right)=\lim_{h\to0}\frac{\varphi\left(x_0\right)\ominus\varphi\left(x_0+h\right)}{-h}=\lim_{h\to0}\frac{\varphi\left(x_0-h\right)\ominus\varphi\left(x_0\right)}{-h} $$

for $h>0$ sufficiently near zero. Distinguishably defining, for the case (a), $\tilde{\varpi}\left(x\right)$ is said to be (1)-CF_H-differentiable, denoted by ${}^{c}\mathbf{D}_{1}^{\nu}\tilde{\varpi}\left(x\right)$ and for the case (b) it is (2)-CF_H-differentiable represented by ${}^{c}\mathbf{D}_{2}^{\nu}\tilde{\varpi}\left(x\right)$, By the same token, for order $1<\nu<2$, take $\tilde{\mathbf{G}}\in\mathbf{E}_f$ into consideration, such that

$$ \tilde{\mathbf{G}}=\lim_{h\to0}\frac{\varphi\left(x_0+h\right)\ominus\varphi\left(x_0\right)}{h}=\lim_{h\to0}\frac{\varphi\left(x_0\right)\ominus\varphi\left(x_0-h\right)}{h}. $$

Then, for $h>0$ sufficiently near zero

$$ (c)\,{}^{c}\mathbf{D}^{\nu}\tilde{\varpi}\left(x_0\right)=\lim_{h\to0}\frac{\tilde{\mathbf{G}}\left(x_0+h\right)\ominus\tilde{\mathbf{G}}\left(x_0\right)}{h}=\lim_{h\to0}\frac{\tilde{\mathbf{G}}\left(x_0\right)\ominus\tilde{\mathbf{G}}\left(x_0-h\right)}{h}, $$

$$ (d)\,{}^{c}\mathbf{D}^{\nu}\tilde{\varpi}\left(x_0\right)=\lim_{h\to0}\frac{\tilde{\mathbf{G}}\left(x_0\right)\ominus\tilde{\mathbf{G}}\left(x_0+h\right)}{-h}=\lim_{h\to0}\frac{\tilde{\mathbf{G}}\left(x_0-h\right)\ominus\tilde{\mathbf{G}}\left(x_0\right)}{-h}, $$

$$ (e)\,{}^{c}\mathbf{D}^{\nu}\tilde{\varpi}\left(x_0\right)=\lim_{h\to0}\frac{\tilde{\mathbf{G}}\left(x_0+h\right)\ominus\tilde{\mathbf{G}}\left(x_0\right)}{h}=\lim_{h\to0}\frac{\tilde{\mathbf{G}}\left(x_0-h\right)\ominus\tilde{\mathbf{G}}\left(x_0\right)}{-h}, $$

$$ (f)\,{}^{c}\mathbf{D}^{\nu}\tilde{\varpi}\left(x_0\right)=\lim_{h\to0}\frac{\tilde{\mathbf{G}}\left(x_0\right)\ominus\tilde{\mathbf{G}}\left(x_0+h\right)}{-h}=\lim_{h\to0}\frac{\tilde{\mathbf{G}}\left(x_0\right)\ominus\tilde{\mathbf{G}}\left(x_0-h\right)}{h}. $$

Numerous research works are found in which generalized Hukuhara differentiability cases for fractional order differentiation are reviewed and illustrated (see e.g. refs. [17,18,22,25,27] and the references therein).

Theorem 2.2.1

Consider $\tilde{\varpi}\left(x\right)\in C^{F}\left[\alpha,\beta\right]\cap L^{F}\left[\alpha,\beta\right]$ with its r-level representation $\left[\varpi\left(x\right)\right]^{r}=\left[\underline{\varpi}^{r}\left(x\right),\overline{\varpi}^{r}\left(x\right)\right]$, for $r\in\left[0,1\right]$, then for $x_0\in\left(0,\beta\right)$

(a) If $\tilde{\varpi}\left(x\right)$ is (1)-CF_H-differentiable function, then for $0<\nu<1$

$$\left[{}^{c}\mathbf{D}_{1}^{\nu}\varpi(x)\right]^{r}=\left[{}^{c}D^{\nu}\underline{\varpi}^{r}(x),{}^{c}D^{\nu}\overline{\varpi}^{r}(x)\right].$$

(b) If $\tilde{\varpi}(x)$ is (2)-CF_{H}-differentiable function, then for $0<\nu<1$

$$\left[{}^{c}\mathbf{D}_{2}^{\nu}\varpi(x)\right]^{r}=\left[{}^{c}D^{\nu}\overline{\varpi}^{r}(x),{}^{c}D^{\nu}\underline{\varpi}^{r}(x)\right].$$

(c) If $\tilde{\varpi}(x)$ and ${}^{c}\mathbf{D}^{\nu}\varpi(x)$ are (1)-CF_{H}-differentiable function, then for $1<\nu<2$

$$\left[{}^{c}\mathbf{D}_{1,1}^{\nu}\varpi(x)\right]^{r}=\left[{}^{c}D^{\nu}\underline{\varpi}^{r}(x),{}^{c}D^{\nu}\overline{\varpi}^{r}(x)\right].$$

(d) If $\tilde{\varpi}(x)$ and ${}^{c}\mathbf{D}^{\nu}\varpi(x)$ are (2)-CF_{H}-differentiable function, then for $1<\nu<2$

$$\left[{}^{c}\mathbf{D}_{2,2}^{\nu}\varpi(x)\right]^{r}=\left[{}^{c}D^{\nu}\underline{\varpi}^{r}(x),{}^{c}D^{\nu}\overline{\varpi}^{r}(x)\right].$$

(e) If $\tilde{\varpi}(x)$ is (1)-CF_{H}-differentiable and ${}^{c}\mathbf{D}^{\nu}\varpi(x)$ is (2)-CF_{H}-differentiable function, then for $1<\nu<2$

$$\left[{}^{c}\mathbf{D}_{1,2}^{\nu}\varpi(x)\right]^{r}=\left[{}^{c}D^{\nu}\overline{\varpi}^{r}(x),{}^{c}D^{\nu}\underline{\varpi}^{r}(x)\right].$$

(f) If $\tilde{\varpi}(x)$ is (2)-CF_{H}-differentiable and ${}^{c}\mathbf{D}^{\nu}\varpi(x)$ is (1)-CF_{H}-differentiable function, then for $1<\nu<2$.

$$\left[{}^{c}\mathbf{D}_{2,1}^{\nu}\varpi(x)\right]^{r}=\left[{}^{c}D^{\nu}\overline{\varpi}^{r}(x),{}^{c}D^{\nu}\underline{\varpi}^{r}(x)\right].$$

${}^{c}D^{\nu}$ being the Caputo fractional derivative of real valued function (for details see e.g. [2,9] and the references therein). Therefore, at this instant,

$${}^{c}D^{\nu}\underline{\varpi}^{r}(x_0)=\begin{cases}\frac{1}{\Gamma(n-\nu)}\int_0^x\frac{\left(\underline{\varpi}^{r}(\xi)\right)^{(n)}}{(x-\xi)^{1-n+\nu}}d\xi, & \nu\in(n-1,n),\ \ n\in\mathbb{N},\\ \frac{d^n}{dx^n}\underline{\varpi}^{r}(x), & \nu=n\in\mathbb{N}.\end{cases}$$

$${}^{c}D^{\nu}\overline{\varpi}^{r}(x_0)=\begin{cases}\frac{1}{\Gamma(n-\nu)}\int_0^x\frac{\left(\overline{\varpi}^{r}(\xi)\right)^{(n)}}{(x-\xi)^{1-n+\nu}}d\xi, & \nu\in(n-1,n),\ \ n\in\mathbb{N},\\ \frac{d^n}{dx^n}\overline{\varpi}^{r}(x), & \nu=n\in\mathbb{N}.\end{cases}$$

2.3 Mittag-Leffler Function

The generalized form of Mittag-Leffler function $\mathscr{E}_{\vartheta}(x)$ has played a vital part in the fractional differential equations. It was introduced by Mittag-Leffler [30] that can be stated as:

$$\mathscr{E}_{\vartheta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(j\vartheta+1)}, \quad \vartheta > 0 \tag{6}$$

and in generalized form $\mathscr{E}_{\theta,\vartheta}(x)$ is defined as:

$$\mathscr{E}_{\theta,\vartheta}(x) = \sum_{j=0}^{\infty} \frac{x^j}{\Gamma(j\vartheta+\theta)}, \quad \theta > 0, \quad \vartheta > 0. \tag{7}$$

3 Linear Fractional Order Fuzzy Differential Equations

In this section, we specifically exemplify an nth order FFDE, and render the proposed operator method through a theorem in conjunction with the illustration of the algorithm for all the cases of CF_H-differentiability of FFDE, as featured in Theorem 2.2.1, consecutively. Subsequently, let the linear nth order FFDE be of the form

$$\mathbf{D}_x^{n\nu}\tilde{\mathrm{y}}(x) + \mathrm{g}\left(x, \tilde{\mathrm{y}}(x), \mathbf{D}_x^{\nu}\tilde{\mathrm{y}}(x), \ldots\right) = \tilde{\mathrm{f}}(x), \tag{8}$$

$$\tilde{\mathrm{y}}(x_0) = \tilde{\mathrm{y}}_0, \quad \tilde{\mathrm{y}}'(x_0) = \tilde{\mathrm{y}}_0', \tag{9}$$

where $\tilde{\mathrm{y}}(x)$ is the ascertaining fuzzy-valued function of x, which can be written in form of r-levels as $[\mathrm{y}(x)]^r = \left[\underline{\mathrm{y}}^r(x), \overline{\mathrm{y}}^r(x)\right]$, $\tilde{\mathrm{f}}(x)$ and $\mathrm{g}\left(x, \tilde{\mathrm{y}}(x), \mathbf{D}_x^{\nu}\tilde{\mathrm{y}}(x), \ldots\right)$ are acquainting continuous nonhomogeneous and linear terms of the FFDE, accordingly.

Theorem 3.1

Suppose $x_0 \in [\alpha, \beta]$ and consider that $\tilde{\mathrm{f}} : [\alpha, \beta] \times \mathbf{E}_f \times \mathbf{E}_f \to \mathbf{E}_f$ is continuous. Moreover, assume that $\tilde{\mathrm{y}}(x), \mathbf{D}_x^{\nu}\tilde{\mathrm{y}}(x), \mathbf{D}_x^{2\nu}\tilde{\mathrm{y}}(x), \ldots \in C^F$, then Eq. 8 can be equated to anyone of the following FRLFI equations:

(1) If $\tilde{\mathrm{y}}(x), \mathbf{D}_x^{\nu}\tilde{\mathrm{y}}(x), \mathbf{D}_x^{2\nu}\tilde{\mathrm{y}}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{\mathrm{y}}(x)$ are (1)-CF_H-differentiable, then

$$\begin{aligned}\tilde{\mathrm{y}}(x) + \Im_x^{n\nu}\mathrm{g}\left(x, \tilde{\mathrm{y}}(x), \mathbf{D}_x^{\nu}\tilde{\mathrm{y}}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{\mathrm{y}}(x)\right) = \\ a_1 + a_2\frac{(x-x_0)^{(n-1)\nu}}{\Gamma((n-1)\nu+1)} + \ldots + a_n\frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)} + \Im_x^{n\nu}\tilde{\mathrm{f}}(x).\end{aligned}$$

(2) If $\tilde{\mathrm{y}}(x)$ is (1)-CF_H-differentiable and $\mathbf{D}_x^{\nu}\tilde{\mathrm{y}}(x), \mathbf{D}_x^{2\nu}\tilde{\mathrm{y}}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{\mathrm{y}}(x)$ are (2)-CF_H-differentiable, then

$$\begin{aligned}\tilde{\mathrm{y}}(x) + \Im_x^{n\nu}\mathrm{g}\left(x, \tilde{\mathrm{y}}(x), \mathbf{D}_x^{\nu}\tilde{\mathrm{y}}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{\mathrm{y}}(x)\right) = \\ a_1 + a_2\frac{(x-x_0)^{(n-1)\nu}}{\Gamma((n-1)\nu+1)} + \ldots + a_n\frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)} \ominus (-1)\Im_x^{n\nu}\tilde{\mathrm{f}}(x).\end{aligned}$$

(3) If $\tilde{y}(x)$ is (2)-CF_H-differentiable and $\mathbf{D}_x^{\nu}\tilde{y}(x), \mathbf{D}_x^{2\nu}\tilde{y}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)$ are (1)-CF_H-differentiable, then

$$\tilde{y}(x) + \Im_x^{n\nu} g\left(x, \tilde{y}(x), \mathbf{D}_x^{\nu}\tilde{y}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right) =$$
$$a_1 \ominus (-1)\left(a_2 \frac{(x-x_0)^{(n-1)\nu}}{\Gamma((n-1)\nu+1)} + \ldots + a_n \frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)} + \Im_x^{n\nu}\tilde{f}(x)\right).$$

(4) If $\tilde{y}(x), \mathbf{D}_x^{\nu}\tilde{y}(x), \mathbf{D}_x^{2\nu}\tilde{y}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)$ are (2)-CF_H-differentiable, then

$$\tilde{y}(x) + \Im_x^{n\nu} g\left(x, \tilde{y}(x), \mathbf{D}_x^{\nu}\tilde{y}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right) =$$
$$a_1 \ominus (-1)\left(a_2 \frac{(x-x_0)^{(n-1)\nu}}{\Gamma((n-1)\nu+1)} + \ldots + a_n \frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)} \ominus (-1)\Im_x^{n\nu}\tilde{f}(x)\right).$$

where $a_1 = \tilde{y}(x_0)$, $a_2 = \mathbf{D}_x^{\nu}\tilde{y}(x_0), \ldots, a_n = \mathbf{D}_x^{n\nu}\tilde{y}(x_0)$.

Proof.

It has been explained by Bede et al. [31] that fuzzy-valued functions are integrable if $\tilde{f} \in C^F[\alpha,\beta] \cap L^F[\alpha,\beta]$. Now, the corresponding FRLFI forms of Eq.8 can be assessed under each type of CF_H-differentiable as:

(1) Let $\tilde{y}(x), \mathbf{D}_x^{\nu}\tilde{y}(x), \mathbf{D}_x^{2\nu}\tilde{y}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)$ are (1)-CF_H-differentiable, then FRLFI of Eq.8 under (1)-CF_H-differentiability of $\mathbf{D}_x^{n\nu}\tilde{y}(x)$ can be measured as:

$$\mathbf{D}_x^{(n-1)\nu}\tilde{y}(x) + \Im_x^{(1)\nu} g\left(x, \tilde{y}(x), \mathbf{D}_x^{\nu}\tilde{y}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right) = a_n + \Im_x^{(1)\nu}\tilde{f}(x),$$

$$\mathbf{D}_x^{(n-2)\nu}\tilde{y}(x) + \Im_x^{(2)\nu} g\left(x, \tilde{y}(x), \mathbf{D}_x^{\nu}\tilde{y}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right) = a_{n-1} + a_n \frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)} + \Im_x^{(2)\nu}\tilde{f}(x)$$

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$$\tilde{y}(x) + \Im_x^{n\nu} g\left(x, \tilde{y}(x), \mathbf{D}_x^{\nu}\tilde{y}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right) =$$
$$a_1 + a_2 \frac{(x-x_0)^{(n-1)\nu}}{\Gamma((n-1)\nu+1)} + \ldots + a_n \frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)} + \Im_x^{n\nu}\tilde{f}(x).$$

(2) Let $\tilde{y}(x)$ be (1)-CF_H-differentiable and $\mathbf{D}_x^{\nu}\tilde{y}(x), \mathbf{D}_x^{2\nu}\tilde{y}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)$ are (2)-CF_H-differentiable, then FRLFI form of Eq.8 under (2)-CF_H-differentiability of $\mathbf{D}_x^{(n)\nu}\tilde{y}(x)$ can be evaluated as below:

$$\mathbf{D}_x^{(n-1)\nu}\tilde{y}(x) + \Im_x^{(1)\nu} g\left(x, \tilde{y}(x), \mathbf{D}_x^{\nu}\tilde{y}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right) = a_n \ominus (-1)\Im_x^{(1)\nu}\tilde{f}(x),$$

$$\mathbf{D}_x^{(n-2)\nu}\tilde{y}(x) + \Im_x^{(2)\nu} g\left(x, \tilde{y}(x), \mathbf{D}_x^{\nu}\tilde{y}(x), \ldots, \mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right) = a_{n-1} + a_n \frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)} \ominus (-1)\Im_x^{(2)\nu}\tilde{f}(x)$$

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$$\tilde{y}(x)+\Im_x^{n\nu}g\left(x,\tilde{y}(x),\mathbf{D}_x^{\nu}\tilde{y}(x),\ldots,\mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right)=$$
$$a_1+a_2\frac{(x-x_0)^{(n-1)\nu}}{\Gamma((n-1)\nu+1)}+\ldots+a_n\frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)}\ominus(-1)\Im_x^{n\nu}\tilde{f}(x).$$

(3) Let $\tilde{y}(x)$ be (2)-CF_H-differentiable and $\mathbf{D}_x^{\nu}\tilde{y}(x),\mathbf{D}_x^{2\nu}\tilde{y}(x),\ldots,\mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)$ are (1)-CF_H-differentiable, then FRLFI form of Eq.8 under (1)-CF_H-differentiability of $\mathbf{D}_x^{(n)\nu}\tilde{y}(x)$ can be measured as:

$$\mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)+\Im_x^{(1)\nu}g\left(x,\tilde{y}(x),\mathbf{D}_x^{\nu}\tilde{y}(x),\ldots,\mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right)=a_n\ominus(-1)\Im_x^{(1)\nu}\tilde{f}(x),$$

$$\mathbf{D}_x^{(n-2)\nu}\tilde{y}(x)+\Im_x^{(2)\nu}g\left(x,\tilde{y}(x),\mathbf{D}_x^{\nu}\tilde{y}(x),\ldots,\mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right)=$$
$$a_{n-1}\ominus(-1)\left(a_n\frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)}+\Im_x^{(2)\nu}\tilde{f}(x)\right)$$

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$$\tilde{y}(x)+\Im_x^{n\nu}g\left(x,\tilde{y}(x),\mathbf{D}_x^{\nu}\tilde{y}(x),\ldots,\mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right)=$$
$$a_1\ominus(-1)\left(a_2\frac{(x-x_0)^{(n-1)\nu}}{\Gamma((n-1)\nu+1)}+\ldots+a_n\frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)}+\Im_x^{n\nu}\tilde{f}(x)\right).$$

(4) Let $\tilde{y}(x),\mathbf{D}_x^{\nu}\tilde{y}(x),\mathbf{D}_x^{2\nu}\tilde{y}(x),\ldots,\mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)$ are (2)-CF_H-differentiable, then FRLFI of Eq.8 under (2)-CF_H-differentiability of $\mathbf{D}_x^{(n)\nu}\tilde{y}(x)$ can be determined as:

$$\mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)+\Im_x^{(1)\nu}g\left(x,\tilde{y}(x),\mathbf{D}_x^{\nu}\tilde{y}(x),\ldots,\mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right)=a_n\ominus(-1)\Im_x^{(1)\nu}\tilde{f}(x),$$

$$\mathbf{D}_x^{(n-2)\nu}\tilde{y}(x)+\Im_x^{(2)\nu}g\left(x,\tilde{y}(x),\mathbf{D}_x^{\nu}\tilde{y}(x),\ldots,\mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right)=$$
$$a_{n-1}\ominus(-1)a_n\frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)}\ominus(-1)\Im_x^{(2)\nu}\tilde{f}(x)$$

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$$\tilde{y}(x)+\Im_x^{n\nu}g\left(x,\tilde{y}(x),\mathbf{D}_x^{\nu}\tilde{y}(x),\ldots,\mathbf{D}_x^{(n-1)\nu}\tilde{y}(x)\right)=$$
$$a_1\ominus(-1)\left(a_2\frac{(x-x_0)^{(n-1)\nu}}{\Gamma((n-1)\nu+1)}+\ldots+a_n\frac{(x-x_0)^{\nu}}{\Gamma(\nu+1)}\ominus(-1)\Im_x^{n\nu}\tilde{f}(x)\right).$$

4 Test Examples

Here, the proposed method defined in previous section is applied on some examples of linear FFDEs for illustration of its efficiency and capability.

Example 1:

Take the following FFDE into consideration subjected to initial conditions as:

$$\mathbf{D}_x^{\nu}\tilde{y}(x)=\tilde{y}(x),\quad 0<\nu<1, \tag{10}$$

$$\tilde{y}(0)=\tilde{y}_0\in\mathbf{E}_f. \tag{11}$$

Let $\tilde{y}(x)$ be (1)-CF_H-differentiable then on taking FRLFI on both sides of Eq.10, equivalent FRLFI form of (1)-CF_H-differentiability is obtained

$$\tilde{y}(x)=\tilde{y}(0)+\mathfrak{I}_x^{\nu}\tilde{y}(x), \tag{12}$$

that can also be written as:

$$\underline{y}^r(x)=\underline{y}^r(0)+\mathbf{I}_x^{\nu}\underline{y}^r(x),\quad \overline{y}^r(x)=\overline{y}^r(x)+\mathbf{I}_x^{\nu}\overline{y}^r(x), \tag{13}$$

where $\underline{y}^r(x)$ and $\overline{y}^r(x)$ are lower and upper functions of $\tilde{y}(x)$, which is further simplified to

$$(1-\mathbf{I}_x^{\nu})\underline{y}^r(x)=\underline{y}^r(0),\quad (1-\mathbf{I}_x^{\nu})\overline{y}^r(x)=\overline{y}^r(0). \tag{14}$$

Taking inverse of $(1-\mathbf{I}_x^{\nu})$ on both sides of the equation

$$\underline{y}^r(x)=(1-\mathbf{I}_x^{\nu})^{-1}\underline{y}^r(0),\quad \overline{y}^r(x)=(1-\mathbf{I}_x^{\nu})^{-1}\overline{y}^r(0) \tag{15}$$

and on substitution of

$$(1-\mathbf{I}_x^{\nu})^{-1}=1+\mathbf{I}_x^{\nu}+\mathbf{I}_x^{2\nu}+\mathbf{I}_x^{3\nu}+\dots \tag{16}$$

in Eq.15 with the assumption $\|\mathbf{I}_x^{\nu}\|<1$, for $r\in(0,1]$, the expanded form of the solution is attained as:

$$\underline{y}^r(x)=\left(1+\mathbf{I}_x^{\nu}+\mathbf{I}_x^{2\nu}+\mathbf{I}_x^{3\nu}+\dots\right)\underline{y}^r(0)=\left(1+\frac{x^{\nu}}{\Gamma(\nu+1)}+\frac{x^{2\nu}}{\Gamma(2\nu+1)}+\frac{x^{3\nu}}{\Gamma(3\nu+1)}+\dots\right)\underline{y}^r(0) \tag{17}$$

$$\overline{y}^r(x) = \left(1+\mathbf{I}_x^{\nu}+\mathbf{I}_x^{2\nu}+\mathbf{I}_x^{3\nu}+\dots\right)\overline{y}^r(0) = \left(1+\frac{x^{\nu}}{\Gamma(\nu+1)}+\frac{x^{2\nu}}{\Gamma(2\nu+1)}+\frac{x^{3\nu}}{\Gamma(3\nu+1)}+\dots\right)\overline{y}^r(0) \tag{18}$$

and in closed form as:

$$\underline{y}^r(x) = \mathscr{E}_{\nu}(x^{\nu})\,\underline{y}^r(0), \quad \overline{y}^r(x) = \mathscr{E}_{\nu}(x^{\nu})\,\overline{y}^r(0), \tag{19}$$

which are the required solutions of Eq.10 under (1)-CF_H-differentiability.

Example 2:

Consider another example under (2)-CF_H-differentiability of $\tilde{y}(x)$

$$\mathbf{D}_x^{\nu}\tilde{y}(x) = -\tilde{y}(x), \quad 0<\nu<1 \tag{20}$$

with initial condition

$$\tilde{y}(0) = \tilde{y}_0 \in \mathbf{E}_f. \tag{21}$$

Then its corresponding FRLFI form of Eq.20 can be articulated as:

$$\tilde{y}(x) = \tilde{y}(0) \ominus \mathfrak{I}_x^{\nu}\tilde{y}(x) \tag{22}$$

that can be simplified and expressed as:

$$(1+\mathbf{I}_x^{\nu})\,\underline{y}^r(x) = \underline{y}^r(0), \quad (1+\mathbf{I}_x^{\nu})\,\overline{y}^r(x) = \overline{y}^r(0) \tag{23}$$

or

$$\underline{y}^r(x) = (1+\mathbf{I}_x^{\nu})^{-1}\,\underline{y}^r(0), \quad \overline{y}^r(x) = (1+\mathbf{I}_x^{\nu})^{-1}\,\overline{y}^r(0). \tag{24}$$

Employing the Binomial expansion

$$(1+\mathbf{I}_x^{\nu})^{-1} = 1-\mathbf{I}_x^{\nu}+\mathbf{I}_x^{2\nu}-\mathbf{I}_x^{3\nu}+\dots \tag{25}$$

on Eq.24 following solutions are achieved:

$$\underline{y}^r(x) = \left(1-\mathbf{I}_x^{\nu}+\mathbf{I}_x^{2\nu}-\mathbf{I}_x^{3\nu}+\dots\right)\underline{y}^r(0) = \left(1-\frac{x^{\nu}}{\Gamma(\nu+1)}+\frac{x^{2\nu}}{\Gamma(2\nu+1)}-\frac{x^{3\nu}}{\Gamma(3\nu+1)}+\dots\right)\underline{y}^r(0) \tag{26}$$

$$\overline{y}^r(x) = \left(1 - \mathbf{I}_x^{\nu} + \mathbf{I}_x^{2\nu} - \mathbf{I}_x^{3\nu} + \ldots\right)\overline{y}^r(0) = \left(1 - \frac{x^{\nu}}{\Gamma(\nu+1)} + \frac{x^{2\nu}}{\Gamma(2\nu+1)} - \frac{x^{3\nu}}{\Gamma(3\nu+1)} + \ldots\right)\overline{y}^r(0) \tag{27}$$

and in closed version it can be represented as:

$$\underline{y}^r(x) = \mathscr{E}_{\nu}(-x^{\nu})\,\underline{y}^r(0), \quad \overline{y}^r(x) = \mathscr{E}_{\nu}(-x^{\nu})\,\overline{y}^r(0). \tag{28}$$

Example 3:

Now, consider the following FFDE

$$\mathbf{D}_x^{\nu}\tilde{y}(x) = -\tilde{y}(x) + x + 1, \quad 0 < \nu < 1 \tag{29}$$

with

$$\tilde{y}(0) = \tilde{y}_0 \in \mathbf{E}_f. \tag{30}$$

Let $\tilde{y}(x)$ be (2)-CF_H-differentiable, then we obtain:

$$\underline{y}^r(x) = (1 + \mathbf{I}_x^{\nu})^{-1}\,\underline{y}^r(0) + (1 + \mathbf{I}_x^{\nu})^{-1}\left(\frac{x^{\nu}}{\Gamma(\nu+1)} + \frac{x^{2\nu}}{\Gamma(2\nu+1)}\right) \tag{31}$$

and

$$\overline{y}^r(x) = (1 + \mathbf{I}_x^{\nu})^{-1}\,\overline{y}^r(0) + (1 + \mathbf{I}_x^{\nu})^{-1}\left(\frac{x^{\nu}}{\Gamma(\nu+1)} + \frac{x^{2\nu}}{\Gamma(2\nu+1)}\right). \tag{32}$$

On exercising Eq.25 on Eqs. 31 and 32 and further simplifying, following results are attained in the compact form, namely

$$\underline{y}^r(x) = \mathscr{E}_{\nu}(-x^{\nu})\,\underline{y}^r(0) + \frac{x^{\nu}}{\Gamma(\nu+1)}, \quad \overline{y}^r(x) = \mathscr{E}_{\nu}(-x^{\nu})\,\overline{y}^r(0) + \frac{x^{\nu}}{\Gamma(\nu+1)}. \tag{33}$$

Example 4:

Next consider the following fractional second order FDE subjected to the initial conditions:

$$\mathbf{D}_x^{2\nu}\tilde{y}(x) = \tilde{\rho}_0, \quad \tilde{\rho}_0 = (r-1, 1-r), \quad r \in [0,1], \quad 1 < \nu < 2 \tag{34}$$

$$[y(0)]^r = (r-1, 1-r), \tag{35}$$

$$\left[\mathbf{D}_x^{\nu}\mathrm{y}(0)\right]^r=(r-1,1-r),\quad \nu\in(0,1]. \tag{36}$$

Taking into account all the cases of CF_H-differentiability, following cases are developed:

Case I:

Let $\tilde{\mathrm{y}}^r(x)$ and $\mathbf{D}_x^{\nu}\tilde{\mathrm{y}}(x)$ are (1)-CF_H-differentiable, then on employing the scheme given in Theorem 3.1, we come down with:

$$\tilde{\mathrm{y}}(x)=\tilde{\mathrm{y}}(0)+\frac{x^{\nu}}{\Gamma(\nu+1)}\mathbf{D}_x^{\nu}\tilde{\mathrm{y}}(0)+\Im_x^{2\nu}\tilde{\rho}_0. \tag{37}$$

After manipulation the solutions in its r-level functions are attained efficiently as:

$$\underline{\mathrm{y}}^r(x)=(r-1)\left(1+\frac{x^{\nu}}{\Gamma(\nu+1)}+\frac{x^{2\nu}}{\Gamma(2\nu+1)}\right), \tag{38}$$

$$\overline{\mathrm{y}}^r(x)=(1-r)\left(1+\frac{x^{\nu}}{\Gamma(\nu+1)}+\frac{x^{2\nu}}{\Gamma(2\nu+1)}\right). \tag{39}$$

Case II:

Let $\tilde{\mathrm{y}}(x)$ is (1)-CF_H-differentiable and $\mathbf{D}_x^{\nu}\tilde{\mathrm{y}}(x)$ are (2)-CF_H-differentiable, then we get:

$$\tilde{\mathrm{y}}(x)=\tilde{\mathrm{y}}(0)+\frac{x^{\nu}}{\Gamma(\nu+1)}\mathbf{D}_x^{\nu}\tilde{\mathrm{y}}(0)\ominus(-1)\Im_x^{2\nu}\tilde{\rho}_0 \tag{40}$$

with its lower and upper expressions:

$$\underline{\mathrm{y}}^r(x)=\underline{\mathrm{y}}^r(0)+\frac{x^{\nu}}{\Gamma(\nu+1)}\mathbf{D}_x^{\nu}\underline{\mathrm{y}}^r(0)+\mathbf{I}_x^{2\nu}\overline{\rho}_0, \tag{41}$$

$$\overline{\mathrm{y}}^r(x)=\overline{\mathrm{y}}^r(0)+\frac{x^{\nu}}{\Gamma(\nu+1)}\mathbf{D}_x^{\nu}\overline{\mathrm{y}}^r(0)+\mathbf{I}_x^{2\nu}\underline{\rho}_0. \tag{42}$$

Consequently, after some simplifications the solutions of r-level functions of Eq.34 in compact form $\forall\chi\in(0,1]$ are attained as:

$$\underline{\mathrm{y}}^r(x)=(r-1)\left(1+\frac{x^{\nu}}{\Gamma(\nu+1)}-\frac{x^{2\nu}}{\Gamma(2\nu+1)}\right). \tag{43}$$

and

$$\overline{y}^r(x) = (1-r)\left(1 + \frac{x^\nu}{\Gamma(\nu+1)} - \frac{x^{2\nu}}{\Gamma(2\nu+1)}\right). \tag{44}$$

Case III:

Now assume that $\tilde{y}(x)$ is (2)-CF_H-differentiable and $\mathbf{D}_x^\nu \tilde{y}(x)$ is (1)-CF_H-differentiable, following the algorithm and applying FRLFI on Eq.34, we procure:

$$\tilde{y}(x) = \tilde{y}(0) \ominus (-1)\left(\frac{x^\nu}{\Gamma(\nu+1)}\mathbf{D}_x^\nu \tilde{y}(0) + \mathfrak{I}_x^{2\nu}\tilde{\rho}_0\right). \tag{45}$$

Expanding in its r-level functions, $\underline{y}^r(x)$ and $\overline{y}^r(x)$, we obtain

$$\underline{y}^r(x) = \underline{y}^r(0) + \frac{x^\nu}{\Gamma(\nu+1)}\mathbf{D}_x^\nu \overline{y}^r(0) + \mathbf{I}_x^{2\nu}\overline{\rho}_0, \tag{46}$$

$$\overline{y}^r(x) = \overline{y}^r(0) + \frac{x^\nu}{\Gamma(\nu+1)}\mathbf{D}_x^\nu \underline{y}^r(0) + \mathbf{I}_x^{2\nu}\underline{\rho}_0. \tag{47}$$

After doing some exercises the solutions of Eq.34 with respect to its lower and upper functions are derived $\forall x \in \left(0, \sqrt{3}-1\right)$ as:

$$\underline{y}^r(x) = (r-1)\left(1 - \frac{x^\nu}{\Gamma(\nu+1)} - \frac{x^{2\nu}}{\Gamma(2\nu+1)}\right) \tag{48}$$

and

$$\overline{y}^r(x) = (1-r)\left(1 - \frac{x^\nu}{\Gamma(\nu+1)} - \frac{x^{2\nu}}{\Gamma(2\nu+1)}\right). \tag{49}$$

Case IV:

Lastly, let $\tilde{y}(x)$ and $\mathbf{D}_x^\nu \tilde{y}(x)$ be (2)-CF_H-differentiable. Then operating the algorithm, we achieve:

$$\tilde{y}(x) = \tilde{y}(0) \ominus (-1)\left(\frac{x^\nu}{\Gamma(\nu+1)}\mathbf{D}_x^\nu \tilde{y}(0) \ominus (-1)\mathfrak{I}_x^{2\nu}\tilde{\rho}_0\right). \tag{50}$$

Expanding in terms of lower and upper functions:

$$\underline{y}^r(x) = \underline{y}^r(0) + \frac{x^\nu}{\Gamma(\nu+1)}\mathbf{D}_x^\nu \overline{y}^r(0) + \mathbf{I}_x^{2\nu}\underline{\rho}_0, \tag{51}$$

$$\overline{y}^r(x) = \overline{y}^r(0) + \frac{x^\nu}{\Gamma(\nu+1)}\mathbf{D}_x^\nu \underline{y}^r(0) + \mathbf{I}_x^{2\nu}\overline{\rho}_0 \tag{52}$$

As a result, the solutions of r-level functions of Eq.34 in the compact form $\forall x \in (0,1)$ are obtained as:

$$\underline{y}^r(x) = (r-1)\left(1 - \frac{x^\nu}{\Gamma(\nu+1)} + \frac{x^{2\nu}}{\Gamma(2\nu+1)}\right) \tag{53}$$

and

$$\overline{y}^r(x) = (1-r)\left(1 - \frac{x^\nu}{\Gamma(\nu+1)} + \frac{x^{2\nu}}{\Gamma(2\nu+1)}\right). \tag{54}$$

Observably, the obtained results are in good agreement with the results in Khastan et al. [32] and Allahviranloo et al. [21] for $\nu = 2$.

5 Conclusions

In this manuscript, we generalized the operator method stated in [21] to amplify its application for the analytical solutions of FFDEs. The concept of the Caputo-type fuzzy fractional Hukuhara differentiability was deliberated on FFDEs to acquire its all feasible systems. We assessed the proposed approach on the systems of FFDEs. Prodigiously getting the solutions of some illustrative examples of FFDEs on employing this approach, it can be established that the method is reliable and efficiently capable of solving the fractional order fuzzy differential equations. Hence, it is concluded to be a practically consistent method for integer and non-integer fuzzy differential equations that appear naturally in different dynamical models.

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