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Hurwitz Type Results for Sum of Two Triangular Numbers

*Chandrashekar Adiga**, *M. S. Surekha and A. Vanitha*

Department of Studies in Mathematics, University of Mysore, Manasagangotri. Mysore 570 006, India

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Abstract: Let $t_2(n)$ denote the number of representations of n as a sum of two triangular numbers and $t_{(a,b)}(n)$ denote number of representations of n as a sum of a times triangular number and b times triangular number. In this paper, we prove number of results in which generating functions of $t_2(n)$ and $t_{(1,3)}(n)$ are infinite product. We also establish relations between $t_{(1,3)}(n)$, $t_{(1,12)}(n)$, $t_{(3,4)}(n)$, $t_2(n)$ and $t_{(1,4)}(n)$.

Keywords: Representation of triangular numbers, generating functions, theta functions

Throughout the paper, we employ the standard notation

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

Ramanujan's general theta function is defined as

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab|<1.$$

For convience, we denote $f(q,q)$ by $\varphi(q)$, $f(q,q^3)$ by $\psi(q)$ and $f(-q,-q^2)$ by $f(-q)$. The Jacobi triple product identity [1] is defined by

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

By Jacobi triple product identity each $\varphi(q)$, $\psi(q)$ and $f(-q)$ is a product. Infact

$$\varphi(q) = (-q;q^2)^2_\infty(q^2;q^2)_\infty,$$
$$\psi(q) = (-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty,$$
$$f(-q) = (q;q^3)_\infty(q^2;q^3)_\infty(q^3;q^3)_\infty.$$

Let $r_k(n)$ denote the number of representations of n as a sum of k squares and $t_k(n)$ denote the number of representations of n as a sum of k triangular numbers. Let $t_{(a,b)}(n)$ denote the number of solutions in non negative integer of the equation

$$a\frac{x_1(x_1+1)}{2} + b\frac{x_2(x_2+1)}{2} = n.$$

There is a remarkable relation between $r_k(n)$ and $t_k(n)$ [2]:

$$r_k(8n+k) = 2^{k-1}\left\{2+\binom{k}{4}\right\}t_k(n), \text{ for } 1\le k\le 7.$$

A. Hurwitz [4] proved several results in which generating function of $r_3(an+b)$ is a simple infinite product. For example

$$\sum_{n\ge 0} r_3(4n+1)q^n = 6\varphi^2(q)\psi(q^2),$$
$$\sum_{n\ge 0} r_3(4n+2)q^n = 12\varphi(q)\psi^2(q^2),$$
$$\sum_{n\ge 0} r_3(8n+1)q^n = 6\varphi^2(q)\psi(q).$$

These results have been proved by S. Cooper and M. D. Hirschhorn [3] and they have also established eighty infinite families of similar results.
The main purpose of this paper is to prove number of results in which generating functions of $t_2(n)$ and $t_{(1,3)}(n)$, when n is restricted to an arithmetic sequence are infinite products.
Infact, we prove the following results.

* Corresponding author e-mail: c_adiga@hotmail.com

Theorem 1. *We have*

$$\sum_{n=0}^{\infty} t_2(8n+1)q^n = 2\psi(q)f(q^7,q^9), \qquad (1)$$

$$\sum_{n=0}^{\infty} t_2(8n+3)q^n = 2\psi(q)f(q^5,q^{11}), \qquad (2)$$

$$\sum_{n=0}^{\infty} t_2(8n+5)q^n = 2q\psi(q)f(q,q^{15}), \qquad (3)$$

$$\sum_{n=0}^{\infty} t_2(8n+7)q^n = 2\psi(q)f(q^3,q^{13}). \qquad (4)$$

Theorem 2. *We have*

$$\sum_{n=0}^{\infty} t_{(1,3)}(16n+2)q^n = 2q\psi(q^3)f(q^3,q^{13}), \qquad (5)$$

$$\sum_{n=0}^{\infty} t_{(1,3)}(16n+3)q^n = 2\psi(q)f(q^{21},q^{27}), \qquad (6)$$

$$\sum_{n=0}^{\infty} t_{(1,3)}(16n+6)q^n = 2\psi(q^3)f(q^7,q^9), \qquad (7)$$

$$\sum_{n=0}^{\infty} t_{(1,3)}(16n+7)q^n = 2q^2\psi(q)f(q^9,q^{39}), \qquad (8)$$

$$\sum_{n=0}^{\infty} t_{(1,3)}(16n+10)q^n = 2\psi(q^3)f(q^5,q^{11}), \qquad (9)$$

$$\sum_{n=0}^{\infty} t_{(1,3)}(16n+11)q^n = 2q^4\psi(q)f(q^3,q^{45}), \qquad (10)$$

$$\sum_{n=0}^{\infty} t_{(1,3)}(16n+14)q^n = 2q\psi(q^3)f(q,q^{15}), \qquad (11)$$

$$\sum_{n=0}^{\infty} t_{(1,3)}(16n+15)q^n = 2\psi(q)f(q^{15},q^{33}). \qquad (12)$$

We also establish the following relations between $t_{(1,3)}(n)$, $t_{(1,12)}(n)$, $t_{(3,4)}(n)$, $t_2(n)$ and $t_{(1,4)}(n)$.

Theorem 3. *We have*

$$t_{(1,3)}(4n+2) = 2t_{(1,12)}(n-1), \qquad n \geq 1, \qquad (13)$$

$$t_{(1,3)}(4n+3) = 2t_{(3,4)}(n), \qquad n \geq 0, \qquad (14)$$

$$t_2(2n+1) = 2t_{(1,4)}(n), \qquad n \geq 0. \qquad (15)$$

1 Proof of Theorem 1

From [1, Entry 25(iv), p. 36], we have

$$\sum_{n=0}^{\infty} t_2(n)q^n = \psi^2(q) = \psi(q^2)\varphi(q). \qquad (16)$$

Adding Entries $30(ii)$ and $30(iii)$ in [1, p. 43], we obtain

$$f(a,b) = f(a^3b,ab^3) + af(b/a,a^5b^3). \qquad (17)$$

Putting a=q and b=q in (17), we obtain

$$\varphi(q) = \varphi(q^4) + 2q\psi(q^8). \qquad (18)$$

Employing (18) in (16), we see that

$$\sum_{n=0}^{\infty} t_2(n)q^n = \psi(q^2)\{\varphi(q^4) + 2q\psi(q^8)\}. \qquad (19)$$

Immediately, it follows that

$$\sum_{n=0}^{\infty} t_2(2n+1)q^n = 2\psi(q)\psi(q^4). \qquad (20)$$

Putting a=q and b=q^3 in (17), we obtain

$$\psi(q) = f(q^6,q^{10}) + qf(q^2,q^{14}). \qquad (21)$$

Employing (21) in (20) and then extracting those terms in which the power of q is $0 \pmod 2$ and replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty} t_2(4n+1)q^n = 2\psi(q^2)f(q^3,q^5). \qquad (22)$$

Putting a=q^3 and b=q^5 in (17), we get

$$f(q^3,q^5) = f(q^{14},q^{18}) + q^3f(q^2,q^{30}). \qquad (23)$$

Employing (23) in (22), it immediately follows that

$$\sum_{n=0}^{\infty} t_2(8n+1)q^n = 2\psi(q)f(q^7,q^9)$$

and

$$\sum_{n=0}^{\infty} t_2(8n+5)q^n = 2q\psi(q)f(q,q^{15}).$$

This completes the proofs of (1) and (3).
The proofs of (2) and (4) are similar.

2 Proof of Theorem 2

We have

$$\sum_{n=0}^{\infty} t_{(1,3)}(n)q^n = \psi(q)\psi(q^3). \qquad (24)$$

From [1, p. 69, Eq. (36.8)], we have

$$\psi(q)\psi(q^3) = \varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12}).$$

Employing the above identity in (24), we obtain

$$\sum_{n=0}^{\infty} t_{(1,3)}(n)q^n = \varphi(q^6)\psi(q^4) + q\varphi(q^2)\psi(q^{12}). \qquad (25)$$

Extracting those terms in which the power of q is 0 $(\mathrm{mod}\ 2)$ and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} t_{(1,3)}(2n)q^n = \varphi(q^3)\psi(q^2) = \psi(q^2)\{\varphi(q^{12}) + 2q^3\psi(q^{24})\}. \qquad (26)$$

Again, extracting those terms in which the power of q is 1 $(\mathrm{mod}\ 2)$, divide by q and replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty} t_{(1,3)}(4n+2)q^n = 2q\psi(q)\psi(q^{12}). \qquad (27)$$

Employing (21) in (27), we immediately see that

$$\sum_{n=0}^{\infty} t_{(1,3)}(8n+2)q^n = 2q\psi(q^6)f(q,q^7), = 2q\psi(q^6)\{f(q^{10},q^{22}) + qf(q^6,q^{26})\}.$$

Hence,

$$\sum_{n=0}^{\infty} t_{(1,3)}(16n+2)q^n = 2q\psi(q^3)f(q^3,q^{13}),$$
$$\sum_{n=0}^{\infty} t_{(1,3)}(16n+10)q^n = 2\psi(q^3)f(q^5,q^{11}).$$

This completes the proofs of (5) and (9).
The proofs of remaining identities are similar to the proofs of (5) and (9).

3 Proof of Theorem 3

By (27), we have

$$\sum_{n=0}^{\infty} t_{(1,3)}(4n+2)q^n = 2q\psi(q)\psi(q^{12}) = 2q\sum_{n=0}^{\infty} t_{(1,12)}(n)q^n.$$

Now, comparing the coefficients of q^n in both sides of the above identity, we get (13).
Proofs of (14) and (15) are similar to that of (13).

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Chandrashekar Adiga is a Professor of Mathematics at the University of Mysore, INDIA. His research interests are in the areas of Special Functions, Number Theory, Partition Theory, Discrete Mathematics and Graph Theory. He has published more than 120 research articles in reputed national and international journals of mathematical sciences. He served as the Chairman of the Department of Studies in Mathematics, University of Mysore during 2010-2012. He is referee and editor of mathematical journals.

M. S. Surekha is research scholar at University of Mysore. Her research interests are in the areas of Special Functions, Number Theory and Partition Theory. She has published research articles in reputed international journals of mathematical sciences.

A. Vanitha is research scholar at University of Mysore. Her research interests are in the areas of Special Functions, Number Theory and Partition Theory. She has published research articles in reputed international journals of mathematical sciences.