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Fractional Wavelet Transform in Terms of Fractional Convolution

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Abstract: Continuity of fractional wavelet transform (FrWT) in terms of fractional convolution operator and its adjoint are obtained. A relation between the FrWT and inverse fractional Fourier transform is established. The FrWT of a test function space is investigated.

Keywords: Fractional wavelet transform, Fractional Fourier transform, Adjoint operator, Schwartz space.

1 Introduction

The word fraction is nowadays very popular in different field of knowledge. We only mention the fractional derivatives in mathematics, the fractional dimension in geometry and fractional transformations. In general, it means that some parameter has a non-integer value. The fractional Fourier transform (FrFT) is a generalization of the conventional Fourier transform with an angle θ. Many years ago, it was proposed in mathematics literature but today many new applications in several areas including Physics, Stochastic Process and Mathematical Analysis are found [1–4]. The one-dimensional FrFT [5–9] with an angle θ of $f(t)\in L^2(\mathbb{R})$ denoted by $(\mathscr{F}^\theta f)(\omega)=\hat{f}^\theta(\omega)$ is given by

$$(\mathscr{F}^\theta f)(\omega)=\hat{f}^\theta(\omega)=\int_{\mathbb{R}} K^\theta(t,\omega)f(t)dt$$

where

$$K^\theta(t,\omega)=\begin{cases} C^\theta\ e^{i(t^2+\omega^2)\frac{\cot\theta}{2}-it\omega\csc\theta}, & \theta\neq n\pi,\\ \frac{1}{\sqrt{2\pi}}\ e^{-it\omega}, & \theta=\frac{\pi}{2},\\ \delta(t-\omega), & \theta=2n\pi,\\ \delta(t+\omega), & \theta=(2n+1)\pi,\ \ n\in\mathbb{Z},\end{cases}$$

δ denotes as the Dirac-delta function and $C^\theta=\sqrt{\frac{1-i\cot\theta}{2\pi}}$.

The inversion formula of FrFT is given by

$$f(t)=\int_{\mathbb{R}}\overline{K^\theta(t,\omega)}\ \hat{f}^\theta(\omega)d\omega,$$

where $\overline{K^\theta(t,\omega)}=K^{-\theta}(t,\omega)$.

Lemma 11 *(Parseval's Relation). If $\hat{f}^\theta$ and $\hat{\psi}^\theta$ are the FrFT of $f(t)$ and $\psi(t)$ respectively, then*

$$\int_{-\infty}^{\infty} f(t)\overline{\psi(t)}dt=\int_{-\infty}^{\infty}\hat{f}^\theta(\omega)\overline{\hat{\psi}^\theta(\omega)}d\omega, \tag{1.1}$$

Proof. See Pathak et al. [9].

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Definition 11 (Pathak [10]) *The Schwartz space $\mathscr{S}$ is the set of rapidly decreasing functions $\phi \in C^\infty(\mathbb{R})$ such that*

$$\gamma_{\alpha,\beta}(\phi) = \sup_{t\in\mathbb{R}} \left| t^\alpha D^\beta \phi(t) \right| < \infty, \ \forall \alpha, \beta \in \mathbb{N}_0. \tag{1.2}$$

If f be a locally integrable function on $\mathbb{R}$. Then f generates a distribution in $\mathscr{S}'$ as follows:

$$\langle f, \phi \rangle = \int_{\mathbb{R}} f(t)\phi(t)dt, \quad \phi \in \mathscr{S}.$$

The space $\mathscr{S}(\mathbb{R})$ is equipped with the topology generated by the collection of semi-norms $\{\gamma_{\alpha,\beta}\}$, it is a Fréchet space. The dual of $\mathscr{S}$ is denoted by $\mathscr{S}'$ and its elements are called tempered distributions.

Definition 12 (Pathak et al. [9]) *The test function space $\mathscr{S}_\theta$ is defined as: ϕ is member of $\mathscr{S}_\theta$ iff it is a complex valued infinitely differentiable function on $\mathbb{R}$, such that*

$$\Gamma^\theta_{\alpha,\beta}(\phi) = \sup_{t\in\mathbb{R}} \left| t^\alpha \ (\Delta_t^*)^\beta \phi(t) \right| < \infty, \ \forall \alpha, \beta \in \mathbb{N}_0, \tag{1.3}$$

where $\Delta_t^* = -\left(\frac{d}{dt} + it\cot\theta\right)$.

The FrWT was introduced first by Mendlovic *et al.* [11] as a way to deal with optical signals. Shi et al. [12] introduced the continuous affine transformation and chirp modulation of mother wavelet $\psi(t) \in L^2(\mathbb{R})$ as

$$\psi_{b,a,\theta}(t) = \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) e^{\frac{-i}{2}(t^2-b^2)\cot\theta},$$

for all $a > 0, \ b \in \mathbb{R}$ and θ as above and defined a novel FrWT of a square integrable function. If $\theta = \frac{\pi}{2}$, then $\psi_{b,a,\theta}(t)$ reduces to conventional mother wavelet defined in [10, 13–15].
Prasad et al. [16] defined the FrFT of $\psi_{b,a,\theta}(t)$ is given by

$$\hat{\psi}^\theta_{b,a,\theta}(\omega) = \sqrt{a}\, e^{\frac{i}{2}(b^2+\omega^2)\cot\theta - ib\omega\csc\theta - \frac{i}{2}a^2\omega^2\cot\theta} \ \mathscr{F}^\theta \left[e^{\frac{-i}{2}(.)^2\cot\theta} \ \psi \right](a\omega), \tag{1.4}$$

and established a generalized continuous fractional wavelet transform of a function $f \in L^2(\mathbb{R})$.
The continuous fractional convolution of two continuous functions $f(t), \psi(t) \in L^2(\mathbb{R})$ is defined as

$$(f \star_\theta \psi)(t) = \int_{-\infty}^{\infty} f(\xi)\psi(t-\xi)\, e^{\frac{-i}{2}(t^2-\xi^2)\cot\theta} d\xi, \tag{1.5}$$

where $\star_\theta$ is known as the continuous fractional convolution operator.

The wavelet transform associated with conventional Fourier transform was studied in [10, 14, 15] and corresponding FrWT involving FrFT was investigated in different way [11, 12, 16, 17]. The square integrable boundedness results for the wavelet transform and its adjoint were proved and established a relation between wavelet and Fourier transform [18]. In this paper continuity of FrWT in terms of fractional convolution operator and its adjoint are obtained. A relation between the FrWT and inverse FrFT is established. The fractional wavelet of a test function space is investigated.

2 Continuity of fractional wavelet transform

The continuous FrWT of $f \in L^2(\mathbb{R})$ w.r.t. the wavelet $\psi \in L^2(\mathbb{R})$ is defined [11, 12, 16] as:

$$\left(W^\theta_\psi f\right)(b,a) = \tilde{f}^\theta(b,a) = \int_{-\infty}^{\infty} f(t)\ \overline{\psi_{b,a,\theta}(t)} dt. \tag{2.1}$$

Using (1.1) of Parseval's identity and (1.4), it follows from (2.1) as:

$$\left(W^\theta_\psi f\right)(b,a) = \tilde{f}^\theta(b,a) = \sqrt{a}\int_{-\infty}^{\infty} e^{\frac{-i}{2}(b^2+\omega^2)\cot\theta + ib\omega\csc\theta + \frac{i}{2}a^2\omega^2\cot\theta}\, \overline{\mathscr{F}^\theta \left[e^{\frac{-i}{2}(.)^2\cot\theta} \psi \right]}(a\omega) \times \hat{f}^\theta(\omega)\, d\omega. \tag{2.2}$$

Definition 21 (Prasad et al. [16]) *A wavelet $\psi \in L^2(\mathbb{R})$ and satisfies the following admissibility condition*

$$C_{\psi,\theta} = \int_{-\infty}^{\infty} \frac{\left| \mathscr{F}^\theta \left(e^{\frac{-i}{2}(.)^2\cot\theta}\psi \right)(v) \right|^2}{v} dv < \infty, \tag{2.3}$$

where $\mathscr{F}^\theta$ denotes the FrFT operator.

Lemma 21 *The continuous fractional convolution transform of function $f \in L^2(\mathbb{R})$ w.r.t. the function $\psi \in L^2(\mathbb{R})$ at b depending on a, is denoted by $(W_b^\theta f)(a)$ and defined as*

$$(W_b^\theta f)(a) = (f \star_\theta \psi_a)(b). \tag{2.4}$$

Then

$$(W_\psi^\theta f)(b,a) = \sqrt{a}\,(W_b^\theta f)(a),$$

where $\psi_a(t) = \frac{1}{a}\overline{\psi}(-\frac{t}{a})$.

Proof. Using (1.5), we have

$$\begin{aligned}(W_b^\theta f)(a) &= \int_{-\infty}^{\infty} f(t)\ \psi_a(b-t)\, e^{\frac{-i}{2}(b^2-t^2)\cot\theta}\ dt\\
&= \int_{-\infty}^{\infty} f(t)\frac{1}{a}\ \overline{\psi}\left(\frac{t-b}{a}\right) e^{\frac{i}{2}(t^2-b^2)\cot\theta}\ dt\\
&= \frac{1}{\sqrt{a}}\int_{-\infty}^{\infty} f(t)\ \overline{\psi_{b,a,\theta}(t)}dt\\
&= \frac{1}{\sqrt{a}}\left(W_\psi^\theta f\right)(b,a).\end{aligned}$$

Remark 21 *Using (2.2), then from Lemma 21 we have*

$$(W_b^\theta f)(a) = \int_{-\infty}^{\infty} e^{\frac{-i}{2}(b^2+\omega^2)\cot\theta+ib\omega\csc\theta+\frac{i}{2}a^2\omega^2\cot\theta}\,\overline{\mathscr{F}^\theta\left[e^{\frac{-i}{2}(.)^2\cot\theta}\psi\right]}(a\omega)\hat{f}^\theta(\omega)\ d\omega. \tag{2.5}$$

As per [18], we have define the operator $(W_b^\theta)^*$ as:

Definition 22 *The operator $(W_b^\theta)^*$ associated with (W_b^θ) is defined by*

$$\begin{aligned}[(W_b^\theta)^* f](t) &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{a}}\ f(a)\ \overline{\psi_{b,a,\theta}(t)}da\\
&= \int_{-\infty}^{\infty}\frac{1}{a}\ f(a)\ \overline{\psi}\left(\frac{t-b}{a}\right) e^{\frac{i}{2}(t^2-b^2)\cot\theta}\ da.\end{aligned} \tag{2.6}$$

Lemma 22 *If $f,g \in L^2(\mathbb{R})$, then the operator $(W_b^\theta)^*$ is adjoint operator of (W_b^θ).*

Proof. Let $f,g \in L^2(\mathbb{R})$. Define

$$\begin{aligned}\langle W_b^\theta g, f\rangle &= \int_{-\infty}^{\infty} f(a)\ (W_b^\theta g)(a)da\\
&= \int_{-\infty}^{\infty} f(a)\ a^{-1}\int_{-\infty}^{\infty} g(t)\ \overline{\psi}\left(\frac{t-b}{a}\right) e^{\frac{i}{2}(t^2-b^2)\cot\theta}\ dt\ da,\end{aligned}$$

and

$$\begin{aligned}\langle g, (W_b^\theta)^* f\rangle &= \int_{-\infty}^{\infty} g(t)\ [(W_b^\theta)^* f](t)dt\\
&= \int_{-\infty}^{\infty} g(t)\ \int_{-\infty}^{\infty} a^{-1}\ f(a)\ \overline{\psi}\left(\frac{t-b}{a}\right) e^{\frac{i}{2}(t^2-b^2)\cot\theta}\ da\ dt\\
&= \int_{-\infty}^{\infty} f(a)\ a^{-1}\int_{-\infty}^{\infty} g(t)\ \overline{\psi}\left(\frac{t-b}{a}\right) e^{\frac{i}{2}(t^2-b^2)\cot\theta}\ dt\ da\\
&= \langle W_b^\theta g, f\rangle.\end{aligned}$$

This completes the proof of Lemma.

Lemma 23 *If fractional convolution operator W_b^θ is self-adjoint, then the generalized fractional wavelets $\overline{\psi}_{b,a,\theta}$ are given by*

$$\overline{\psi}_{b,a,\theta}(t) = \sqrt{a/t}\ \overline{\psi}_{b,t,\theta}(a), \qquad \forall\, t \in \mathbb{R}\setminus\{0\}. \tag{2.7}$$

Proof. From Lemma 21 and Definition 22, it follows that operator W_b^θ is self-adjoint if
$(W_b^\theta f)(a) = [(W_b^\theta)^* f](a)$.
Now

$$(W_b^\theta f)(a) = \frac{1}{\sqrt{a}}\left(W_\psi^\theta f\right)(b,a)$$
$$= \int_{-\infty}^{\infty} \frac{1}{a} f(t)\,\overline{\psi}\left(\frac{t-b}{a}\right) e^{\frac{i}{2}(t^2-b^2)\cot\theta}\,dt. \tag{2.8}$$

Similarly

$$[(W_b^\theta)^* f](a) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{t}}\, f(t)\, \overline{\psi_{b,t,\theta}(a)}dt$$
$$= \int_{-\infty}^{\infty} \frac{1}{t} f(t)\,\overline{\psi}\left(\frac{a-b}{t}\right) e^{\frac{i}{2}(a^2-b^2)\cot\theta}\,dt. \tag{2.9}$$

From (2.8) and (2.9), we have the desired result.

Remark 22 *If setting $x=(a-b)/t$ and $\beta = 1-b/a$, we find that operator W_b^θ will be self-adjoint for ψ defined by*

$$\psi(x) = (\beta/x)\ \psi(\beta/x - b/a)e^{\frac{i}{2}a^2[(\beta/x)^2-1]\cot\theta}, \quad x\in\mathbb{R}\setminus\{0\}. \tag{2.10}$$

This requires the ratio b/a to be constant. Further, (2.7) holds if $a=t$.

The following theorem yields the L^2-continuity of the operator W_b^θ and $(W_b^\theta)^*$.

Theorem 21 *If ψ satisfies the admissibility condition (2.3). Then adjoint $(W_b^\theta)^* \in L^2(\mathbb{R})$ and $||(W_b^\theta)^* f||_2 \le \sqrt{a\,C_{\psi,\theta}}\ ||f||_2$. Similarly, if W_b^θ is self-adjoint operator, then $||W_b^\theta f||_2 \le \sqrt{t\,C_{\psi,\theta}}\ ||f||_2$.*

Proof. First, we shall show that $(W_b^\theta)^* f \in L^2(\mathbb{R})$ and $||(W_b^\theta)^* f||_2 \le \sqrt{a\,C_{\psi,\theta}}\ ||f||_2$. For any $h\in L^2(\mathbb{R})$ using Schwartz inequality, we have

$$\int_{-\infty}^{\infty} |[(W_b^\theta)^* f](t) h(t)|dt \le \int_{-\infty}^{\infty} |h(t)|\ |[(W_b^\theta)^* f](t)|dt$$
$$\le \int_{-\infty}^{\infty} |h(t)| \left(\left|\int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} f(a)\ \overline{\psi_{b,a,\theta}(t)}da\right|\right) dt$$
$$\le \int_{-\infty}^{\infty} |f(a)| \left(\left|\int_{-\infty}^{\infty} \frac{1}{\sqrt{a}} h(t)\ \overline{\psi_{b,a,\theta}(t)}dt\right|\right) da$$
$$= \int_{-\infty}^{\infty} |f(a)||(W_b^\theta h)(a)|da$$
$$\le \left(\int_{-\infty}^{\infty} |f(a)|^2 da\right)^{1/2} \left(\int_{-\infty}^{\infty} |(W_b^\theta h)(a)|^2 da\right)^{1/2}$$
$$\le \left(\int_{-\infty}^{\infty} |f(a)|^2 da\right)^{1/2} \left(\int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} e^{\frac{-i}{2}(b^2+\omega^2)\cot\theta + ib\omega\csc\theta + \frac{i}{2}a^2\omega^2\cot\theta}\right.\right.$$
$$\left.\left.\times\ \overline{\mathscr{F}^\theta\left[e^{\frac{-i}{2}(.)^2\cot\theta}\psi\right]}(a\omega)\hat{h}^\theta(\omega)\,d\omega\right|^2 da\right)^{1/2}$$
$$\le ||f||_2 \left(\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} |\mathscr{F}^\theta\left[e^{\frac{-i}{2}(.)^2\cot\theta}\psi\right](a\omega)|^2 da\right] |\hat{h}^\theta(\omega)|^2\,d\omega\right)^{1/2}$$
$$\le \sqrt{a\,C_{\psi,\theta}}\ ||f||_2 \left(\int_{-\infty}^{\infty} |\hat{h}^\theta(\omega)|^2\,d\omega\right)^{1/2}.$$

Now, using Parseval's relation, we have
$||[(W_b^\theta)^* f]\ h||_1 \le \sqrt{a\,C_{\psi,\theta}}\ ||h||_2||f||_2$.
By converse of Schwartz inequality [19, p. 385], if $h\in L^2(\mathbb{R})$ and $||[(W_b^\theta)^* f]\ h||_1 \le \sqrt{a\,C_{\psi,\theta}}\ ||h||_2||f||_2$, then we get desired result.
Anologously, we can prove the second part.

3 Relation between the fractional wavelet transform and inverse fractional Fourier transform

Let a measurable function ψ defined on $\mathbb{R}$ which satisfies

$$\int_{-\infty}^{\infty} |\psi(v)||v|^{-1/2}dv = A < \infty, \tag{3.1}$$

instead of the admissibility condition (2.3).

Theorem 31 *If* $E^{\theta}_{b,a} = e^{\frac{-i}{2}a^2\cot\theta+iab\csc\theta}$, $\psi \in L^1(\mathbb{R})$ *and* $\int_{-\infty}^{\infty}|\psi(v)||v|^{-1/2}dv < \infty$. *Then*

$$E^{\theta}_{b,a}W^{\theta}_{b}\mathscr{F}^{-\theta} = sign(a)\mathscr{F}^{-\theta}[(W^{\theta}_{b})^{*}E^{\theta}_{b,a}],$$

where

$$sign(a) = \begin{cases} 1, & a > 0 \\ -1, & a < 0. \end{cases}$$

Proof. Let $f \in L^2(\mathbb{R})$ and $g = \mathscr{F}^{-\theta}\hat{f}^{\theta}$. Then

$$\begin{aligned}
(W^{\theta}_{b}g)(a) &= |a|^{-1}\int_{-\infty}^{\infty}\overline{\psi}\left(\frac{y-b}{a}\right)e^{\frac{i}{2}(y^2-b^2)\cot\theta}\, g(y)dy \\
&= |a|^{-1}\int_{-\infty}^{\infty}\overline{\psi}\left(\frac{y-b}{a}\right)e^{\frac{i}{2}(y^2-b^2)\cot\theta}\left(\int_{-\infty}^{\infty}\overline{K^{\theta}(y,t)}\;\hat{f}^{\theta}(t)dt\right)dy \\
&= \frac{a}{|a|}\int_{-\infty}^{\infty}\overline{\psi}(v)\, e^{\frac{i}{2}[(av+b)^2-b^2]\cot\theta}\left(\int_{-\infty}^{\infty}\overline{K^{\theta}(av+b,t)}\;\hat{f}^{\theta}(t)dt\right)dv \\
&= sign(a)\int_{-\infty}^{\infty}\overline{K^{\theta}(b,t)}\hat{f}^{\theta}(t)\left(\int_{-\infty}^{\infty}e^{itav\csc\theta}\;\overline{\psi}(v)dv\right)dt \\
&= sign(a)\int_{-\infty}^{\infty}\overline{K^{\theta}(b,t)}\hat{f}^{\theta}(t)t^{-1}\left(\int_{-\infty}^{\infty}e^{iau\csc\theta}\;\overline{\psi}(u/t)du\right)dt \\
&= sign(a)\, e^{\frac{i}{2}a^2\cot\theta-iab\csc\theta}\int_{-\infty}^{\infty}\overline{K^{\theta}(x,a)}\Big(\int_{-\infty}^{\infty}\frac{1}{t}\overline{\psi}\left(\frac{x-b}{t}\right)e^{\frac{i}{2}(x^2-b^2)\cot\theta} \\
&\quad\times\hat{f}^{\theta}(t)e^{\frac{-i}{2}t^2\cot\theta+itb\csc\theta}dt\Big)dx
\end{aligned}$$

$$= sign(a)\, e^{\frac{i}{2}a^2\cot\theta-iab\csc\theta}\int_{-\infty}^{\infty}\overline{K^{\theta}(x,a)}\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{t}}\overline{\psi_{b,t,\theta}(x)}\hat{f}^{\theta}(t)e^{\frac{-i}{2}t^2\cot\theta+itb\csc\theta}dt\right)dx. \tag{3.2}$$

Therefore

$$(W^{\theta}_{b}\mathscr{F}^{-\theta}\hat{f}^{\theta})(a) = sign(a)\, e^{\frac{i}{2}a^2\cot\theta-iab\csc\theta}\int_{-\infty}^{\infty}\overline{K^{\theta}(x,a)}[(W^{\theta}_{b})^{*}\hat{f}^{\theta}(t)e^{\frac{-i}{2}t^2\cot\theta+itb\csc\theta}](x)dx,$$

or

$$e^{\frac{-i}{2}a^2\cot\theta+iab\csc\theta}W^{\theta}_{b}\mathscr{F}^{-\theta}\hat{f}^{\theta} = sign(a)\mathscr{F}^{-\theta}[(W^{\theta}_{b})^{*}\hat{f}^{\theta}(t)e^{\frac{-i}{2}t^2\cot\theta+itb\csc\theta}].$$

Hence

$$E^{\theta}_{b,a}W^{\theta}_{b}\mathscr{F}^{-\theta} = sign(a)\mathscr{F}^{-\theta}[(W^{\theta}_{b})^{*}E^{\theta}_{b,a}].$$

Theorem 32 *Let* ψ *be a periodic function with period* b/a *defined on* $[-\sigma,\sigma]$, *where* $\sigma = max(1,|\beta|)$, *and satisfies*

$$\int_{|x|\le 1}|\psi(x)||x|^{-1/2}dx < \infty.$$

Define $\psi(x) = (\beta/x)\;\psi(\beta/x - b/a)e^{\frac{i}{2}a^2[(\beta/x)^2-1]\cot\theta}$ *for* $|x| > \sigma$. *Then*

$$E^{\theta}_{b,a}W^{\theta}_{b}\mathscr{F}^{-\theta} = sign(a)\mathscr{F}^{-\theta}[W^{\theta}_{b}E^{\theta}_{b,a}].$$

Proof. Using (2.10) for $|\beta|<1$, we have

$$\int_{|v|\geq 1}|\psi(v)||v|^{-1/2}dv = \int_{|v|\geq 1}|\psi(\beta/v-b/a)e^{\frac{i}{2}a^2[(\beta/v)^2-1]\cot\theta}||\beta/v||v|^{-1/2}dv$$
$$= |\beta|^{1/2}\int_{|x|\leq|\beta|}\left|\psi\left(\frac{x-b}{a}\right)\right||x|^{-1/2}dx$$
$$\leq |\beta|^{1/2}\int_{|x|\leq 1}|\psi(x)||x|^{-1/2}dx<\infty.$$

Now again, using (2.10) for $|\beta|\geq 1$, we have

$$\int_{|t|\geq|\beta|}|\psi(t)||t|^{-1/2}dt = \int_{|t|\geq|\beta|}|\psi(\beta/t-b/a)e^{\frac{i}{2}a^2[(\beta/t)^2-1]\cot\theta}||\beta/t||t|^{-1/2}dt$$
$$= |\beta|^{1/2}\int_{|x|\leq 1}\left|\psi\left(\frac{x-b}{a}\right)\right||x|^{-1/2}dx$$
$$\leq |\beta|^{1/2}\int_{|x|\leq 1}|\psi(x)||x|^{-1/2}dx<\infty.$$

Since W_b^θ is self-adjoint, so by using (3.2) and (2.7), we get

$$(W_b^\theta g)(a) = sign(a)\, e^{\frac{i}{2}a^2\cot\theta-iab\csc\theta}\int_{-\infty}^{\infty}\overline{K^\theta(x,a)}\left(\int_{-\infty}^{\infty}\frac{1}{\sqrt{x}}\overline{\psi_{b,x,\theta}(t)}\hat{f}^\theta(t)e^{\frac{-i}{2}t^2\cot\theta+itb\csc\theta}dt\right)dx$$
$$= sign(a)\, e^{\frac{i}{2}a^2\cot\theta-iab\csc\theta}\int_{-\infty}^{\infty}\overline{K^\theta(x,a)}[W_b^\theta\hat{f}^\theta(t)e^{\frac{-i}{2}t^2\cot\theta+itb\csc\theta}](x)dx.$$

Therefore

$$e^{\frac{-i}{2}a^2\cot\theta+iab\csc\theta}W_b^\theta\mathscr{F}^{-\theta}\hat{f}^\theta = sign(a)\mathscr{F}^{-\theta}[W_b^\theta\hat{f}^\theta(t)e^{\frac{-i}{2}t^2\cot\theta+itb\csc\theta}].$$

Hence

$$E_{b,a}^\theta W_b^\theta\mathscr{F}^{-\theta} = sign(a)\mathscr{F}^{-\theta}[W_b^\theta E_{b,a}^\theta].$$

4 Fractional wavelet transform of generalized functions

As per [18, 20], we define the fractional wavelet transform of $\psi,\phi\in\mathscr{S}_\theta(\mathbb{R})$ by

$$\left(W_\psi^\theta f\right)(b,a) = \int_{-\infty}^{\infty}f(t)\,\frac{1}{\sqrt{a}}\,\overline{\psi}\left(\frac{t-b}{a}\right)e^{\frac{i}{2}(t^2-b^2)\cot\theta}dt.$$

Definition 41 (Pathak [18]) *A complex valued smooth function $\eta(b,a)$ belongs to the space $B(\mathbb{R}^2)$ if and only if*

$$\gamma_{m,k}^{l,n}(\eta) = \sup_{(b,a)\in\mathbb{R}^2}\left|\left(\frac{b}{1+|a|}\right)^l a^n\left(\frac{\partial}{\partial b}\right)^{m+n}\left(a\frac{\partial}{\partial a}+1\right)^k\frac{1}{\sqrt{a}}\eta(b,a)\right|<\infty, \tag{4.1}$$

for all $l,n,m,k\in\mathbb{N}_0$.

Lemma 41 *Let $\psi_{b,a,\theta}$ be wavelet. Then*

$$(\Delta_t)^m\left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right) = \left(\frac{d}{dt}\right)^m\left[\frac{1}{\sqrt{a}}\overline{\psi}_{b,a}(t)\right]e^{\frac{i}{2}(t^2-b^2)\cot\theta},\quad\forall\, m\in\mathbb{N}_0,$$

where $\Delta_t = \left(\frac{d}{dt}-it\cot\theta\right)$.

Proof. The proof is very easy and left to the eager reader.

Lemma 42 *If $f(t),\ \phi(t)\in\mathscr{S}_\theta(\mathbb{R})$. Then*

$$(\Delta_t^*)^k[f(t)\,\phi(t)] = \sum_{r=0}^{k}A_{k,r}\,\frac{\partial^r f}{\partial t^r}\,(\Delta_t^*)^{k-r}\phi(t),\qquad k\in\mathbb{N}_0,$$

where Δ_t^* is the same defined in (1.3) and $A_{k,r}$ are constants.

Proof. Since $f,\phi \in \mathscr{S}_\theta(\mathbb{R})$. Then

$$\begin{aligned}(\Delta_t^*)[f(t)\ \phi(t)] &= -\left(\frac{\partial}{\partial t} + it\cot\theta\right) f(t)\phi(t)\\ &= -\frac{\partial f}{\partial t}\phi(t) - f(t)\frac{\partial \phi}{\partial t} - it\cot\theta\, f(t)\phi(t)\\ &= -\frac{\partial f}{\partial t}\phi(t) - f(t)\left(\frac{\partial}{\partial t} + it\cot\theta\right)\phi(t)\\ &= -\frac{\partial f}{\partial t}\phi(t) + f(t)(\Delta_t)^*\phi(t)\\ &= \sum_{r=0}^{1} A_{1,r}\ \frac{\partial^r f}{\partial t^r}\ (\Delta_t^*)^{1-r}\phi(t).\end{aligned}$$

Similarly,

$$\begin{aligned}(\Delta_t^*)^2[f(t)\ \phi(t)] &= \Delta_t^*[\Delta_t^*(f(t).\phi(t))]\\ &= \Delta_t^*\left[-\frac{\partial f}{\partial t}\phi(t) + f(t)\Delta_t^*\phi(t)\right]\\ &= -\left(\frac{\partial}{\partial t} + it\cot\theta\right)\left[-\frac{\partial f}{\partial t}\phi(t)\right] - \left(\frac{\partial}{\partial t} + it\cot\theta\right)[f(t).\Delta_t^*\phi(t)]\\ &= \frac{\partial^2 f}{\partial t^2}\phi(t) - 2\frac{\partial f}{\partial t}\Delta_t^*\phi(t) + f(t)(\Delta_t^*)^2\phi(t)\\ &= \sum_{r=0}^{2} A_{2,r}\ \frac{\partial^r f}{\partial t^r}\ (\Delta_t^*)^{2-r}\phi(t).\end{aligned}$$

In general, for $k \in \mathbb{N}_0$, we get

$$(\Delta_t^*)^k[f(t)\ \phi(t)] = \sum_{r=0}^{k} A_{k,r}\ \frac{\partial^r f}{\partial t^r}\ (\Delta_t^*)^{k-r}\phi(t).$$

Remark 41 *If $f(t),\phi(t) \in \mathscr{S}_\theta(\mathbb{R})$. Then*

$$\Delta_t^k[f(t)\phi(t)] = \sum_{r=0}^{k} B_{k,r}\ \frac{\partial^r f}{\partial t^r}\ \Delta_t^{k-r}\phi(t), \qquad k \in \mathbb{N}_0, \tag{4.2}$$

where Δ_t is the same as defined in Lemma 41 and $B_{k,r}$ are constants.

Lemma 43 *If wavelet $\psi_{b,a,\theta}(t)$ is differentiable. Then*

$$\left(a\frac{\partial}{\partial a} + 1\right)^k \left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right) = (b-t)^k \Delta_t^k \left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right), \qquad k \in \mathbb{N}_0,$$

where Δ_t is the same as defined in Lemma 41.

Proof. We have

$$\begin{aligned}\left(a\frac{\partial}{\partial a} + 1\right)\left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right) &= \left(a\frac{\partial}{\partial a} + 1\right)\frac{1}{a}\overline{\psi}\left(\frac{t-b}{a}\right)\ e^{\frac{i}{2}(t^2-b^2)\cot\theta}\\ &= a\left[\frac{-1}{a^2}\overline{\psi}\left(\frac{t-b}{a}\right)\ e^{\frac{i}{2}(t^2-b^2)\cot\theta} - \frac{(t-b)}{a^3}\overline{\psi}'\left(\frac{t-b}{a}\right)\ e^{\frac{i}{2}(t^2-b^2)\cot\theta}\right]\\ &\quad + \frac{1}{a}\overline{\psi}\left(\frac{t-b}{a}\right)\ e^{\frac{i}{2}(t^2-b^2)\cot\theta}\\ &= \frac{(b-t)}{a^2}\overline{\psi}'\left(\frac{t-b}{a}\right)\ e^{\frac{i}{2}(t^2-b^2)\cot\theta}\\ &= (b-t)\Delta_t\left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right).\end{aligned}$$

Similarly,

$$\begin{aligned}\left(a\frac{\partial}{\partial a}+1\right)^2\left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right) &= \left(a\frac{\partial}{\partial a}+1\right)\left(a\frac{\partial}{\partial a}+1\right)\left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right)\\ &= \left(a\frac{\partial}{\partial a}+1\right)\left[(b-t)\Delta_t\left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right)\right]\\ &= (b-t)\Delta_t\Big[a\Big(\frac{-1}{a^2}\overline{\psi}\Big(\frac{t-b}{a}\Big)\ e^{\frac{i}{2}(t^2-b^2)\cot\theta}\\ &\quad -\frac{(t-b)}{a^3}\overline{\psi}'\left(\frac{t-b}{a}\right)\ e^{\frac{i}{2}(t^2-b^2)\cot\theta}\Big)+\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\Big]\\ &= (b-t)^2\Delta_t^2\left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right).\end{aligned}$$

In general, by induction on $k\in\mathbb{N}_0$, we get

$$\left(a\frac{\partial}{\partial a}+1\right)^k\left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right)=(b-t)^k\Delta_t^k\left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right).$$

Remark 42 *If $\psi,\phi\in\mathscr{S}_\theta(\mathbb{R})$ and $k\in\mathbb{N}_0$. Then*

$$\int_{-\infty}^{\infty}(b-t)^k\phi(t)\Delta_t^k\left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right)dt=\int_{-\infty}^{\infty}(\Delta_t^*)^k[\phi(t)(b-t)^k]\ \frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}dt. \tag{4.3}$$

Theorem 41 *If mother wavelet $\psi\in\mathscr{S}_\theta(\mathbb{R})$. Then the FrWT W_ψ^θ is a continuous linear mapping of $\mathscr{S}_\theta(\mathbb{R})$ into $B(\mathbb{R}^2)$.*

Proof. Using Lemma 41, 42 and 43 and integrating by parts, we have

$$\begin{aligned}\left(a\frac{\partial}{\partial a}+1\right)^k\frac{1}{\sqrt{a}}(W_\psi^\theta\phi)(b,a) &= \left(a\frac{\partial}{\partial a}+1\right)^k\int_{-\infty}^{\infty}\phi(t)\ \frac{1}{a}\ \overline{\psi}\left(\frac{t-b}{a}\right)\ e^{\frac{i}{2}(t^2-b^2)\cot\theta}dt\\ &= \int_{-\infty}^{\infty}\phi(t)\ (b-t)^k\Delta_t^k\left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right)dt\\ &= \int_{-\infty}^{\infty}(-1)^k(\Delta_t^*)^k[\phi(t)(t-b)^k]\ \frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}dt.\end{aligned}$$

Now

$$\begin{aligned}\left(\frac{\partial}{\partial b}\right)^m\left(a\frac{\partial}{\partial a}+1\right)^k\frac{1}{\sqrt{a}}(W_\psi^\theta\phi)(b,a) &= \int_{-\infty}^{\infty}(-1)^{m+k}(\Delta_t^*)^k[\phi(t)(t-b)^k]\\ &\quad\times\left(\frac{\partial}{\partial t}\right)^m\left[\frac{1}{\sqrt{a}}\overline{\psi}_{b,a}(t)\right]e^{\frac{i}{2}(t^2-b^2)\cot\theta}dt\\ &= \int_{-\infty}^{\infty}(-1)^{m+k}(\Delta_t^*)^k[\phi(t)(t-b)^k]\ \Delta_t^m\left(\frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}\right)dt\\ &= \int_{-\infty}^{\infty}(-1)^{m+k}(\Delta_t^*)^{m+k}[\phi(t)(t-b)^k]\ \frac{\overline{\psi}_{b,a,\theta}(t)}{\sqrt{a}}dt.\end{aligned}$$

Similarly

$$\begin{aligned}&\left(\frac{\partial}{\partial b}\right)^{m+n}\left(a\frac{\partial}{\partial a}+1\right)^k\frac{1}{\sqrt{a}}(W_\psi^\theta\phi)(b,a)\\ &=\int_{-\infty}^{\infty}(-1)^{m+n+k}(\Delta_t^*)^{m+k}[\phi(t)(t-b)^k]\left(\frac{\partial}{\partial t}\right)^n\left[\frac{1}{\sqrt{a}}\overline{\psi}_{b,a}(t)\right]e^{\frac{i}{2}(t^2-b^2)\cot\theta}dt.\end{aligned}$$

Now, using the inequality

$$|b|^l\le(|t-b|+|t|)^l\le|a|^l2^l\left(\left|\frac{b-t}{a}\right|^l+|t/a|^l\right),\qquad l>0,$$

we can write

$$\left| b^l a^n \left(\frac{\partial}{\partial b}\right)^{m+n} \left(a\frac{\partial}{\partial a}+1\right)^k \frac{1}{\sqrt{a}} (W_\psi^\theta \phi)(b,a) \right|$$
$$\leq (2|a|)^l \int_{-\infty}^{\infty} \left[\left|\frac{b-t}{a}\right|^l + |t/a|^l] \; (\Delta_t^*)^{m+k} [\phi(t)(t-b)^k \right] \left| \frac{1}{a} \overline{\psi}^{(n)} \left(\frac{t-b}{a}\right) e^{\frac{i}{2}(t^2-b^2)\cot\theta} \right| dt$$
$$\leq (2|a|)^l \int_{-\infty}^{\infty} \left|\frac{b-t}{a}\right|^l \sum_{r=0}^{k} A_{k,m,r} \; (t-b)^{k-r} \; (\Delta_t^*)^{m+k-r} \phi(t) \left| \frac{1}{a} \overline{\psi}^{(n)} \left(\frac{t-b}{a}\right) \right| dt$$
$$+(2|a|)^l \int_{-\infty}^{\infty} |t/a|^l \sum_{r=0}^{k} A_{k,m,r} \; (t-b)^{k-r} \; (\Delta_t^*)^{m+k-r} \phi(t) \left| \frac{1}{a} \overline{\psi}^{(n)} \left(\frac{t-b}{a}\right) \right| dt$$
$$\leq \sum_{r=0}^{k} A_{k,m,r} \; (2|a|)^l \sup_{t\in\mathbb{R}} |(t-b)^{k-r} \; (\Delta_t^*)^{m+k-r} \phi(t)| \int_{-\infty}^{\infty} \left|\frac{t-b}{a}\right|^l \frac{1}{a} \left| \overline{\psi}^{(n)} \left(\frac{t-b}{a}\right) \right| dt$$
$$+\sum_{r=0}^{k} A_{k,m,r} \; 2^l \sup_{t\in\mathbb{R}} |t^l (t-b)^{k-r} \; (\Delta_t^*)^{m+k-r} \phi(t)| \int_{-\infty}^{\infty} \frac{1}{a} \left| \overline{\psi}^{(n)} \left(\frac{t-b}{a}\right) \right| dt$$
$$\leq \sum_{r=0}^{k} A_{k,m,r} \; 2^l \Big[(1+|a|)^l \sup_{t\in\mathbb{R}} |(t-b)^{k-r} \; (\Delta_t^*)^{m+k-r} \phi(t)| \int_{-\infty}^{\infty} |u^l \; \overline{\psi}^{(n)}(u)| du$$
$$+\sup_{t\in\mathbb{R}} |t^l (t-b)^{k-r} \; (\Delta_t^*)^{m+k-r} \phi(t)| \int_{-\infty}^{\infty} |\overline{\psi}^{(n)}(u)| du \Big].$$

Therefore

$$\sup_{(b,a)\in\mathbb{R}^2} \left| \left(\frac{b}{1+|a|}\right)^l a^n \left(\frac{\partial}{\partial b}\right)^{m+n} \left(a\frac{\partial}{\partial a}+1\right)^k \frac{1}{\sqrt{a}} (W_\psi^\theta \phi)(b,a) \right|$$
$$\leq \sum_{r=0}^{k} A_{k,m,r} \; 2^l \Big[\Gamma_{k-r,m+k-r}(\phi) \int_{-\infty}^{\infty} |u^l \; \overline{\psi}^{(n)}(u)| du + \Gamma_{l,k-r,m+k-r}(\phi) \int_{-\infty}^{\infty} |\overline{\psi}^{(n)}(u)| du \Big]$$
$$\leq \sum_{r=0}^{k} A_{k,m,r} \; 2^l \Big[\Gamma_{k-r,m+k-r}(\phi) \sup_{u\in\mathbb{R}} |(1+u^2) \; u^l \; D^n \psi(u)| \int_{-\infty}^{\infty} \frac{1}{1+u^2} du$$
$$+\Gamma_{l,k-r,m+k-r}(\phi) \sup_{u\in\mathbb{R}} |(1+u^2) \; D^n \psi(u)| \int_{-\infty}^{\infty} \frac{1}{1+u^2} du \Big],$$

where $A_{k,m,r} > 0$. Since $\phi, \psi \in \mathscr{S}_\theta(\mathbb{R})$ each term of the right-hand side is convergent. Hence, W_ψ^θ is a continuous linear mapping of $\mathscr{S}_\theta(\mathbb{R})$ into $B(\mathbb{R}^2)$.

5 Examples

In this section we shall illustrate, by means of some examples, the advantages of the fractional wavelet transform in terms of fractional convolution as:

If $\psi(t) = e^{\frac{i}{2}t^2\cot\theta}$ and $\phi(t) = \frac{1}{\sqrt{2\pi}} t^2 e^{-\frac{t^2}{2}(1+2i)\cot\theta}$. Then

$$(\phi \star_\theta \psi)(t) = \int_0^{\infty} \phi(\xi)\psi(t-\xi) e^{-\frac{i}{2}(t^2-\xi^2)\cot\theta} d\xi$$
$$= \tan^{\frac{3}{2}}\theta \; (1-t^2\cot\theta) e^{-\frac{t^2}{2}\cot\theta},$$

which is fractional Mexican hat wavelet. If $\theta = \pi/4$, then it reduce to the conventional Mexican hat wavelet.

Similarly if $\psi(t) = e^{\frac{i}{2}t^2\cot\theta}$ and $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-\left[\frac{1}{2}(t+\xi_0)^2+it^2\right]\cot\theta}$. Then

$$(\phi \star_\theta \psi)(t) = \tan^{\frac{1}{2}}\theta \; e^{\left[i\xi_0 t - \frac{t^2}{2}\right]\cot\theta},$$

which is fractional Morlet wavelet. If $\theta = \pi/4$, then it reduce to the conventional Morlet wavelet.

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