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Numerical Solutions of the Sine-Gordon Equation by Collocation Method

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Abstract: In the present study, a B-spline collocation method has been applied to obtain a numerical solution of the sine-Gordon equation. Then, the obtained numerical results have been compared with those given in the literature. The error norms L_2 and L_∞ are computed and they have been found out small enough to be accepted.

Keywords: sine-Gordon equation, Finite element method, Collocation method, B-Spline

1 Introduction

The sine-Gordon equation is a nonlinear hyperbolic partial difefrential equation appearing in a number of physical applications such as relativistic field theory, Josephson junctons, mechanical transmission lines and modern physics. Hence it is of great interest for many scientists and mathematicians. Therefore, its analytical and numerical solutions are found by many authors using various methods. In this paper, we will deal with the sine-Gordon equation given in the form

$$u_{tt} = u_{xx} - \sin u, \qquad x \in (L_0, L_1), \qquad t > 0 \qquad (1)$$

where u is the dependent variable, and t and x are the independent time and space parameters, respectively. In the present study, the numerical solutions of Eq. (1) will be sought with the following boundary conditions

$$\begin{aligned} u(L_0,t) &= f_0(t), \\ u(L_1,t) &= f_1(t), \end{aligned} \quad t \geq 0.$$

The initial conditions will be taken as follows

$$\begin{aligned} u(x,0) &= \phi(x), \\ u_t(x,0) &= \psi(x), \end{aligned} \quad x \in [L_0, L_1].$$

The main purpose of this study is to apply the cubic B-spline collocation finite element method to develop a numerical technique for solving the sine-Gordon equation. Eq.(1) has been solved by several authors using various methods and techniques. For example, Rashidinia and Mohammadi [1] have developed two implicit finite difference schemes for the numerical solution of one-dimensional sine-Gordon equation by using spline function approximation and also given stability analysis of the method. Dehghan and Shokri [2] have proposed a numerical scheme to solve the one-dimensional undamped Sine-Gordon equation using collocation points and approximating the solution using Thin Plate Splines(TPS) radial basis function(RBF). Rigge [3] has presented several numerical solutions to the 1D, 2D, and 3D sine-Gordon equation and given comments on the nature of the solutions. Sheng et al.[4] have concerned an adaptive splitting scheme for the numerical solution of two dimensional sine-Gordon equation. Uddin et al [5] have proposed a numerical method based on radial basis functions for the numerical solution of nonlinear sine-Gordon equation. Keskin et al. [6] have implemented reduced differential transform method (RDTM), which does not need small parameter in the equation, for solving the sine-Gordon equation. Kaya [7] has implemented the decomposition method for solving the sine-Gordon equation by using a number of initial values in the form of convergent power series with easily computable components. Soori and Aminataei [8] have applied the spectral method with a basis of a new orthogonal polynomial which is orthogonal over the interval [0,1] with weighting function one. Akgül and Inc [9] have proposed a reproducing kernel Hilbert space

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method(RKHSM) for solving the sine-Gordon equation with initial and boundary conditions based on the reproducing kernel theory. In this paper, first of all, we will deal with the sine-Gordon equation (1) in terms of a system composed of two equations

$$u_t = v, \qquad v_t = u_{xx} - \sin u \tag{2}$$

to obtain the numerical solution of the sine-Gordon equation. The performance of the method has been tested on numerical examples.

2 The Finite Element Solution

Let's assume that the interval $[a,b]$ is divided into N finite elements having uniform equal length by the knots $x_m,\ m = 0(1)N$ such that $a = x_0 < x_1 \cdots < x_{N-1} < x_N = b$ and $h = x_{m+1} - x_m$. The cubic B-splines $\phi_m(x)$, $(i = -1(1)N+1)$ at the knots x_m are defined over the interval $[a,b]$ as [10]

$$\phi_m(x) = \frac{1}{h^3}\begin{cases} (x-x_{m-2})^3, & x \in [x_{m-2}, x_{m-1}], \\ h^3 + 3h^2(x-x_{m-1}) + 3h(x-x_{m-1})^2 - 3(x-x_{m-1})^3, & x \in [x_{m-1}, x_m], \\ h^3 + 3h^2(x_{m+1}-x) + 3h(x_{m+1}-x)^2 - 3(x_{m+1}-x)^3, & x \in [x_m, x_{m+1}], \\ (x_{m+2}-x)^3, & x \in [x_{m+1}, x_{m+2}], \\ 0, & otherwise. \end{cases}$$

The set of cubic B-splines $\{\phi_{-1}(x), \phi_0(x), \ldots, \phi_{N+1}(x)\}$ constitutes a basis for the functions to be defined over the interval $[a,b]$. Thus, an approximation solution $U_N(x,t)$ to analytical solution $U(x,t)$ and an approximation solution $V_N(x,t)$ to analytical solution $V(x,t)$ on this interval can be written in terms of these cubic B- splines as

$$U_N(x,t) = \sum_{m=-1}^{N+1} \delta_m(t)\phi_m(x), \tag{3}$$

$$V_N(x,t) = \sum_{m=-1}^{N+1} \sigma_m(t)\phi_m(x) \tag{4}$$

in which $\delta_m(t)$'s and $\sigma_m(t)$'s are unknown time dependent element parameters to be determined. Because of the fact that each cubic B-spline covers four elements, on the other hand, each element $[x_m, x_{m+1}]$ is covered by four cubic B-splines. In this paper, the finite elements are identified with the interval $[x_m, x_{m+1}]$ and the elements knots x_m and x_{m+1}. In terms of the local coordinate transformation $\xi = x - x_m$, the cubic B-splines can now be expressed in terms of the local variable ξ as follows

$$\begin{matrix} \phi_{m-1} \\ \phi_m \\ \phi_{m+1} \\ \phi_{m+2} \end{matrix} = \frac{1}{h^3}\begin{cases} (h-\xi)^3, \\ h^3 + 3h^2(h-\xi) + 3h(h-\xi)^2 - 3(h-\xi)^3, \\ h^3 + 3h^2\xi + 3h\xi^2 - 3\xi^3, \\ \xi^3, \end{cases} \tag{5}$$

where $0 \le \xi \le h$. Since all other cubic B-splines are identically zero over the element $[x_m, x_{m+1}]$, the variations of $U_N(x,t)$ and $V_N(x,t)$ in Eqs. (3)-(4) over a typical element $[x_m, x_{m+1}]$ is written as

$$U_N(\xi,t) = \sum_{j=m-1}^{m+2} \delta_j(t)\phi_j(\xi) \tag{6}$$

and

$$V_N(\xi,t) = \sum_{j=m-1}^{m+2} \sigma_j(t)\phi_j(\xi). \tag{7}$$

If we use the Eqs. (5) and (6), then the nodal values of U_m and U_m'' at the knots $x = x_m$ can be easily found in terms of element parameter δ_m as follows

$$\begin{aligned} U_m &= U(x_m) = \delta_{m-1} + 4\delta_m + \delta_{m+1}, \\ U_m'' &= U''(x_m) = \tfrac{6}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}). \end{aligned} \tag{8}$$

If we use the Eqs. (5) and (7), then the nodal value of V_m at the knots $x = x_m$ can be easily found in terms of element parameter σ_m as follows

$$V_m = V(x_m) = \sigma_{m-1} + 4\sigma_m + \sigma_{m+1}. \tag{9}$$

If we put the nodal values given by Eqs. (8)-(9) into Eq. (1), we obtain the following systems of equations:

$$\dot{\delta}_{m-1} + 4\dot{\delta}_m + \dot{\delta}_{m+1} = \sigma_{m-1} + 4\sigma_m + \sigma_{m+1} \tag{10}$$

and

$$\dot{\sigma}_{m-1} + 4\dot{\sigma}_m + \dot{\sigma}_{m+1} = \frac{6}{h^2}(\delta_{m-1} - 2\delta_m + \delta_{m+1}) - \sin\hat{u}. \tag{11}$$

In Eq. (10), if we take the

$$\dot{\delta} = \frac{\delta^{n+1} - \delta^n}{\Delta t},$$

$$\sigma = \frac{\sigma^{n+1} + \sigma^n}{2}$$

and put them in their places, we obtain

$$\begin{aligned} &\frac{1}{\Delta t}\delta_{m-1}^{n+1} + \frac{4}{\Delta t}\delta_m^{n+1} + \frac{1}{\Delta t}\delta_{m+1}^{n+1} - \frac{1}{2}\sigma_{m-1}^{n+1} - 2\sigma_m^{n+1} - \frac{1}{2}\sigma_{m+1}^{n+1} = \\ &\frac{1}{\Delta t}\delta_{m-1}^{n} + \frac{4}{\Delta t}\delta_m^{n} + \frac{1}{\Delta t}\delta_{m+1}^{n} + \frac{1}{2}\sigma_{m-1}^{n} + 2\sigma_m^{n} + \frac{1}{2}\sigma_{m+1}^{n} \end{aligned} \tag{12}$$

where $m = 0(1)M$. In Eq. (11), if we take the

$$\dot{\sigma} = \frac{\sigma^{n+1} - \sigma^n}{\Delta t},$$

$$\delta = \frac{\delta^{n+1} + \delta^n}{2}$$

and put them in their places, we obtain

$$\begin{aligned} &-\frac{3}{h^2}\delta_{m-1}^{n+1} + \frac{6}{h^2}\delta_m^{n+1} - \frac{3}{h^2}\delta_{m+1}^{n+1} + \frac{1}{\Delta t}\sigma_{m-1}^{n+1} + \frac{4}{\Delta t}\sigma_m^{n+1} \\ &+\frac{1}{\Delta t}\sigma_{m+1}^{n+1} = \frac{3}{h^2}\delta_{m-1}^{n} - \frac{6}{h^2}\delta_m^{n} + \frac{3}{h^2}\delta_{m+1}^{n} \\ &+\frac{1}{\Delta t}\sigma_{m-1}^{n} + \frac{4}{\Delta t}\sigma_m^{n} + \frac{1}{\Delta t}\sigma_{m+1}^{n} - \sin\hat{u}, m = 0(1)M. \end{aligned} \tag{13}$$

Both the systems (12) and (13) consist of $N+1$ linear equations including $N+3$ unknown parameters $(\delta_{-1},\ldots,\delta_{N+1})^T$ and $(\sigma_{-1},\ldots,\sigma_{N+1})^T$. To obtain a unique solution to these systems, we need four additional constraints. These are obtained from the boundary conditions and can be used to eliminate δ_{-1}. δ_{N+1}, σ_{-1} and σ_{N+1} from the systems.

2.1 Initial state

To proceed with the iterative formula (12) and (13), first of all, we do need the initial vectors d^0 and $\tilde{d}^0$ which is going to be determined from the initial and boundary conditions. To achieve this, the approximations (3) and (4) ought to be rewritten for the initial condition as

$$U_N(x,t_0)=\sum_{m=-1}^{N+1}\delta_m(t_0)\phi_m(x)$$

and

$$V_N(x,t_0)=\sum_{m=-1}^{N+1}\sigma_m(t_0)\phi_m(x)$$

where the δ_m's and σ_m's are unknown element parameters. Now, if we force the initial numerical approximations $U_N(x,t_0)$ and $V_N(x,t_0)$ comply with the following boundary conditions to discard δ_{-1}, δ_{N+1}, σ_{-1} and σ_{N+1}

$$\begin{aligned} &U_N(x,t_0)=U(x_m,t_0), \quad m=0,1,...,N\\ &(U_N)_x(a,t_0)=0, \quad (U_N)_x(b,t_0)=0, \end{aligned}$$

and

$$\begin{aligned} &V_N(x,t_0)=V(x_m,t_0), \quad m=0,1,...,N\\ &(V_N)_x(a,t_0)=0, \quad (V_N)_x(b,t_0)=0, \end{aligned}$$

we obtain the matrix form for the initial vector $\mathbf{d}^0$ as

$$\mathbf{W}\mathbf{d}^0=\mathbf{b}$$

where

$$\mathbf{W}=\begin{bmatrix} 4 & 2 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 2 & 4 \end{bmatrix}$$

$$\mathbf{d}^0=(\delta_0,\delta_1,\delta_2,\ldots,\delta_{N-2},\delta_{N-1},\delta_N)^T$$

and

$$\mathbf{b}=(U(x_0,t_0)+\tfrac{h}{3}U'(x_0,t_0),U(x_1,t_0),U(x_2,t_0),\ldots,U(x_{N-2},t_0),U(x_{N-1},t_0),U(x_N,t_0)-\tfrac{h}{3}U'(x_N,t_0))^T.$$

and $\tilde{\mathbf{d}}^0$ as

$$\mathbf{W}\tilde{\mathbf{d}}^0=\tilde{\mathbf{b}}$$

where

$$\mathbf{W}=\begin{bmatrix} 4 & 2 & & & & \\ 1 & 4 & 1 & & & \\ & 1 & 4 & 1 & & \\ & & & \ddots & & \\ & & & 1 & 4 & 1 \\ & & & & 2 & 4 \end{bmatrix}$$

$$\tilde{\mathbf{d}}^0=(\sigma_0,\sigma_1,\sigma_2,\ldots,\sigma_{N-2},\sigma_{N-1},\sigma_N)^T$$

and

$$\tilde{\mathbf{b}}=(V(x_0,t_0+\tfrac{h}{3}V'(x_0,t_0)),V(x_1,t_0),V(x_2,t_0),\ldots,V(x_{N-2},t_0),V(x_{N-1},t_0),V(x_N,t_0)-\tfrac{h}{3}V'(x_N,t_0))^T.$$

3 Numerical examples and results

Numerical results for sine-Gordon problem are obtained by Collocation finite element method using cubic B-spline base functions. The accuracy of the method is measured by the error norm L_2

$$L_2=\left\|U^{exact}-U_N\right\|_2\simeq\sqrt{h\sum_{j=0}^{N}\left|U_j^{exact}-(U_N)_j\right|^2}$$

and the error norm L_∞

$$L_\infty=\left\|U^{exact}-U_N\right\|_\infty\simeq\max_j\left|U_j^{exact}-(U_N)_j\right|.$$

To show how good the method, we have considered the following two problems into consideration.

Problem 1

First of all, we will deal with the sine-Gordon equation (1) in terms of a system composed of two equations

$$u_t=v, \qquad v_t=u_{xx}-\sin u.$$

The exact solutions of the above equations have been obtained as [12]

$$u(x,t)=4\tan^{-1}(\exp[\gamma(x-Ct)+\beta]), \tag{14}$$

$$v(x,t)=\frac{-4\gamma C(\exp[\gamma(x-Ct)+\beta])}{1+\exp[\gamma(x-Ct)+\beta]^2} \tag{15}$$

where $\gamma=(1-C^2)^{-1/2}$. The initial conditions $u(x,0)$, $v(x,0)$ and boundary conditions $u(a,t)$, $u(b,t)$,$v(a,t)$, $v(b,t)$ of the problem have been taken from those exact solutions (14) and (15). We have solved the current problem in the solution interval $-2\le x\le 58$. The results of Problem 1 are tabulated in

Tables 1 and 2. In Table 1, we have compared the error norms L_2 and L_∞ at different times with those in [12,13]. As it is seen from the table, the results obtained in the present study are in agrement with those in other studies and become better as time increases. In table 2, it is seen that as the number of elements increase,that is mesh sizes decrease, the error norms decrease. In Fig. 1, the numerical solutions of the problem have been drawn at times $t = 9, 36, 108$.

Table 1: Error norms L_2 and L_∞ when solution for $\Delta t = 0.001$, $N = 120$, $C = 0.5$, $\beta = 0$ in $[-2, 58]$.

t	Present		[12]		[13]
	L_2	L_∞	L_2	L_∞	L_∞
9	0.169580	0.133033	2.683E-002	6.836E-003	4.518E-001
36	0.680536	0.500513	3.113E-001	8.032E-002	2.293E+000
108	2.025488	1.480139	2.378E+000	6.253E-001	5.120E+000

Table 2: Error norms L_2 and L_∞ when solution for $\Delta t = 0.001$, $C = 0.5$, $\beta = 0$ in $[-2, 58]$ for different values of N.

t	$N = 120$		$N = 240$		$N = 360$		$N = 480$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞	L_2	L_∞
9	0.169580	0.133033	0.044859	0.036136	0.020158	0.016270	0.011383	0.009170
18	0.345557	0.254219	0.091444	0.070693	0.041090	0.031807	0.023203	0.017988
36	0.680536	0.500513	0.180242	0.135790	0.080998	0.061487	0.045739	0.034824
54	1.017154	0.737338	0.269906	0.203068	0.121319	0.091966	0.068512	0.052105
72	1.354803	0.978039	0.359861	0.270816	0.161747	0.122360	0.091341	0.069204
108	2.025488	1.480139	0.539357	0.403595	0.242441	0.182980	0.136907	0.103556

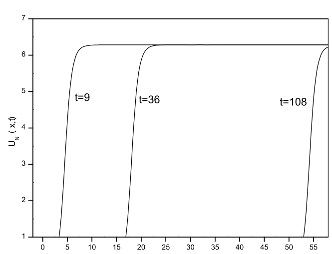

Fig. 1: Numerical solutions of Problem I at $t = 9, 36, 108$ for $x \in [-2, 58], \beta = 0, C = 0.5, N = 240$ and $\Delta t = 0.001$.

Problem 2

Secondly, we will deal with the sine-Gordon equation (1) in terms of a system composed of the following two equations

$$u_t = v, \qquad v_t = u_{xx} - \sin u.$$

The exact solutions of the above equations have been obtained as[14]

$$u(x,t) = 4\tan^{-1}(\operatorname{sech}(x)t), v(x,t) = \frac{4\operatorname{sech}(x)}{1 + t^2\operatorname{sech}^2(x)} \quad (16)$$

with the initial conditions

$$u(x,0) = 0, \quad v(x,0) = 4\operatorname{sech}(x), \quad -10 \leq x \leq 10. \quad (17)$$

Boundary conditions $u(a,t)$, $u(b,t)$, $v(a,t)$, $v(b,t)$ have been taken from those exact solutions (17). We have solved the present problem in the solution interval $-10 \leq x \leq 10$. The results of Problem 2 are evaluated in Tables 3,4 and 5. In Table 3, for $t > 10$, the error norm L_∞ of the present study are better than those in [14].

Table 3: Error norms L_2 and L_∞ when solution for $\Delta t = 0.01$, $N = 400$ in $[-10, 10]$.

t	Present		[14]
	L_2	L_∞	L_∞
1	1.115×10^{-3}	2.281×10^{-4}	1.678×10^{-4}
2	3.251×10^{-3}	4.258×10^{-4}	4.237×10^{-4}
5	3.028×10^{-3}	3.269×10^{-4}	3.257×10^{-4}
10	8.092×10^{-3}	9.512×10^{-4}	4.829×10^{-3}
15	1.7508×10^{-2}	1.953×10^{-3}	1.124×10^{-2}
20	3.0830×10^{-2}	3.541×10^{-3}	1.155×10^{-2}

In Table 4, the error norms L_2 and L_∞ are tabulated for different values of mesh sizes. It is clear that as the number of elements increase, as it is expected, both of the error norms decrease. Finally, the error norms are computed for different values of time steps. Again, the error norms have become smaller as time step decreases. We have also depicted the numerical solutions of the problem in Fig. 2 at times $t = 5, 20$.

Table 4: Error norms L_2 and L_∞ when solution for $\Delta t = 0.01$ in $[-10, 10]$ for different values of N.

t	$N = 200$		$N = 250$		$N = 500$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
1	3.394×10^{-3}	1.012×10^{-3}	2.396×10^{-3}	6.360×10^{-4}	7.546×10^{-4}	0.134×10^{-4}
2	9.814×10^{-3}	2.141×10^{-3}	6.935×10^{-3}	1.215×10^{-3}	2.229×10^{-3}	0.277×10^{-4}
5	1.127×10^{-2}	1.849×10^{-3}	7.534×10^{-3}	1.076×10^{-3}	2.314×10^{-3}	0.220×10^{-4}
10	4.858×10^{-2}	7.787×10^{-3}	3.119×10^{-2}	4.489×10^{-3}	2.551×10^{-3}	0.187×10^{-4}
15	1.113×10^{-1}	1.746×10^{-2}	7.122×10^{-2}	9.993×10^{-3}	2.175×10^{-3}	0.181×10^{-4}
20	0.199	3.175×10^{-2}	0.127	1.817×10^{-2}	1.686×10^{-3}	0.203×10^{-4}

Table 5: Error norms L_2 and L_∞ when solution for $N = 400$ in $[-10, 10]$ for different values of Δt.

t	$\Delta t = 0.1$		$\Delta t = 0.05$		$\Delta t = 0.01$	
	L_2	L_∞	L_2	L_∞	L_2	L_∞
1	1.395×10^{-2}	3.103×10^{-3}	2.862×10^{-3}	5.808×10^{-4}	1.115×10^{-3}	2.281×10^{-4}
2	5.570×10^{-2}	8.094×10^{-3}	1.269×10^{-2}	1.642×10^{-3}	3.251×10^{-3}	4.258×10^{-4}
5	2.861×10^{-1}	3.104×10^{-2}	6.940×10^{-2}	7.445×10^{-3}	3.028×10^{-3}	3.269×10^{-4}
10	1.222	1.351×10^{-1}	2.945×10^{-1}	3.253×10^{-2}	8.092×10^{-3}	9.512×10^{-4}
15	2.764	3.071×10^{-1}	6.744×10^{-1}	7.495×10^{-2}	1.7508×10^{-2}	1.953×10^{-3}
20	4.846	5.396×10^{-1}	1.204	1.341×10^{-1}	3.0830×10^{-2}	3.541×10^{-3}

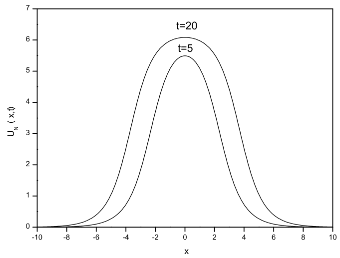

Fig. 2: Numerical solutions of Problem 2 at $t = 5, 20$ for $x \in [-10, 10], N = 250$ and $\Delta t = 0.01$.

4 Conclusions

In this paper, numerical solutions of the sine-Gordon equation based on the cubic B-spline finite element method have been calculated and presented. Two test problems have been worked out to examine the performance of the present algorithm. The performance and efficiency of the method are shown by calculating the error norms L_2 and L_∞. The obtained results show that the error norms are sufficiently small during all computer runs. The obtained results indicate that the present method is a particularly successful numerical scheme to solve the sine-Gordon equation. As a conclusion, the method can be efficiently applied to this type of non-linear problems arising in physics and mathematics with success.

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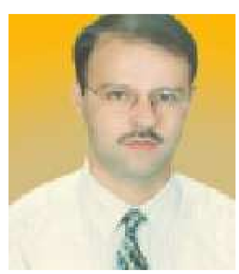

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