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On Characterizing Probability Distributions by Conditional Expectation of Two Order Statistics

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Abstract: The order statistics have been extensively studied in the literature to characterize some particular distributions as well as family of distributions. The problem of characterizing distributions through conditional expectation of adjacent and non-adjacent order statistics has been of increasing interest due to its several applications. Several approaches are available in literature. In this paper, two general classes of distributions $F(x) = 1-e^{-ah(x)}$ and $F(x) = 1-[ah(x)+b]^c$, where $h(x)$ is a continuous, differentiable and monotonic function of $x\varepsilon(\alpha,\beta)$ have been characterized through the conditional expectation of k^{th} power of difference of two order statistics. Further, several deductions and particular cases are discussed.

Keywords: Order statistics, conditional expectation, continuous distributions, characterization.

1 Introduction

Let $X_1, X_2, \ldots X_n$ be a random sample of size n from a continuous population with the distribution function (df) $F(x)$ and the probability density function (pdf) $f(x)$ and let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the corresponding order statistics. Then the conditional pdf of $X_{s:n}$ given $X_{r:n} = x$, $1 \le r < s \le n$, is $\left[\text{David and Nagaraja [1]}\right]$

$$\frac{(n-r)!}{(s-r-1)!(n-s)!}\frac{[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}}{[1-F(x)]^{n-r}}f(y),\ x<y \tag{1}$$

Conditional moments of order statistics are extensively used in characterizing the probability distributions. Various approaches are available in the literature. For detailed survey one may refer to Khan and Ali [2], Khan and Abu-Salih [3], Franco and Ruiz [4,5], López-Blázquez and Moreno-Rebello [6], Wesolowski and Ahsanullah [7], Dembińska and Wesolowski [8], Khan and Abouammoh [9], Khan and Athar [10] and references therein.

Khan and Abu-Salih [3] characterized some general family of distributions through conditional expectation of functions of order statistics fixing adjacent order statistic. Further, Khan and Abouammoh [9] extended the result of Khan and Abu-Salih [3] where the conditioned order statistic may not be adjacent one. Khan *et al.* [11] characterized a general form of distribution by conditional spacing of order statistics. Here in this chapter, an attempt is being made to characterize two general forms of distributions $F(x) = 1-e^{-ah(x)}$ and $F(x) = 1-[ah(x)+b]^c$, $a \ne 0$ through k^{th} conditional moment of difference between functions of two order statistics.

2 Characterization theorems

Before coming to the main result, we shall prove the following lemma:

Lemma 2.1. *For any positive integers μ and ν with $n\ \varepsilon\ N$*

$$\int_0^1 (\ln u)^n (1-u)^{\nu-1} u^{\mu-1} du$$

$$= (-1)^n\ n!\beta(\mu,\nu)\sum_{i_1=0}^{\nu-1}\sum_{i_2=i_1}^{\nu-1}\cdots\sum_{i_n=i_{n-1}}^{\nu-1}\frac{1}{\mu+i_1}\frac{1}{\mu+i_2}\cdots\frac{1}{\mu+i_n} \tag{2}$$

where $\beta(\mu,\nu)$ is complete beta function.

Proof. Consider $\int_0^1 (\ln u)^n (1-u)^{\nu-1} u^{\mu-1} du$.

For $n=1$, we have

$$\int_0^1 (\ln u)(1-u)^{\nu-1} u^{\mu-1} du$$

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$$= \beta(\mu,\nu)[\psi(\mu)-\psi(\mu+\nu)] \quad \text{see [11]}$$

$$= -\beta(\mu,\nu)\sum_{i_1=0}^{\nu-1}\frac{1}{\mu+i_1} \tag{3}$$

Again for $n=2$, we have

$$\int_0^1 (\ln u)^2(1-u)^{\nu-1}u^{\mu-1}du$$

$$= \beta(\mu,\nu)[\{\psi(\mu)-\psi(\mu+\nu)\}^2+\psi'(\mu)-\psi'(\mu+\nu)] \text{ see [11]}$$

$$= 2!\beta(\mu,\nu)\sum_{i_1=0}^{\nu-1}\sum_{i_2=i_1}^{\nu-1}\frac{1}{\mu+i_1}\frac{1}{\mu+i_2} \tag{4}$$

where $\psi(x) = \frac{d}{dx}\ln\Gamma(x)$ with functional relation $\psi(x+n)=\psi(x)+\sum_{k=0}^{n-1}\frac{1}{x+k}$.

Now we assume that (2) holds for $n=k$, then

$$\int_0^1 (\ln u)^k(1-u)^{\nu-1}u^{\mu-1}du$$

$$= (-1)^k k!\,\beta(\mu,\nu)\sum_{i_1=0}^{\nu-1}\sum_{i_2=i_1}^{\nu-1}\cdots\sum_{i_k=i_{k-1}}^{\nu-1}\frac{1}{\mu+i_1}\frac{1}{\mu+i_2}\cdots\frac{1}{\mu+i_k}. \tag{5}$$

Then, the statement should be true for $n=k+1$.

Therefore, integrating by parts, we get

$$\int_0^1 (\ln u)^{k+1}(1-u)^{\nu-1}u^{\mu-1}du$$

$$= -\frac{(k+1)}{\mu}\int_0^1(\ln u)^k(1-u)^{\nu-1}u^{\mu-1}du + \frac{(\nu-1)}{\mu}\int_0^1(\ln u)^{k+1}(1-u)^{\nu-2}u^{\mu}du. \tag{6}$$

In view of (5), (6) reduces to

$$= (-1)^{k+1}\frac{(k+1)!}{\mu}\beta(\mu,\nu) \times\sum_{i_1=0}^{\nu-1}\sum_{i_2=i_1}^{\nu-1}\cdots\sum_{i_k=i_{k-1}}^{\nu-1}\frac{1}{\mu+i_1}\frac{1}{\mu+i_2}\cdots\frac{1}{\mu+i_k} +\frac{(\nu-1)}{\mu}\int_0^1(\ln u)^{k+1}(1-u)^{\nu-2}u^{\mu}du \tag{7}$$

Similarly integrating (7) by parts $(\nu-1)$ times and using (5), we get

$$= (-1)^{k+1}(k+1)!\,\beta(\mu,\nu) \times\frac{1}{\mu}\sum_{i_1=0}^{\nu-1}\sum_{i_2=i_1}^{\nu-1}\cdots\sum_{i_k=i_{k-1}}^{\nu-1}\frac{1}{\mu+i_1}\frac{1}{\mu+i_2}\cdots\frac{1}{\mu+i_k}$$

$$+(-1)^{k+1}(k+1)!\,\beta(\mu,\nu) \times\frac{1}{\mu+1}\sum_{i_1=0}^{\nu-1}\sum_{i_2=i_1}^{\nu-1}\cdots\sum_{i_k=i_{k-1}}^{\nu-1}\frac{1}{\mu+i_1}\frac{1}{\mu+i_2}\cdots\frac{1}{\mu+i_k}$$

$$+(-1)^{k+1}(k+1)!\,\beta(\mu,\nu) \times\frac{1}{\mu+2}\sum_{i_1=0}^{\nu-1}\sum_{i_2=i_1}^{\nu-1}\cdots\sum_{i_k=i_{k-1}}^{\nu-1}\frac{1}{\mu+i_1}\frac{1}{\mu+i_2}\cdots\frac{1}{\mu+i_k}$$

$$+\cdots\cdots\cdots$$

$$+(-1)^{k+1}(k+1)!\,\beta(\mu,\nu) \times\frac{1}{\mu+\nu-1}\sum_{i_1=0}^{\nu-1}\sum_{i_2=i_1}^{\nu-1}\cdots\sum_{i_k=i_{k-1}}^{\nu-1}\frac{1}{\mu+i_1}\frac{1}{\mu+i_2}\cdots\frac{1}{\mu+i_k}$$

$$=(-1)^{k+1}(k+1)!\,\beta(\mu,\nu) \times\frac{1}{\mu+1}\sum_{i_1=0}^{\nu-1}\sum_{i_2=i_1}^{\nu-1}\cdots\sum_{i_{k+1}=i_k}^{\nu-1}\frac{1}{\mu+i_1}\frac{1}{\mu+i_2}\cdots\frac{1}{\mu+i_{k+1}}.$$

Therefore (2) holds for $n=k+1$.

Hence the Lemma.

Theorem 2.1: *Let X be an absolutely continuous random variable with the df $F(x)$ and the pdf $f(x)$ in the interval (α,β), where α and β may be finite or infinite, then for $1\le r<s\le n$,*

$$E[(h(X_{s:n})-h(X_{r:n}))^k|X_{r:n}=x] = k!\,\frac{1}{a^k}\sum_{i_1=r}^{s-1}\sum_{i_2=i_1}^{s-1}\cdots\sum_{i_k=i_{k-1}}^{s-1}\frac{1}{(n-i_1)}\frac{1}{(n-i_2)}\cdots\frac{1}{(n-i_k)} \tag{8}$$

if and only if

$$F(x)=1-e^{-ah(x)},\ \ a\neq 0 \tag{9}$$

where $h(x)$ is a continuous, differentiable and non-decreasing function of x and k is a positive integer.

Proof. To prove the necessary part, we have

$$E[(h(X_{s:n})-h(X_{r:n}))^k|X_{r:n}=x] = \frac{(n-r)!}{(s-r-1)!(n-s)!}\int_x^\beta (h(y)-h(x))^k \times\left[1-\frac{1-F(y)}{1-F(x)}\right]^{s-r-1}\left[\frac{1-F(y)}{1-F(x)}\right]^{n-s}\frac{f(y)}{1-F(x)}dy.$$

Assuming

$$\frac{1-F(y)}{1-F(x)}=u, \text{ which implies } (h(y)-h(x))^k=(-1)^k\frac{1}{a^k}(\ln u)^k.$$

Thus, RHS of the above expression reduces to

$$=\frac{(n-r)!\,(-1)^k}{a^k(s-r-1)!(n-s)!}\int_0^1(\ln u)^k(1-u)^{s-r-1}u^{n-s}du.$$

Now, on application of Lemma 2.1, we get

$$E[(h(X_{s:n})-h(X_{r:n}))^k|X_{r:n}=x]$$

$$=\frac{(n-r)!(-1)^{2k}}{a^k\,(s-r-1)!(n-s)!}\,k!\,\beta(n-s+1,s-r)\times\sum_{i_1=0}^{s-r-1}\sum_{i_2=i_1}^{s-r-1}$$

$$\cdots\sum_{i_k=i_{k-1}}^{s-r-1}\frac{1}{(n-s+1+i_1)}\frac{1}{(n-s+1+i_2)}\cdots\frac{1}{(n-s+1+i_k)}$$

$$=k!\,\frac{1}{a^k}\sum_{i_1=r}^{s-1}\sum_{i_2=i_1}^{s-1}\cdots\sum_{i_k=i_{k-1}}^{s-1}\frac{1}{(n-i_1)}\frac{1}{(n-i_2)}\cdots\frac{1}{(n-i_k)}.$$

Hence the (8).

To prove the sufficiency part, consider

$$E[(h(X_{s:n})-h(X_{r:n}))^k|X_{r:n}=x]=g_{r,s,k}$$

or

$$\frac{(n-r)!}{(s-r-1)!(n-s)!}\int_x^\beta(h(y)-h(x))^k$$

$$\times[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(y)dy$$

$$=g_{r,s,k}[1-F(x)]^{n-r}. \tag{10}$$

Differentiating (10) *w.r.t.* x, we have

$$-\,k\,h'(x)\,\frac{(n-r)!}{(s-r-1)!(n-s)!}$$

$$\times\int_x^\beta(h(y)-h(x))^{k-1}\,\frac{[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}}{[1-F(x)]^{n-r}}\,f(y)dy$$

$$-\,(n-r)\frac{f(x)}{1-F(x)}\;\frac{(n-r-1)!}{(s-r-2)!(n-s)!}$$

$$\times\int_x^\beta(h(y)-h(x))^k\,\frac{[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s}}{[1-F(x)]^{n-r-1}}\,f(y)dy$$

$$=\;-\;(n-r)\frac{f(x)}{1-F(x)}\;g_{r,s,k}. \tag{11}$$

Rearranging the terms of (11), we get

$$\frac{f(x)}{1-F(x)}=\frac{1}{(n-r)}\cdot\frac{k\,h'(x)g_{r,s,k-1}}{[g_{r,s,k}-g_{r+1,s,k}]}$$

$$=ah'(x).$$

Hence the theorem.

Remark 2.1: At $k=1$, (8) reduces to

$$E[h(X_{s:n})|X_{r:n}=x]=h(x)+\frac{1}{a}\sum_{j=r}^{s-1}\frac{1}{(n-j)}$$

as obtained by Khan and Abouammoh [9].

Theorem 2.2: *Let X be an absolutely continuous random variable with the df $F(x)$ and the pdf $f(x)$ in the interval (α,β), where α and β may be finite or infinite, then for $1\le r<s\le n$,*

$$E[(h(X_{s:n})-h(X_{r:n}))^k|X_{r:n}=x]=g_{r,s,k}(x)$$

$$=\left(\frac{ah(x)+b}{a}\right)^k\sum_{i=0}^{k}(-1)^{i+k}\binom{k}{i}\prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right) \tag{12}$$

if and only if

$$F(x)=1-[ah(x)+b]^c \tag{13}$$

where a,b and c are so chosen that $F(x)$ is a df and $h(x)$ is a monotonic and differentiable function of x over the support (α,β).

Proof. First we shall prove (13) implies (12).

In view of (1), we have

$$E[(h(X_{s:n})-h(X_{r:n}))^k|X_{r:n}=x]$$

$$=\frac{(n-r)!}{(s-r-1)!(n-s)!}\int_x^\beta(h(y)-h(x))^k$$

$$\times\left[1-\frac{1-F(y)}{1-F(x)}\right]^{s-r-1}\left[\frac{1-F(y)}{1-F(x)}\right]^{n-s}\frac{f(y)}{1-F(x)}dy.$$

Assuming

$$\frac{1-F(y)}{1-F(x)}=u,$$

implies

$$(h(y)-h(x))^k=(-1)^k\left(\frac{ah(x)+b}{a}\right)^k(1-u^{1/c})^k.$$

Thus, we have

$$E[(h(X_{s:n})-h(X_{r:n}))^k|X_{r:n}=x]$$

$$=(-1)^k\left(\frac{ah(x)+b}{a}\right)^k\frac{(n-r)!}{(s-r-1)!(n-s)!}$$

$$\times\int_0^1(1-u^{1/c})^k(1-u)^{s-r-1}u^{n-s}du,$$

$$=(-1)^k\left(\frac{ah(x)+b}{a}\right)^k\frac{(n-r)!}{(s-r-1)!(n-s)!}$$

$$\times\sum_{i=0}^{k}\binom{k}{i}(-1)^i\int_0^1(1-u)^{s-r-1}u^{(i/c)+(n-s)}du,$$

$$=(-1)^k\left(\frac{ah(x)+b}{a}\right)^k\frac{(n-r)!}{(s-r-1)!(n-s)!}$$

$$\times\sum_{i=0}^{k}\binom{k}{i}(-1)^i\beta(s-r,(i/c)+(n-s)+1),$$

$$=(-1)^k\left(\frac{ah(x)+b}{a}\right)^k\frac{(n-r)!}{(n-s)!}$$

$$\times\sum_{i=0}^{k}\binom{k}{i}(-1)^i\frac{\Gamma[(i/c)+(n-s)+1]}{\Gamma[(i/c)+(n-r)+1]},$$

$$= (-1)^k \left(\frac{ah(x)+b}{a}\right)^k \sum_{i=0}^{k} \binom{k}{i} (-1)^i \prod_{j=r}^{s-1} \left(\frac{c(n-j)}{i+c(n-j)}\right).$$

Hence the (12).

Now to prove the sufficiency part, let

$$E[(h(X_{s:n})-h(X_{r:n}))^k)|X_{r:n}=x] = \xi_{r,s,k}(x)$$

or

$$\frac{(n-r)!}{(s-r-1)!(n-s)!}$$
$$\times \int_x^\beta (h(y)-h(x))^k [F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s} f(y)dy$$
$$= \xi_{r,s,k}(x)[1-F(x)]^{n-r}. \quad (14)$$

Differentiating (14) *w.r.t.* x, we have

$$-kh'(x)\frac{(n-r)!}{(s-r-1)!(n-s)!}$$
$$\times \int_x^\beta (h(y)-h(x))^{k-1} \frac{[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}}{[1-F(x)]^{n-r}} f(y)dy$$
$$-(n-r)\frac{f(x)}{1-F(x)}\frac{(n-r-1)!}{(s-r-2)!(n-s)!}$$
$$\times \int_x^\beta (h(y)-h(x))^k \frac{[F(y)-F(x)]^{s-r-2}[1-F(y)]^{n-s}}{[1-F(x)]^{n-r-1}} f(y)dy$$
$$= \xi'_{r,s,k}(x) - (n-r)\frac{f(x)}{[1-F(x)]}\xi_{r,s,k}(x). \quad (15)$$

Rearranging the terms of (15), we get

$$\frac{f(x)}{1-F(x)} = -\frac{1}{(n-r)} \frac{\xi'_{r,s,k}(x)+k\,h'(x)\xi_{r,s,k-1}(x)}{[\xi_{r+1,s,k}(x)-\xi_{r,s,k}(x)]}. \quad (16)$$

Consider

$$\xi'_{r,s,k}(x)+k\,h'(x)\xi_{r,s,k-1}(x)$$
$$= (-1)^k k\,h'(x)\left(\frac{ah(x)+b}{a}\right)^{k-1}$$
$$\times \sum_{i=0}^{k}\binom{k}{i}(-1)^i \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)$$
$$+(-1)^{k-1}k\,h'(x)\left(\frac{ah(x)+b}{a}\right)^{k-1}$$
$$\times \sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)$$
$$= (-1)^k k\,h'(x)\left(\frac{ah(x)+b}{a}\right)^{k-1}$$
$$\times \left[\sum_{i=0}^{k}\binom{k}{i}(-1)^i \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)\right.$$
$$\left. -\sum_{i=0}^{k-1}\binom{k-1}{i}(-1)^i \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)\right]$$

$$= (-1)^k k\,h'(x)\left(\frac{ah(x)+b}{a}\right)^{k-1}$$
$$\times \left[\sum_{i=0}^{k}\binom{k}{i}(-1)^i \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)\right.$$
$$+(-1)^k \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)$$
$$\left. -\sum_{i=0}^{k-1}\binom{k}{i}\left(1-\frac{i}{k}\right)(-1)^i \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)\right]$$
$$= (-1)^k h'(x)\left(\frac{ah(x)+b}{a}\right)^{k-1}$$
$$\times \left[(-1)^k k \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{k+c(n-j)}\right)\right.$$
$$\left. +\sum_{i=0}^{k-1}\binom{k}{i}(-1)^i\, i \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)\right]$$
$$= (-1)^k h'(x)\left(\frac{ah(x)+b}{a}\right)^{k-1}$$
$$\times \sum_{i=0}^{k}\binom{k}{i}(-1)^i\, i \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)$$

and

$$\xi_{r+1,s,k}(x)-\xi_{r,s,k}(x)$$
$$= (-1)^k \left(\frac{ah(x)+b}{a}\right)^k \sum_{i=0}^{k}\binom{k}{i}(-1)^i \prod_{j=r+1}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)$$
$$-(-1)^k \left(\frac{ah(x)+b}{a}\right)^k \sum_{i=0}^{k}\binom{k}{i}(-1)^i \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)$$
$$= (-1)^k \left(\frac{ah(x)+b}{a}\right)^k \sum_{i=0}^{k}\binom{k}{i}(-1)^i$$
$$\times \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)\left(\frac{i+c(n-r)}{c(n-r)}-1\right)$$
$$= (-1)^k \left(\frac{ah(x)+b}{a}\right)^k \frac{1}{c(n-r)}$$
$$\times \sum_{i=0}^{k}\binom{k}{i}(-1)^i\, i \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right).$$

Therefore in view of (16), we get

$$= -\frac{a\,c\,h'(x)}{(ah(x)+b)} \frac{\sum_{i=0}^{k}\binom{k}{i}(-1)^i\, i \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)}{\sum_{i=0}^{k}\binom{k}{i}(-1)^i\, i \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{i+c(n-j)}\right)}$$

$$= -\frac{a\,c\,h'(x)}{(ah(x)+b)}.$$

Hence the theorem.

Remark 2.2: At $k=1$, (12) reduces to

$$E[h(X_{s:n})|X_{r:n}=x] = a^* h(x)+b^*$$

where

$$a^* = \prod_{j=r}^{s-1}\left(\frac{c(n-j)}{1+c(n-j)}\right) \text{ and } b^* = -\frac{b}{a}(1-a^*).$$

3 Some Examples

Examples based on Theorem 2.1

a) Pareto Distribution

$F(x) = 1 - (x/\alpha)^{-\theta}, \alpha < x < \infty, \quad \theta > 0$

with $a = \theta, \quad h(x) = \log(x/\alpha)$ and

$$g_{r,s,k} = k! \, \frac{1}{\theta^k} \sum_{i_1=r}^{s-1} \sum_{i_2=i_1}^{s-1} \cdots \sum_{i_k=i_{k-1}}^{s-1} \frac{1}{(n-i_1)} \frac{1}{(n-i_2)} \cdots \frac{1}{(n-i_k)}.$$

b) Weibull Distribution

$F(x) = 1 - e^{-\theta \, x^p}, 0 < x < \infty, \quad p, \theta > 0$

with $a = \theta, \quad h(x) = x^p$ and

$$g_{r,s,k} = k! \, \frac{1}{\theta^k} \sum_{i_1=r}^{s-1} \sum_{i_2=i_1}^{s-1} \cdots \sum_{i_k=i_{k-1}}^{s-1} \frac{1}{(n-i_1)} \frac{1}{(n-i_2)} \cdots \frac{1}{(n-i_k)}.$$

c) Log Logistic Distribution

$F(x) = 1 - (1 + \theta \, x^p)^{-1}, 0 < x < \infty, \quad p, \theta > 0$

with $a = 1, \quad h(x) = \log(1 + \theta \, x^p)$ and

$$g_{r,s,k} = k! \sum_{i_1=r}^{s-1} \sum_{i_2=i_1}^{s-1} \cdots \sum_{i_k=i_{k-1}}^{s-1} \frac{1}{(n-i_1)} \frac{1}{(n-i_2)} \cdots \frac{1}{(n-i_k)}.$$

Similarly, with proper choice of a and $h(x)$ characterization results for other distributions based on Theorem 2.1 can be obtained. For more distributions one may refer Noor and Athar [13].

Examples based on Theorem 2.2

a) Power Function Distribution

$F(x) = \alpha^{-p} x^p, \quad 0 < x < \alpha, \quad \alpha, p > 0$

with $a = -\alpha^{-p}, b = 1, c = 1, \quad h(x) = x^p$ and

$$g_{r,s,k}(x) = (\alpha^p - x^p)^k \sum_{i=0}^{k} (-1)^i \binom{k}{i} \prod_{j=r}^{s-1} \left(\frac{(n-j)}{i + (n-j)} \right).$$

b) Pareto Distribution

$F(x) = 1 - \alpha^p x^{-p}, \quad \alpha \le x < \infty, \quad \alpha, p > 0$

with $a = \alpha^p, b = 0, c = 1, \quad h(x) = x^{-p}$ and

$$g_{r,s,k}(x) = x^{-p\,k} \sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} \prod_{j=r}^{s-1} \left(\frac{(n-j)}{i + (n-j)} \right).$$

c) Weibull Distribution

$F(x) = 1 - e^{-\theta x^p}, \quad 0 \le x < \infty, \quad \theta, p > 0$

with $a = 1, b = 0, c = \theta, \quad h(x) = e^{-x}$ and

$$g_{r,s,k}(x) = e^{-x\,k} \sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} \prod_{j=r}^{s-1} \left(\frac{\theta(n-j)}{i + \theta(n-j)} \right).$$

d) Inverse Weibull Distribution

$F(x) = e^{-\theta x^{-p}}, \quad 0 \le x < \infty, \quad \theta, p > 0$

with $a = -1, b = 1, c = 1, \quad h(x) = e^{-\theta x^{-p}}$ and

$$g_{r,s,k}(x) = (1 - e^{-\theta x^{-p}})^k \sum_{i=0}^{k} (-1)^i \binom{k}{i} \prod_{j=r}^{s-1} \left(\frac{(n-j)}{i + (n-j)} \right).$$

Similarly with proper choice of a, b, c and $h(x)$, results based on Theorem 2.2, can be obtained for various other distributions. One may refer to Noor and Athar [13].

4 Conclusion

The study of ordered random variables and its application have always been interesting topic among the researchers, particularly in characterization of probability distributions, reliability theory, and estimation theory. In this paper, we have proposed a new approach to characterize two general form of distributions through conditional expectation of k^{th} power of difference of two order statistics. These new characterization results are then applied to characterize some well known probability distributions.

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