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Constant-Stress Partially Accelerated Life Testing for Log-Logistic Distribution with Censored Data

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Abstract: In order to quantify the life characteristics of a product, partially accelerated life tests are used when the data obtained from accelerated conditions cannot be extrapolated to normal use conditions. This study considers constant-stress partially accelerated life tests for censored lifetime data, where the lifetime distribution is assumed to follow log-logistic distribution. The maximum likelihood estimates are obtained for the distribution parameters and acceleration factor. Simulation studies are conducted to illustrate the statistical properties of the estimates and evaluate the performance of confidence intervals.

Keywords: Partially accelerated life tests, Constant-stress, Log-logistic distribution, Maximum likelihood estimates, Type-I censoring, Type-II censoring, Confidence intervals.

1 Introduction

Traditional lifetime data analyses are used to obtain information on the life characteristics of a product, system or component at normal use conditions. Nowadays, products and equipments are, however, well designed and giving good satisfaction with very long lifetimes to customers. As a result of great reliability of today's products, obtaining such lifetime data (or times-to-failure data) is becoming more and more difficult under normal use conditions. To obtain failures quickly, reliability practitioners have attempted to force the products to fail more quickly than they would under normal use conditions. That is, a sample of the items is tested at accelerated conditions than normal ones. These conditions are often referred to as stresses which may be in the form of temperature, pressure, vibrations, and so on. The phrase accelerated life testing (ALT) has been used to describe all such practices, and the lifetime data from accelerated conditions are extrapolated to estimate the life distribution at normal operating conditions. The types of stress loadings in ALT are generally classified as constant-stress, step-stress or random-stress. The constant-stress loading is a time-independent test setting where the stress remains unchanged until an item fails. The constant-stress loading has several advantages over time-dependent stress loadings because most of real products are operated at a constant-stress condition. For more details about ALTs, see [1], [2], and [3] among others.

In ALT the main assumption is that a life-stress relationship is known or can be assumed so that the data obtained from accelerated conditions can be extrapolated to normal use conditions. In some cases, such relationship can not be known or assumed. So, partially accelerated life tests (PALT) are often used in such cases. In a constant-stress PALT, each test item is run at a constant-stress under either normal use condition or accelerated condition only until the test is terminated, and the analysis of PALT has been extensively studied in recent years. Bai and Chung [4] studied the problem of estimation and optimal constant-stress PALT design for an exponential lifetime distribution. Bai *et al.* [5] also considered PALT design for items having lognormal distribution. Abdel-Ghani [6] considered the estimation problem in constant-stress PALT for the Weibull distribution, Abdel-Ghaly *et al.* [7] discussed parameter estimation for Pareto distribution under PALT, and Abd-Elfattah *et al.* [8] considered estimation in step-stress partially accelerated life tests for the Burr Type XII distribution. Zarrin *et al.* [9] studied the maximum likelihood method for estimating the acceleration factor and the parameters of Rayleigh distribution for constant-stress PALT. Kamal *et al.* [10] dealt with constant stress partially accelerated life test assuming that the lifetimes of test item follow Inverted Weibull distribution.

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As indicated by Bennett [11], the log-logistic distribution is found to be a good reliability model as it fits well in many practical situations of reliability data analyses. For example, Chiodo and Mazzanti [12] used the log-logistic distribution for describing the degradation rate for highly reliable products, Kantam *et al.* [13] used the log-logistic distribution for the basic probability model of the life of the product, and Akhtar and Khan [14] utilized log-logistic distribution as a reliability model using a Bayesian method. Another important feature with the log-logistic distribution is that its reliability and hazard functions can be written in closed forms. Thus the log-logistic distribution is convenient in handling censored data. In this paper we consider constant-stress partially accelerated life tests for log-logistic lifetime distribution with Type-I and Type-II censored data.

The rest of this paper is organized as follows. Section 2.1 introduces the notations and model assumptions. Section 2.2 presents the maximum likelihood estimators of underlying parameters with Type-I censored data, and Type-II censored data are considered in Section 2.3. Section 3 contains the simulation results that demonstrate and evaluate the performance of the estimators based on the proposed censoring schemes. Section 4 concludes the paper and suggests some future ideas in this area.

2 Model description and maximum likelihood estimates

2.1 Model assumptions

In a constant-stress PALT, all of the n items are divided into two groups. $n\pi$ items are randomly chosen from the n items and are allocated to accelerated conditions where π is the proportion of the sample items allocated to accelerated conditions, while the remaining $n-n\pi$ items are placed to normal use conditions. Some assumptions are made in a constant-stress PALT.

- Each test item is run until the censoring time τ.
- The test condition is not changed.
- The lifetimes X_i, $i=1,\ldots,n(1-\pi)$ and Y_j, $j=1,\ldots,n\pi$ of items allocated at normal use conditions and accelerated conditions, respectively, are i.i.d. random variables.
- The lifetimes X_i and Y_j are mutually independent.

In this study the lifetimes of test items are assumed to follow a log-logistic distribution. The probability density function of an item at use conditions is given by

$$f(x) = \frac{\alpha x^{\alpha-1}\lambda}{(1+\lambda x^{\alpha})^2}, \quad x \geq 0, \alpha > 0, \lambda > 0,$$

where α is a shape parameter and λ is a scale parameter. Its cumulative distribution function is

$$F(x) = \Pr(X \leq x) = \int_0^x \frac{\alpha u^{\alpha-1}\lambda}{(1+\lambda u^{\alpha})^2}\,du = \int_1^{1+\lambda x^{\alpha}} \frac{1}{t^2}\,dt = \frac{\lambda x^{\alpha}}{1+\lambda x^{\alpha}},$$

the reliability function is

$$S(x) = \Pr(X > x) = \frac{1}{1+\lambda x^{\alpha}},$$

and its hazard function is obtained by

$$h(x) = f(x)/S(x) = \frac{\alpha x^{\alpha-1}\lambda}{1+\lambda x^{\alpha}}.$$

In a constant-stress PALT where $Y = \beta^{-1}X$ with β being the acceleration factor which is the ratio of mean lifetime at use conditions to that at accelerated conditions, the probability density function for an item tested at acceleration conditions is given by

$$f(y) = \frac{\beta\alpha(\beta y)^{\alpha-1}\lambda}{\{1+\lambda(\beta y)^{\alpha}\}^2}, \quad y \geq 0, \alpha > 0, \lambda > 0.$$

In many cases when lifetime data are collected, all items in the sample may not fail. Because of time or cost considerations, the practitioners will terminate the testing and report the results before all items realize their failures. First, we will consider Type-I censoring where the failure is observed only if it occurs prior to some prespecified time. In this instance, all censored items have times equal to the length of the study period τ. Type-I censored data are usually obtained when censoring time is fixed, and then the number of failures in that fixed time is a random variable.

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2.2 Type-I censored data

In a constant-stress PALT with Type-I censoring, the data consist of a random sample of $n(1-\pi)$ lifetimes $x_1, x_2, \ldots, x_{n(1-\pi)}$ under normal use conditions and a random sample of $n\pi$ lifetimes $y_1, y_2, \ldots, y_{n\pi}$ at accelerated conditions respectively. Let δ_{ui} and δ_{aj} denote the failure indicators such that

$$\delta_{ui} = \begin{cases} 1 & x_i \leq \tau \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, \ldots, n(1-\pi)$ and

$$\delta_{aj} = \begin{cases} 1 & y_j \leq \tau \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1, \ldots, n\pi$. We consider the maximum likelihood procedure to estimate the parameters α, λ and β of the model.

The likelihood function for $\{(x_i, \delta_{ui}) : i = 1, \ldots, n(1-\pi)\}$ at normal use conditions is given by

$$L_u(x_i, \delta_{ui}|\alpha, \lambda) = \prod_{i=1}^{n(1-\pi)} \left[\left\{ \frac{\alpha x_i^{\alpha-1}\lambda}{(1+\lambda x_i^{\alpha})^2} \right\}^{\delta_{ui}} \left\{ \frac{1}{1+\lambda\tau^{\alpha}} \right\}^{1-\delta_{ui}} \right]$$

and the likelihood function for $\{(y_j, \delta_{aj}) : j = 1, \ldots, n\pi\}$ at accelerated conditions is given by

$$L_a(y_j, \delta_{aj}|\alpha, \lambda, \beta) = \prod_{j=1}^{n\pi} \left[\left\{ \frac{\beta\alpha(\beta y_j)^{\alpha-1}\lambda}{(1+\lambda(\beta y_j)^{\alpha})^2} \right\}^{\delta_{aj}} \left\{ \frac{1}{1+\lambda(\beta\tau)^{\alpha}} \right\}^{1-\delta_{aj}} \right].$$

Thus, the total likelihood function for $\{(x_i, \delta_{ui}), (y_j, \delta_{aj}) : i = 1, \ldots, n(1-\pi), j = 1, \ldots, n\pi\}$ is

$$L(\alpha, \lambda, \beta) = \prod_{i=1}^{n(1-\pi)} \left[\left\{ \frac{\alpha x_i^{\alpha-1}\lambda}{(1+\lambda x_i^{\alpha})^2} \right\}^{\delta_{ui}} \left\{ \frac{1}{1+\lambda\tau^{\alpha}} \right\}^{1-\delta_{ui}} \right] \times \prod_{j=1}^{n\pi} \left[\left\{ \frac{\beta\alpha(\beta y_j)^{\alpha-1}\lambda}{(1+\lambda(\beta y_j)^{\alpha})^2} \right\}^{\delta_{aj}} \left\{ \frac{1}{1+\lambda(\beta\tau)^{\alpha}} \right\}^{1-\delta_{aj}} \right].$$

Let n_u and n_a be the number of items failed at normal and accelerated conditions respectively. Similarly, let c_u and c_a be the number of items censored at normal and accelerated conditions respectively. That is, $c_u = n(1-\pi) - n_u$ and $c_a = n\pi - n_a$. To obtain the maximum likelihood estimates, the natural logarithm of the likelihood function is usually considered. Then, the log likelihood function is

$$\ln L = \sum_{i=1}^{n(1-\pi)} \delta_{ui} \{\ln\alpha + (\alpha-1)\ln x_i + \ln\lambda - 2\ln(1+\lambda x_i^{\alpha})\} - c_u \ln(1+\lambda\tau^{\alpha}) + \sum_{j=1}^{n\pi} \delta_{aj} \{\ln\alpha + \ln\beta + (\alpha-1)\ln(\beta y_j) + \ln\lambda - 2\ln(1+\lambda(\beta y_j)^{\alpha})\} - c_a \ln\{1+\lambda(\beta\tau)^{\alpha}\}$$

The maximum likelihood estimates (MLE) of the parameters α, λ and β are solutions to the system of likelihood equations obtained by

$$\frac{\partial \ln L}{\partial \alpha} = \frac{n_u}{\alpha} + \sum_{i=1}^{n(1-\pi)} \delta_{ui} \ln x_i \left(1 - 2\frac{\lambda x_i^{\alpha}}{1+\lambda x_i^{\alpha}}\right) - c_u \frac{\lambda\tau^{\alpha}}{1+\lambda\tau^{\alpha}} \ln\tau + \frac{n_a}{\alpha} + \sum_{j=1}^{n\pi} \delta_{aj} \ln(\beta y_j) \left(1 - 2\frac{\lambda(\beta y_j)^{\alpha}}{1+\lambda(\beta y_j)^{\alpha}}\right) - c_a \frac{\lambda(\beta\tau)^{\alpha}}{1+\lambda(\beta\tau)^{\alpha}} \ln(\beta\tau) = 0$$

$$\frac{\partial \ln L}{\partial \lambda} = \frac{n_u}{\lambda} - 2\sum_{i=1}^{n(1-\pi)} \delta_{ui} \frac{x_i^{\alpha}}{1+\lambda x_i^{\alpha}} - c_u \frac{\tau^{\alpha}}{1+\lambda\tau^{\alpha}} + \frac{n_a}{\lambda} - 2\sum_{j=1}^{n\pi} \delta_{aj} \frac{(\beta y_j)^{\alpha}}{1+\lambda(\beta y_j)^{\alpha}} - c_a \frac{(\beta\tau)^{\alpha}}{1+\lambda(\beta\tau)^{\alpha}} = 0$$

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$$\frac{\partial \ln L}{\partial \beta} = n_a \frac{\alpha}{\beta} - 2\sum_{j=1}^{n\pi} \delta_{aj} \frac{\lambda\alpha(\beta y_j)^\alpha y_j}{1+\lambda(\beta y_j)^\alpha} - c_a \frac{\lambda\alpha(\beta\tau)^\alpha \tau}{1+\lambda(\beta\tau)^\alpha} = 0$$

Here, it is difficult to obtain a closed form solution to nonlinear score equations, so an iterative method such as the Newton-Raphson method is used to solve the equations to obtain MLEs. Since the MLEs of parameters are not in closed forms, it is not possible to obtain the Fisher information matrix and construct exact confidence intervals. So asymptotic confidence intervals based on the asymptotic normal distribution of MLEs are obtained here. Hence the asymptotic variance of the maximum likelihood estimates can be obtained by the inverse of observed Fisher information matrix which is evaluated at the MLE

$$\widehat{\Sigma} = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \alpha^2} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} & -\frac{\partial^2 \ln L}{\partial \lambda^2} & -\frac{\partial^2 \ln L}{\partial \lambda \partial \beta} \\ -\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ln L}{\partial \beta \partial \lambda} & -\frac{\partial^2 \ln L}{\partial \beta^2} \end{bmatrix}^{-1}.$$

The elements of the observed Fisher information matrix are as follows:

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \alpha^2} = &-\frac{n_u}{\alpha^2} - 2\sum_{i=1}^{n(1-\pi)} \delta_{ui} \frac{\lambda x_i^\alpha}{(1+\lambda x_i^\alpha)^2} (\ln x_i)^2 - c_u \frac{\lambda\tau^\alpha}{(1+\lambda\tau^\alpha)^2} (\ln\tau)^2 \\ &-\frac{n_a}{\alpha^2} - 2\sum_{j=1}^{n\pi} \delta_{aj} \frac{\lambda(\beta y_j)^\alpha}{\{1+\lambda(\beta y_j)^\alpha\}^2} \{\ln(\beta y_j)\}^2 - c_a \frac{\lambda(\beta\tau)^\alpha}{\{1+\lambda(\beta\tau)^\alpha\}^2} \{\ln(\beta\tau)\}^2 \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = \frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} = &-2\sum_{i=1}^{n(1-\pi)} \delta_{ui} \frac{x_i^\alpha}{(1+\lambda x_i^\alpha)^2} \ln x_i - c_u \frac{\tau^\alpha}{(1+\lambda\tau^\alpha)^2} \ln\tau \\ &-2\sum_{j=1}^{n\pi} \delta_{aj} \frac{(\beta y_j)^\alpha}{\{1+\lambda(\beta y_j)^\alpha\}^2} \ln(\beta y_j) - c_a \frac{(\beta\tau)^\alpha}{\{1+\lambda(\beta\tau)^\alpha\}^2} \ln(\beta\tau) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} = &\frac{n_a}{\beta} - 2\sum_{j=1}^{n\pi} \delta_{aj} \left[\frac{\lambda\alpha(\beta y_j)^{\alpha-1} y_j}{\{1+\lambda(\beta y_j)^\alpha\}^2} \ln(\beta y_j) + \frac{\lambda(\beta y_j)^\alpha}{1+\lambda(\beta y_j)^\alpha} \frac{1}{\beta} \right] \\ &- c_a \left[\frac{\lambda\alpha(\beta\tau)^{\alpha-1}\tau}{\{1+\lambda(\beta\tau)^\alpha\}^2} \ln(\beta\tau) + \frac{\lambda(\beta\tau)^\alpha}{1+\lambda(\beta\tau)^\alpha} \frac{1}{\beta} \right] \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \lambda^2} = &-\frac{n_u}{\lambda^2} + 2\sum_{i=1}^{n(1-\pi)} \delta_{ui} \left(\frac{x_i^\alpha}{1+\lambda x_i^\alpha} \right)^2 + c_u \left(\frac{\tau^\alpha}{1+\lambda\tau^\alpha} \right)^2 \\ &-\frac{n_a}{\lambda^2} + 2\sum_{j=1}^{n\pi} \delta_{aj} \left(\frac{(\beta y_j)^\alpha}{1+\lambda(\beta y_j)^\alpha} \right)^2 + c_a \left(\frac{(\beta\tau)^\alpha}{1+\lambda(\beta\tau)^\alpha} \right)^2 \end{aligned}$$

$$\frac{\partial^2 \ln L}{\partial \lambda \partial \beta} = \frac{\partial^2 \ln L}{\partial \beta \partial \lambda} = -2\sum_{j=1}^{n\pi} \delta_{aj} \frac{\alpha(\beta y_j)^{\alpha-1} y_j}{\{1+\lambda(\beta y_j)^\alpha\}^2} - c_a \frac{\alpha(\beta\tau)^{\alpha-1}\tau}{\{1+\lambda(\beta\tau)^\alpha\}^2}$$

$$\begin{aligned} \frac{\partial^2 \ln L}{\partial \beta^2} = &-n_a \frac{\alpha}{\beta^2} - 2\sum_{j=1}^{n\pi} \delta_{aj} \frac{\lambda\alpha(\beta y_j)^{\alpha-2} y_j^2 \{\alpha - 1 - \lambda(\beta y_j)^\alpha\}}{\{1+\lambda(\beta y_j)^\alpha\}^2} \\ &- c_a \frac{\lambda\alpha(\beta\tau)^{\alpha-2}\tau^2\{\alpha-1-\lambda(\beta\tau)^\alpha\}}{\{1+\lambda(\beta\tau)^\alpha\}^2} \end{aligned}$$

Thus, an asymptotic $(1-c) \times 100\%$ confidence intervals for α, λ, and β are given by following

$$\widehat{\alpha} \pm z_{c/2}\sqrt{\widehat{\Sigma}_{11}}, \widehat{\lambda} \pm z_{c/2}\sqrt{\widehat{\Sigma}_{22}}, \text{ and } \widehat{\beta} \pm z_{c/2}\sqrt{\widehat{\Sigma}_{33}},$$

where z_c is the $100c$ upper percentage point of the standard normal distribution and $\widehat{\Sigma}_{kk}$ is the (k,k) component of $\widehat{\Sigma}$.

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2.3 Type-II censored data

Another censoring often used in testing of equipments is Type-II censoring where all items are put on test at the same time, and the test is terminated when the predetermined r of the items have failed. Such an experiment may save time and money because it could take a very long time for all items to fail. It is also true that the statistical treatment of Type-II censored data is simpler because the data consists of the r smallest lifetimes in a random sample of lifetimes, so that the theory of order statistics is directly applicable to determining the likelihood and any inferential technique employed.

For a constant-stress PALT with Type-II censoring, the failure times consist of rth smallest lifetimes $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(r)}$ out of a random sample of $n(1-\pi)$ lifetimes $X_1,\ldots,X_{n(1-\pi)}$ under normal use conditions and $y_{(1)} \leq y_{(2)} \leq \ldots \leq y_{(r)}$ out of a random sample of $n\pi$ lifetimes $Y_1,\ldots,Y_{n\pi}$ at accelerated conditions respectively. Let δ_{ui} and δ_{aj} denote the failure indicators such that

$$\delta_{ui} = \begin{cases} 1 & x_i \leq x_{(r)} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1,\ldots,n(1-\pi)$ and

$$\delta_{aj} = \begin{cases} 1 & y_j \leq y_{(r)} \\ 0 & \text{otherwise} \end{cases}$$

for $j = 1,\ldots,n\pi$. The likelihood function for $\{(x_i,\delta_{ui}) : i = 1,\ldots,n(1-\pi)\}$ at normal use conditions is given by

$$L_u(x_i,\delta_{ui}|\alpha,\lambda) = \frac{(n-n\pi)!}{(n-n\pi-r)!}\left[\prod_{i=1}^{r}\frac{\alpha x_{(i)}{}^{\alpha-1}\lambda}{(1+\lambda x_{(i)}{}^{\alpha})^2}\right]\left\{\frac{1}{1+\lambda x_{(r)}{}^{\alpha}}\right\}^{n-n\pi-r}$$

and the likelihood function for $\{(y_j,\delta_{aj}) : j = 1,\ldots,n\pi\}$ at accelerated conditions is given by

$$L_a(y_j,\delta_{aj}|\alpha,\lambda,\beta) = \frac{(n\pi)!}{(n\pi-r)!}\left[\prod_{j=1}^{r}\frac{\beta\alpha(\beta y_{(j)})^{\alpha-1}\lambda}{\{1+\lambda(\beta y_{(j)})^{\alpha}\}^2}\right]\left\{\frac{1}{1+\lambda(\beta y_{(r)})^{\alpha}}\right\}^{n\pi-r}.$$

Let c_u and c_a be the number of items censored at normal and accelerated conditions respectively. That is, $c_u = n(1-\pi)-r$ and $c_a = n\pi - r$. Then, the log likelihood function for $\{(x_i,\delta_{ui}),(y_j,\delta_{aj}) : i = 1,\ldots,n(1-\pi), j = 1,\ldots,n\pi\}$ is written as

$$\begin{aligned}\ln L =& \ln(n-n\pi)! - \ln c_u! + \ln(n\pi)! - \ln c_a! \\ &+ \sum_{i=1}^{r}\left\{\ln\alpha + (\alpha-1)\ln x_{(i)} + \ln\lambda - 2\ln(1+\lambda x_{(i)}{}^{\alpha})\right\} - c_u\ln(1+\lambda x_{(r)}{}^{\alpha}) \\ &+ \sum_{j=1}^{r}\left\{\ln\alpha + \ln\beta + (\alpha-1)\ln(\beta y_{(j)}) + \ln\lambda - 2\ln(1+\lambda(\beta y_{(j)})^{\alpha})\right\} - c_a\ln\{1+\lambda(\beta y_{(r)})^{\alpha}\}.\end{aligned}$$

The maximum likelihood estimates of the parameters α, λ and β are solutions to the system of likelihood equations obtained by

$$\begin{aligned}\frac{\partial\ln L}{\partial\alpha} =& \frac{2r}{\alpha} + \sum_{i=1}^{r}\ln x_{(i)}\left(1 - 2\frac{\lambda x_{(i)}{}^{\alpha}}{1+\lambda x_{(i)}{}^{\alpha}}\right) - c_u\frac{\lambda x_{(r)}{}^{\alpha}}{1+\lambda x_{(r)}{}^{\alpha}}\ln x_{(r)} \\ &+ \sum_{j=1}^{r}\ln(\beta y_{(j)})\left(1 - 2\frac{\lambda(\beta y_{(j)})^{\alpha}}{1+\lambda(\beta y_{(j)})^{\alpha}}\right) - c_a\frac{\lambda(\beta y_{(r)})^{\alpha}}{1+\lambda(\beta y_{(r)})^{\alpha}}\ln(\beta y_{(r)}) = 0\end{aligned}$$

$$\begin{aligned}\frac{\partial\ln L}{\partial\lambda} =& \frac{2r}{\lambda} - 2\sum_{i=1}^{r}\frac{x_{(i)}{}^{\alpha}}{1+\lambda x_{(i)}{}^{\alpha}} - c_u\frac{x_{(r)}{}^{\alpha}}{1+\lambda x_{(r)}{}^{\alpha}} \\ &- 2\sum_{j=1}^{r}\frac{(\beta y_{(j)})^{\alpha}}{1+\lambda(\beta y_{(j)})^{\alpha}} - c_a\frac{(\beta y_{(r)})^{\alpha}}{1+\lambda(\beta y_{(r)})^{\alpha}} = 0\end{aligned}$$

$$\frac{\partial\ln L}{\partial\beta} = r\frac{\alpha}{\beta} - 2\sum_{j=1}^{r}\frac{\lambda\alpha(\beta y_{(j)})^{\alpha}y_{(j)}}{1+\lambda(\beta y_{(j)})^{\alpha}} - c_a\frac{\lambda\alpha(\beta y_{(r)})^{\alpha}y_{(r)}}{1+\lambda(\beta y_{(r)})^{\alpha}} = 0$$

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Similarly, an iterative method such as the Newton-Raphson method is used to solve the equations to obtain MLEs and asymptotic confidence intervals based on the asymptotic normal distribution of MLEs are obtained here. Hence the asymptotic variance of the maximum likelihood estimates can be obtained by the inverse of observed Fisher information matrix which is evaluated at the MLE

$$\widehat{\Omega} = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \alpha^2} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} & -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} & -\frac{\partial^2 \ln L}{\partial \lambda^2} & -\frac{\partial^2 \ln L}{\partial \lambda \partial \beta} \\ -\frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & -\frac{\partial^2 \ln L}{\partial \beta \partial \lambda} & -\frac{\partial^2 \ln L}{\partial \beta^2} \end{bmatrix}^{-1}.$$

The elements of the observed Fisher information matrix are as follows:

$$\frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{2r}{\alpha^2} - 2\sum_{i=1}^{r} \frac{\lambda {x_{(i)}}^{\alpha}}{(1+\lambda {x_{(i)}}^{\alpha})^2} (\ln x_{(i)})^2 - c_u \frac{\lambda {x_{(r)}}^{\alpha}}{(1+\lambda {x_{(r)}}^{\alpha})^2} (\ln x_{(r)})^2$$
$$- 2\sum_{j=1}^{r} \frac{\lambda (\beta y_{(j)})^{\alpha}}{\{1+\lambda(\beta y_{(j)})^{\alpha}\}^2} \{\ln(\beta y_{(j)})\}^2 - c_a \frac{\lambda (\beta y_{(r)})^{\alpha}}{\{1+\lambda(\beta y_{(r)})^{\alpha}\}^2} \{\ln(\beta y_{(r)})\}^2$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = \frac{\partial^2 \ln L}{\partial \lambda \partial \alpha} = -2\sum_{i=1}^{r} \frac{{x_{(i)}}^{\alpha}}{(1+\lambda {x_{(i)}}^{\alpha})^2} \ln x_{(i)} - c_u \frac{{x_{(r)}}^{\alpha}}{(1+\lambda {x_{(r)}}^{\alpha})^2} \ln x_{(r)}$$
$$- 2\sum_{j=1}^{r} \frac{(\beta y_{(j)})^{\alpha}}{\{1+\lambda(\beta y_{(j)})^{\alpha}\}^2} \ln(\beta y_{(j)}) - c_a \frac{(\beta y_{(r)})^{\alpha}}{\{1+\lambda(\beta y_{(r)})^{\alpha}\}^2} \ln(\beta y_{(r)})$$

$$\frac{\partial^2 \ln L}{\partial \alpha \partial \beta} = \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} = \frac{r}{\beta} - 2\sum_{j=1}^{r} \left[\frac{\lambda \alpha (\beta y_{(j)})^{\alpha-1} y_{(j)}}{\{1+\lambda(\beta y_{(j)})^{\alpha}\}^2} \ln(\beta y_{(j)}) + \frac{\lambda (\beta y_{(j)})^{\alpha}}{1+\lambda(\beta y_{(j)})^{\alpha}} \frac{1}{\beta} \right]$$
$$- c_a \left[\frac{\lambda \alpha (\beta y_{(r)})^{\alpha-1} y_{(r)}}{\{1+\lambda(\beta y_{(r)})^{\alpha}\}^2} \ln(\beta y_{(r)}) + \frac{\lambda (\beta y_{(r)})^{\alpha}}{1+\lambda(\beta y_{(r)})^{\alpha}} \frac{1}{\beta} \right]$$

$$\frac{\partial^2 \ln L}{\partial \lambda^2} = -\frac{2r}{\lambda^2} + 2\sum_{i=1}^{r} \left(\frac{{x_{(i)}}^{\alpha}}{1+\lambda {x_{(i)}}^{\alpha}} \right)^2 + c_u \left(\frac{{x_{(r)}}^{\alpha}}{1+\lambda {x_{(r)}}^{\alpha}} \right)^2$$
$$+ 2\sum_{j=1}^{r} \left(\frac{(\beta y_{(j)})^{\alpha}}{1+\lambda(\beta y_{(j)})^{\alpha}} \right)^2 + c_a \left(\frac{(\beta y_{(r)})^{\alpha}}{1+\lambda(\beta y_{(r)})^{\alpha}} \right)^2$$

$$\frac{\partial^2 \ln L}{\partial \lambda \partial \beta} = \frac{\partial^2 \ln L}{\partial \beta \partial \lambda} = -2\sum_{j=1}^{r} \frac{\alpha (\beta y_{(j)})^{\alpha-1} y_{(j)}}{\{1+\lambda(\beta y_{(j)})^{\alpha}\}^2} - c_a \frac{\alpha (\beta y_{(r)})^{\alpha-1} y_{(r)}}{\{1+\lambda(\beta y_{(r)})^{\alpha}\}^2}$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -r\frac{\alpha}{\beta^2} - 2\sum_{j=1}^{r} \frac{\lambda \alpha (\beta y_{(j)})^{\alpha-2} {y_{(j)}}^2 \{\alpha - 1 - \lambda(\beta y_{(j)})^{\alpha}\}}{\{1+\lambda(\beta y_{(j)})^{\alpha}\}^2}$$
$$- c_a \frac{\lambda \alpha (\beta y_{(r)})^{\alpha-2} {y_{(r)}}^2 \{\alpha - 1 - \lambda(\beta y_{(r)})^{\alpha}\}}{\{1+\lambda(\beta y_{(r)})^{\alpha}\}^2}$$

Thus, an asymptotic $(1-c)\times 100\%$ confidence intervals for α, λ, and β are given by following

$$\widehat{\alpha} \pm z_{c/2}\sqrt{\widehat{\Omega}_{11}}, \widehat{\lambda} \pm z_{c/2}\sqrt{\widehat{\Omega}_{22}}, \text{ and } \widehat{\beta} \pm z_{c/2}\sqrt{\widehat{\Omega}_{33}},$$

where z_c is the $100c$ upper percentage point of the standard normal distribution and $\widehat{\Omega}_{kk}$ is the (k,k) component of $\widehat{\Sigma}$.

3 Simulations

To evaluate the statistical properties of the estimates and the performance of confidence intervals, simulation studies are conducted. For Type-I censored data, a random sample of $x_1, x_2, \ldots, x_{n(1-\pi)}$ under normal use conditions is generated

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Table 1: Summary statistics for $\alpha = 1$, $\lambda = 2.5$, $\beta = 1.5$, and $\pi = 25\%$. MLE is the mean of the estimates, SE is the standard error of the estimates, SEE is the mean of the standard error estimates, and CP is the coverage probability of the proposed 95% confidence interval.

Sample size	Parameter	MLE	SE	SEE	CP
	$\widehat{\alpha}$	1.0113	0.0829	0.0816	94.2
150	$\widehat{\lambda}$	2.5879	0.4922	0.4885	95.4
	$\widehat{\beta}$	1.6020	0.5263	0.5287	94.4
	$\widehat{\alpha}$	1.0090	0.0702	0.0703	95.6
200	$\widehat{\lambda}$	2.5841	0.4328	0.4208	96.2
	$\widehat{\beta}$	1.5488	0.4354	0.4394	94.4
	$\widehat{\alpha}$	1.0061	0.0581	0.0573	95.0
300	$\widehat{\lambda}$	2.5496	0.3501	0.3380	94.2
	$\widehat{\beta}$	1.5476	0.3743	0.3590	94.0
	$\widehat{\alpha}$	1.0035	0.0454	0.0443	94.5
500	$\widehat{\lambda}$	2.5282	0.2747	0.2590	94.1
	$\widehat{\beta}$	1.5210	0.2813	0.2738	94.0

Table 2: Summary statistics for $\alpha = 1$, $\lambda = 2.5$, $\beta = 1.5$, and $\pi = 50\%$. MLE is the mean of the estimates, SE is the standard error of the estimates, SEE is the mean of the standard error estimates, and CP is the coverage probability of the proposed 95% confidence interval.

Sample size	Parameter	MLE	SE	SEE	CP
	$\widehat{\alpha}$	1.0114	0.0784	0.0801	95.7
150	$\widehat{\lambda}$	2.6119	0.5973	0.5793	94.6
	$\widehat{\beta}$	1.5698	0.4652	0.4473	93.2
	$\widehat{\alpha}$	1.0090	0.0694	0.0692	95.3
200	$\widehat{\lambda}$	2.5908	0.5285	0.4964	93.8
	$\widehat{\beta}$	1.5534	0.3851	0.3832	95.0
	$\widehat{\alpha}$	1.0058	0.0569	0.0564	94.9
300	$\widehat{\lambda}$	2.5586	0.3984	0.3992	95.0
	$\widehat{\beta}$	1.5269	0.3023	0.3075	93.6
	$\widehat{\alpha}$	1.0035	0.0447	0.0436	94.3
500	$\widehat{\lambda}$	2.5244	0.3174	0.3044	93.8
	$\widehat{\beta}$	1.5268	0.2396	0.2384	95.1

from the log-logistic distribution with the parameters $\alpha = 1$ and $\lambda = 2.5$. For accelerated lifetimes $y_1, y_2, \ldots, y_{n\pi}$ we consider the log-logistic distribution with acceleration factor $\beta = 1.5$. The lifetimes from both conditions are censored at $\tau = 1$. With acceleration factor $\beta = 1.5$, about 29% and 21% of lifetimes are censored under normal use condition and acceleration condition, respectively. We consider three different proportion π of the sample items allocated to accelerated conditions, $\pi = 25\%$, 50% and 75%. For Type-II censored data, the experiment continues until 40% of lifetimes at normal use condition occur with different sample sizes $n = 150$, 200, 300, and 500. The results give the mean of the estimates (MLE), standard error of the estimates (SE), mean of the standard error estimates (SEE), and coverage probability (CP) of the proposed 95% confidence interval based on 1000 replications.

The results from Tables 1–3 for Type-I censored data and Tables 4–5 for Type-II censored data indicate that the parameter estimates perform well. The bias of the maximum likelihood estimate decreases as the sample size increases and the asymptotic variances of the estimators are decreasing as the sample size increases. The standard error estimates are based on $\widehat{\Sigma}$ and $\widehat{\Omega}$ respectively and the estimates provides a fairly accurate of true variance of the estimates, and the corresponding confidence intervals have reasonable coverage probabilities. Simulation studies were also conducted with acceleration factor $\beta = 2$ and the results, not provided here, showed the same pattern.

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Table 3: Summary statistics for $\alpha = 1$, $\lambda = 2.5$, $\beta = 1.5$, and $\pi = 75\%$. MLE is the mean of the estimates, SE is the standard error of the estimates, SEE is the mean of the standard error estimates, and CP is the coverage probability of the proposed 95% confidence interval.

Sample size	Parameter	MLE	SE	SEE	CP
150	$\widehat{\alpha}$	1.0114	0.0805	0.0792	94.7
	$\widehat{\lambda}$	2.7005	0.8814	0.8194	93.8
	$\widehat{\beta}$	1.5792	0.5714	0.5246	92.4
200	$\widehat{\alpha}$	1.0091	0.0688	0.0682	94.7
	$\widehat{\lambda}$	2.6267	0.7551	0.6835	94.6
	$\widehat{\beta}$	1.5731	0.4474	0.4500	94.7
300	$\widehat{\alpha}$	1.0058	0.0557	0.0556	95.3
	$\widehat{\lambda}$	2.5825	0.5574	0.5473	95.1
	$\widehat{\beta}$	1.5429	0.3642	0.3602	94.3
500	$\widehat{\alpha}$	1.0034	0.0440	0.0430	94.0
	$\widehat{\lambda}$	2.5502	0.4331	0.4182	94.2
	$\widehat{\beta}$	1.5217	0.2744	0.2752	95.0

Table 4: Summary statistics for $\alpha = 1$, $\lambda = 2.5$, $\beta = 1.5$, $\pi = 30\%$ and $r = 40\%$. MLE is the mean of the estimates, SE is the standard error of the estimates, SEE is the mean of the standard error estimates, and CP is the coverage probability of the proposed 95% confidence interval.

Sample size	Parameter	MLE	SE	SEE	CP
150	$\widehat{\alpha}$	1.0248	0.0976	0.0969	95.0
	$\widehat{\lambda}$	2.7633	0.7863	0.7104	96.7
	$\widehat{\beta}$	1.5352	0.5146	0.5012	91.5
200	$\widehat{\alpha}$	1.0202	0.0853	0.0835	95.3
	$\widehat{\lambda}$	2.7021	0.6324	0.5952	95.9
	$\widehat{\beta}$	1.5280	0.4356	0.4316	92.9
300	$\widehat{\alpha}$	1.0141	0.0708	0.0678	94.1
	$\widehat{\lambda}$	2.6313	0.5121	0.4691	95.8
	$\widehat{\beta}$	1.5345	0.3670	0.3549	92.8
500	$\widehat{\alpha}$	1.0069	0.0541	0.0521	93.7
	$\widehat{\lambda}$	2.5663	0.3842	0.3514	94.2
	$\widehat{\beta}$	1.5163	0.2800	0.2728	94.6

Table 5: Summary statistics for $\alpha = 1$, $\lambda = 2.5$, $\beta = 1.5$, $\pi = 50\%$ and $r = 40\%$. MLE is the mean of the estimates, SE is the standard error of the estimates, SEE is the mean of the standard error estimates, and CP is the coverage probability of the proposed 95% confidence interval.

Sample size	Parameter	MLE	SE	SEE	CP
150	$\widehat{\alpha}$	1.0422	0.1181	0.1219	95.7
	$\widehat{\lambda}$	2.8958	1.0131	0.9151	96.1
	$\widehat{\beta}$	1.5764	0.5291	0.4897	92.1
200	$\widehat{\alpha}$	1.0301	0.1059	0.1043	95.1
	$\widehat{\lambda}$	2.7865	0.8390	0.7507	96.3
	$\widehat{\beta}$	1.5605	0.4333	0.4238	93.6
300	$\widehat{\alpha}$	1.0202	0.0858	0.0843	95.3
	$\widehat{\lambda}$	2.6869	0.6200	0.5822	96.4
	$\widehat{\beta}$	1.5310	0.3398	0.3414	93.6
500	$\widehat{\alpha}$	1.0124	0.0681	0.0648	93.5
	$\widehat{\lambda}$	2.6000	0.4659	0.4312	94.3
	$\widehat{\beta}$	1.5315	0.2720	0.2660	94.5

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4 Conclusions

In this study we have considered a constant-stress PALT for log-logistic lifetime distribution with Type-I or Type-II censored data. The maximum likelihood estimates of the model parameters and acceleration factor are obtained using the Newton-Raphson iterative method and their performances are discussed. The asymptotic confidence intervals of model parameters and acceleration factor are also obtained. From the simulation results it is easy to find that the maximum likelihood estimates have good statistical properties. Although the lifetime distribution is assumed to follow log-logistic distribution with Type-I or Type-II censoring, most of the methods can be applied to other distributions and other censoring schemes. This work is in progress and will be reported elsewhere.

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