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On The Diophantine Equation $(p^q-1)^x+p^{qy}=z^2$

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Abstract: We find non-negative integer solutions of the title equation,where p is a prime and $q>1$ is an integer.

Keywords: Diophantine equation

1 Introduction

The Diophantine equation of the type $a^x+b^y=c^z$ has been studied by many author's over the several years. Cao [2] proved that this equation has at most one solution under certain conditions. Acu [1] proved that the Diophantine equation $2^x+5^y=z^2$ has only two solutions in non-negative integers x,y and z. In 2011, Suvarnamani et al. [12,13] studied the Diophantine equation $2^x+p^y=z^2$ where p is a prime and $x,\ y,\ z$ are non-negative integers. Peker et al. [5] gave the non-negative integer solutions of the Diophantine equation of the form $(4^n)^x+p^y=z^2$, where p is an odd prime. In 2012, Sroysang [6] established that $(x,y,z)=(1,0,3)$ is the only non-negative integer solution of the Diophantine equation $8^x+19^y=z^2$. He [7] also established that $(x,y,z)=(1,0,2)$ is the only non-negative integer solution of the Diophantine equation $3^x+5^y=z^2$. Moreover he [8,11] showed that the Diophantine equation $31^x+32^y=z^2$ has no non-negative integer solution, but the Diophantine equation $2^x+3^y=z^2$ has three non-negative integer solutions. In 2013, Sroysang [10] showed that the Diophantine equation $23^x+32^y=z^2$ has no non-negative integer solution. In the same year, he [9] showed that the Diophantine equation $7^x+8^y=z^2$ has only one solution which is $(x,y,z)=(0,1,3)$ and he introduced an open problem regarding the set of all solutions (x,y,z) for the Diophantine equation $p^x+(p+1)^y=z^2$, where $x,\ y$ and z are non-negative integers. By attempting this open problem, Chotchaisthit [3] proved that $(x,y,z,p)\in\{(0,1,3,7),\ (2,2,5,3)\}$ are the only non-negative integer solutions of the Diophantine equation $p^x+(p+1)^y=z^2$ where p is a Mersenne prime.

In this paper we find the solutions of the Diophantine equation $(p^q-1)^x+p^{qy}=z^2$ in the non-negative integers $x,\ y,\ z,\ q$ and a prime p.

2 Main Results

We first state the Catalan's conjecture as a proposition which was proved by Mihailescu [4].

Proposition 2.1 [4]. $(a,\ b,\ x,\ y)=(3,\ 2,\ 2,\ 3)$ is the only solution of the Diophantine equation $a^x-b^y=1$, where $a,\ b,\ x$ and y are integers with $\min\{a,\ b,\ x,\ y\}>1$.

We now solve the Diophantine equation

$$(p^q-1)^x+p^{qy}=z^2 \tag{1}$$

where $x,\ y,\ z$, and $q(>1)$ are non-negative integers and p is a prime.
We find the solutions of the Diophantine equation (1) via the following theorems.

Theorem 2.2. The Diophantine equation

$$(2^q-1)^x+2^{qy}=z^2 \tag{2}$$

has only three solutions $(x,\ y,\ z,\ q)=(1,\ 0,\ 2,\ 2)$, $(x,\ y,\ z,\ q)=(0,\ 1,\ 3,\ 3)$ and $(x,\ y,\ z,\ q)=(2,\ 2,\ 5,\ 2)$.

Proof. We prove this theorem by dividing it into two parts.
Part-I: $y=0$.
In this case the equation (2) becomes

$$z^2-(2^q-1)^x=1 \tag{3}$$

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If $\min\{x,\ z\} > 1$ then by Proposition 2.1, the equation (3) has no solution.
Again the equation (3) has no solution whenever either $z = 0,\ 1$ or $x = 0$.
Now for $x = 1$, the equation (3) has only one solution which is given by $(x,\ z,\ q) = (1,\ 2,\ 2)$
Part-II: $y \geq 1$.
In the equation (2), we observed that z is odd and thus $z^2 \equiv 1 (\bmod\ 4)$.
Let $x = 0$, then the equation (2) becomes

$$2^{qy} = z^2 - 1 \tag{4}$$

Thus $2^{qy} = (z+1)(z-1)$ and hence there exist two integers m and n such that $2^m = z+1$ and $2^n = z-1$, where $m > n$ and

$$m + n = qy \tag{5}$$

Now $2^n(2^{m-n} - 1) = 2^m - 2^n = 2$.
This gives $m = 2$ and $n = 1$.
Since $q > 1$, equation (5) gives $q = 3$ and $y = 1$. Therefore $z = 2^n + 1 = 3$ and thus $(x,\ y,\ z,\ q) = (0,\ 1,\ 3,\ 3)$ is a solution of the equation (2).
Now let $x \geq 1$.
since $2^{qy} \equiv 0 (\bmod\ 4)$ and $z^2 \equiv 1 (\bmod\ 4)$, the equation (2) gives

$$(2^q - 1)^x \equiv 1 (\bmod\ 4) \tag{6}$$

Again

$$2^q - 1 \equiv 3 (\bmod\ 4) \tag{7}$$

Congruences (6) and (7) imply x is even.
Let $x = 2k$ for some integer $k \geq 1$. Then the equation (2) becomes

$$2^{qy} = z^2 - (2^q - 1)^{2k}$$

$$\Rightarrow 2^{qy} = (z + (2^q - 1)^k)(z - (2^q - 1)^k)$$

Thus we can find two non-negative integers r and s such that $2^r = z + (2^q - 1)^k$ and $2^s = z - (2^q - 1)^k$ with $r > s$ and

$$r + s = qy \tag{8}$$

Now $2^s(2^{r-s} - 1) = 2^r - 2^s = 2(2^q - 1)^k$.
This implies $s = 1$ and

$$2^{r-1} - (2^q - 1)^k = 1 \tag{9}$$

If $r > 2$ and $k > 1$, then by Proposition 2.1, the equation (9) has no solution.
Since $r \geq 0,\ q > 1$ and $k \geq 1$, it is remaining to examine when $r = 0,1,2$ or $k = 1$.
Clearly for $r = 0,1,2$, the equation (9) has no solution.
Now for $k = 1$, the equation (8) becomes

$$2^{r-1} = 2^{q-1} \tag{10}$$

From equations (8) and (10), we get

$$2^{qy-2} = 2^q$$

This gives $q = 2$ and $y = 2$ as $q > 1$.
Also $z = 2^s + (2^q - 1)^k = 5$.
Thus $(x,y,z,q) = (2,2,5,2)$ is a solution of the equation (2).

Theorem 2.3. Let p be an odd prime and x be an even integer. Then the equation (1) has no solution.

Proof. From the equation (1), we see that z is odd and hence $z^2 \equiv 1 (\bmod\ 4)$.
Let $y = 0$. Then the equation (1) becomes

$$z^2 - (p^q - 1)^x = 1 \tag{11}$$

If $\min\{x,\ z\} > 1$, then by Proposition 2.1, the equation (11) has no solution.
It is clear that the equation (11) has no solution when $z = 0,1$ or $x = 0$.
Let $y \geq 1$ and let $x = 2t$ for some integer $t \geq 1$. Then the equation (2) can be written as

$$p^{qy} = z^2 - (p^q - 1)^{2t}$$

$$\Rightarrow p^{qy} = (z + (p^q - 1)^t)(z - (p^q - 1)^t)$$

Thus we can find two non-negative integers a and b such that $p^a = z + (p^q - 1)^t$ and $p^b = z - (p^q - 1)^t$ with $a > b$ and $a + b = qy$.
Now
$p^b(p^{a-b} - 1) = p^a - p^b = 2(p^q - 1)^t$
$\Rightarrow 0 \equiv 2(-1)^t (\bmod\ p)$
Which is an absurd.
Hence the equation (1) has no solution.

Theorem 2.4. Let $p\ (\neq z)$ be an odd prime and $q \geq 1$ be a integer. Then $(x,z,p,q) = (3,3,3,1)$ is the only solution of the Diophantine equation

$$(p^q - 1)^x + 1 = z^2 \tag{12}$$

Proof. By Proposition 2.1, the equation (12) has a unique solution $(x,z,p,q) = (3,3,3,1)$ if $\min\{x,z\} > 1$.
It is remaining to examine when $x = 0,1$ or $z = 0,1$.
Clearly The equation (12) has no solution when $x = 0$ or $z = 0,1$
Again if $x = 1$, then the equation (12) gives

$$z^2 = p^q$$

This implies $q = 2$ and $p = z$.
This contradicts to $p \neq z$.
Thus once again the equation (12) has no solution.
Remark: For $p = 3$ and $q = 1$, the equation (1) becomes

$$2^x + 3^y = z^2 \tag{13}$$

Suvarnamani [13] showed that $(x,y,z) \in \{(0,1,2),\ (3,0,3),\ (4,2,5)\}$ are the only solutions of the equation (13) in the non-negative integers x,y and z. Sroysang [11] also found the same solutions of this equation.

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