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Solution to a Class of First-Order Fuzzy Cauchy-Euler Differential Equations

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Abstract: In this paper, we study the solution related to a class of fuzzy differential equations which are called fuzzy Cauchy-Euler differential equations of first-order. We first, investigate the proper spaces which are contained of generalized differentiable fuzzy-valued functions, based on the solution functions representation in the crisp case of equation. Next, we obtain the solution function and the existence conditions of it, in details. Finally, it is illustrated by solving two numerical examples.

Keywords: Fuzzy differential equations, Generalized differentiability, Fuzzy-valued functions

1 Introduction

Solving differential equations in the conditions of uncertainty is interesting and applicable in mathematics and engineering sciences. Up to now, many studies have been done on such equations (for examples [1,2,3,4,5,6,7,8]). Also, some numerical methods can be seen in [9,10,11,12]. The first-order linear fuzzy differential equations are samples of such equations that are presented more in application. Such an equation may be appeared in one of three forms, with fuzzy initial value $y(0) = y_0$
(a). $y'(t) = a(t)y(t) + b(t)$,
(b). $y'(t) + (-a(t))y(t) = b(t)$ or
(c). $y'(t) + (-b(t)) = a(t)y(t)$.

The possibility solutions of equation (a) are obtained in [2] and [13], by assuming $a(t)$ is a continuous positive or negative real function, defined on time interval $[0,T]$ and using the generalized differentiability concept (G-differentiability), which is introduced by Bede et al. [1]. Recently, we obtain all solutions of equations (a), (b) and (c), under generalized differentiability concept, by using length function properties on fuzzy-valued functions [14]. A class of linear differential equations are known to Cauchy-Euler equations which appear in number of physics and engineering applications. In this paper, we consider a Cauchy-Euler equation of first-order under uncertainty and in the following form

$$\begin{cases} (t-\alpha)y'(t) + u(t) = \beta y(t), \\ y(a) = y_0, \quad t \in I = [a,b], \ \ t \neq \alpha. \end{cases} \tag{1}$$

Where y_0 is a fuzzy number, and α and β are real numbers. We wish to study the existence of solution to the equation (1), when the function $u(t)$ is a fuzzy polynomial of degree at most n, around point $t = \alpha$, i.e.

$$u(t) = \sum_{i=0}^{n} (t-\alpha)^i u_i \tag{2}$$

where u_i for $i = 0, 1, \ldots, n$ are fuzzy numbers. For this end, we first, point out the structure of solutions related to the crisp form of the equation. Accordingly, we introduce the proper spaces of fuzzy-valued functions, which can be included the solution of the problem (1). On these spaces, we give some results of generalized differentiability, ((i) or (ii)-differentiability). Next, we explain the details attaining to solution formula and express the conditions of it's existence, given as Theorem 4.1. Finally, the solution expression is applied for solving two numerical examples.

2 Preliminaries

An arbitrary fuzzy number can be represented by an ordered pair of functions $u = (u_r^-, u_r^+)$ on interval $[0,1]$ as parametric form such that u_r^- is a left continuous, bounded and non-decreasing function, u_r^+ is a left continuous, bounded and non-increasing function and $u_r^- \leq u_r^+$. We denote the r-cut form of a fuzzy number u as $[u]^r = [u_r^-, u_r^+]$. The set all fuzzy numbers is denoted by

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$\mathbb{R}_F$. If $u,v\in\mathbb{R}_F$ and $\lambda\in\mathbb{R}$, then $u+v$ and λu are defined by

$$\begin{aligned}[u+v]^r &= [u]^r+[v]^r\\ &= [u_r^-+v_r^-,u_r^++v_r^+],\end{aligned}$$

and

$$\begin{aligned}[\lambda u]^r &= \lambda[u]^r\\ &= [\min(\lambda u_r^-,\lambda u_r^+),\max(\lambda u_r^-,\lambda u_r^+)],\end{aligned}$$

for all $r\in[0,1]$ (see [15]).

Definition 2.1.[1,4]. Let $u,v\in\mathbb{R}_F$. If there exists $w\in\mathbb{R}_F$ such that, $u=v+w$ then w is called the H-difference of u,v and it is denoted as $u\ominus v$.
We note that $u\ominus v\neq u-v=u+(-1)v$.

Definition 2.2.[1,14]. Let $f:(a,b)\to\mathbb{R}_F$. Fix $t_0\in(a,b)$. We say f is G-differentiable at t_0, if the H-differences $f(t_0+h)\ominus f(t_0)$, $f(t_0)\ominus f(t_0-h)$ for all $h>0$ or all $h<0$, sufficiently close to 0 exist, and an element $f'(t_0)\in\mathbb{R}_F$ exists, such that either
(i):

$$\lim_{h\to0^+}\frac{f(t_0+h)\ominus f(t_0)}{h}=\lim_{h\to0^+}\frac{f(t_0)\ominus f(t_0-h)}{h}=f'(t_0)$$

or (ii):

$$\lim_{h\to0^-}\frac{f(t_0+h)\ominus f(t_0)}{h}=\lim_{h\to0^-}\frac{f(t_0)\ominus f(t_0-h)}{h}=f'(t_0)$$

or (iii):

$$\lim_{h\to0^+}\frac{f(t_0+h)\ominus f(t_0)}{h}=\lim_{h\to0^-}\frac{f(t_0+h)\ominus f(t_0)}{h}=f'(t_0)$$

or (iv):

$$\lim_{h\to0^+}\frac{f(t_0)\ominus f(t_0-h)}{h}=\lim_{h\to0^-}\frac{f(t_0)\ominus f(t_0-h)}{h}=f'(t_0).$$

A straightforward way to calculating $[f'(t)]^r$ is visible in [14], Theorem 2.13, also see [16], Theorem 5.

Definition 2.3.[17]. We say that a point $t_0\in(a,b)$ is a switching point for the differentiability of f, if G-differentiability changes from type (i) to type (ii), or from type (ii) to type (i), in Definition 2.2.

Chalco et al. [18], have been demonstrated if t_0 be a switching point, then f at t_0 is a differentiable function in the sense (iii) or (iv). Moreover, if f is differentiable on over (a,b) in the sense (iii) (or (iv)) then $f'(t)=\{c\}$, where $c\in\mathbb{R}$ is a real number.

3 Structure of solution space

In this section, we attempt to obtain a proper space of fuzzy-valued functions which can be included the solution of equation (1). For this purpose, let us point out the structure of solution function in the crisp case of the equation. In the crisp case, the solution is considered as exponential function $y(t)=(t-\alpha)^r$, where the unknown number r should be found such that y satisfies in the equation (1). Since the behaviour of solution in the uncertainty case should reflects the behaviour of solution in the crisp case of the equation, namely 1-cut equation of (1), then we define a space of requirement fuzzy-valued functions as follows:
Consider $\alpha\in\mathbb{R}$ and $I=[a,b]$. We set

$$F_\alpha(I)=\Big\{f:I\to\mathbb{R}_F|\ f(t)=\sum_{i=0}^{n}|t-\alpha|^{\alpha_i}\,v_i,\ \alpha_i\in\mathbb{R},\ v_i\in\mathbb{R}_F\Big\}.$$

The following result shows that the set $F_\alpha(I)$ includes some G-differentiable functions on (a,b), except presumably at point $t=\alpha$.

Theorem 3.1. Consider $f(t)=\sum_{i=0}^{n}|t-\alpha|^{\alpha_i}\ v_i\in F_\alpha(I)$ such that numbers α_i for $i=0,1,\ldots,n$ have same sign. In this case
(a). For $\alpha\le a$, the function f is (i)-differentiable on (a,b), if $\alpha_i\ge0$ and is (ii)-differentiable on (a,b), if $\alpha_i\le0$.
(b). For $\alpha\ge b$, the function f is (ii)-differentiable on (a,b), if $\alpha_i\ge0$ and is (i)-differentiable on (a,b), if $\alpha_i\le0$.
(c). For $a<\alpha<b$, if $\alpha_i\ge0$ then f is (ii)-differentiable on (a,α) and (i)-differentiable on (α,b) and if $\alpha_i\le0$ then f is (i)-differentiable on (a,α) and (ii)-differentiable on (α,b).
Also, for each cases (a), (b) and (c), we have

$$f'(t)=\sum_{i=0}^{n}\alpha_i|t-\alpha|^{\alpha_i-1}\,v_i' \tag{3}$$

where $v_i'=\begin{cases}-v_i; & a<t<\alpha<b\ \ or\ \ \alpha\ge b,\\ v_i; & a<\alpha<t<b,\ \ or\ \ \alpha\le a\end{cases}$.

Proof. Let us, denote $f(t)=\sum_{i=0}^{n}f_i(t)$, where $f_i(t)=g_i(t)\,v_i$ and $g_i(t)=|t-\alpha|^{\alpha_i}$. According to Lemma 4 in [2], it is sufficient to show that the functions $f_i(t)$ have the same type of G-differentiability. For case (a), we get $g_i(t)=(t-\alpha)^{\alpha_i}$, for $i\in\{0,1,\ldots,n\}$ and then

$$g_i(t)g_i'(t)=\alpha_i(t-\alpha)^{2\alpha_i-1},\ \ t\in(a,b). \tag{4}$$

Therefore, the product $g_i(t)g_i'(t)$ has the same sign as α_i. Since the numbers α_i have the same sign, then by cases (a) and (b) of Theorem 5, in [2], we find that the functions $f_i(t)$ are (i)-differentiable, if $\alpha_i\ge0$, $i=0,1,\ldots,n$ and (ii)-differentiable if $\alpha_i\le0$, $i=0,1,\ldots,n$.
(b). Since $\alpha\ge b$, we get for $i=0,1,\ldots,n$, following

$$g_i(t)g_i'(t)=-\alpha_i(\alpha-t)^{2\alpha_i-1},\ \ t\in(a,b). \tag{5}$$

Therefore, the reasoning is similar to case (a), with $-\alpha_i$ instead of α.
(c). Since $\alpha \in (a,b)$, then the functions $g_i(t)$ are satisfy in equations (5) and (4) on intervals (a,α) and (α,b), respectively. Therefore, the reasoning is straightforward for this case, by considering the reasons (a) and (b).
Finally, by Theorem 2 and Lemma 4 in [2], we get $f'(t) = \sum_{i=0}^{n} g_i'(t)\ v_i$, where $g_i'(t) = \alpha_i(t-\alpha)^{\alpha_i - 1}$, for when $a < \alpha < t < b$ or $\alpha \le a$ and $g_i'(t) = -\alpha_i(\alpha - t)^{\alpha_i - 1}$, for when $a < t < \alpha < b$ or $\alpha \ge b$, which simply gives us the equality (3). □.

As a consequently, we should consider the special class of functions belong to $F_\alpha(I)$. In fact, Theorem 3.1 shows that, if the values $\alpha_0, \alpha_1, ..., \alpha_n$, associated to function $f \in F_\alpha(I)$, are all non-negative or all non-positive, then f is appropriate for using from the point of view G-differentiability. We thus, consider two the following family of functions

$$F_\alpha^+(I) = \Big\{ f \in F_\alpha(I) |\ \alpha_i \ge 0,\ i = 0, 1, ..., n \Big\}$$

and

$$F_\alpha^-(I) = \Big\{ f \in F_\alpha(I) |\ \alpha_i \le 0,\ i = 0, 1, ..., n \Big\}.$$

One is note that, if $f \in F_\alpha^-(I)$ and $\alpha \in (a,b)$, then the functions f and f' are discontinues at point $t = \alpha$. While if $f \in F_\alpha^+(I)$ and $\alpha \in (a,b)$, then by (3), the functions f and f' are continuous on interval I, provided that $\alpha_i \ge 1$ for $i = 0, 1, ..., n$. Furthermore, we get the following result.

Corollary 3.2. Let $f(t) = \sum_{i=0}^{n} |t-\alpha|^{\alpha_i}\ v_i \in F_\alpha^+(I)$ and $\alpha \in (a,b)$. If $\alpha_i \in \{0\} \cup (1, +\infty)$ for $i = 0, 1, ..., n$, then f has a switching point at point $t = \alpha$ and further $f'(\alpha) = 0$.

Proof. Considering Theorem 3.1 case (c), it is sufficient that, we show f is (iii)-differentiable at point $t = \alpha$. For h sufficiently close to zero, such that $\alpha + h \in (a,b)$, we obtain

$$\lim_{h\to 0^+} \frac{f(\alpha+h) \ominus f(\alpha)}{h} = \lim_{h\to 0^+} \frac{1}{h} \sum_{i=0}^{n} h^{\alpha_i} v_i = 0$$

and

$$\lim_{h\to 0^-} \frac{f(\alpha+h) \ominus f(\alpha)}{h} = \lim_{h\to 0^-} \frac{1}{h} \sum_{i=0}^{n} (-h)^{\alpha_i} v_i = 0$$

that gives us the required conclusion.□.

In final this section, we consider fuzzy-values functions which are appeared in the fuzzy polynomial form. A fuzzy polynomial around point α and considered on interval I, can be seen as a function with two criteria belong to $F_\alpha^+(I)$. Indeed

$$\sum_{i=0}^{n} (t-\alpha)^i u_i = \sum_{i=0}^{n} |t-\alpha|^i u_i', \qquad (6)$$

where $u_i' = \begin{cases} -u_i; & t < \alpha,\ i \text{ is odd} \\ u_i; & Otherwise \end{cases}.$

Remark 3.3. Consider the fuzzy polynomial $f(t) = \sum_{i=0}^{n} (t-\alpha)^i u_i$ on the real axes t. By Theorem 3.1 and equality (6), we find that if $t > \alpha$ then f is (i)-differentiable and if $t < \alpha$ then f is (ii)-differentiable and also, for both cases (i) or (ii)-differentiability, we have

$$f'(t) = \sum_{i=1}^{n} i(t-\alpha)^{i-1} u_i. \qquad (7)$$

Furthermore, it is easy to deduce that

$$\lim_{h\to 0^+} \frac{f(\alpha+h) \ominus f(\alpha)}{h} = \lim_{h\to 0^-} \frac{f(\alpha+h) \ominus f(\alpha)}{h} = v_1,$$

that results that f has a switching point at $t = \alpha$.

4 Solution of fuzzy Cauchy-Euler equation

In this section, obtaining solution for fuzzy Cauchy-Euler equation in form (1), is discussed where $u(t)$ is given as (2). Since $u(t)$ is a fuzzy polynomial around α, then we focus on the space $F_\alpha^+(I)$ for sake of finding solution. Suppose that the equation

$$(t-\alpha)y'(t) + u(t) = \beta y(t) \qquad (8)$$

has solution as fuzzy polynomial $y_p(t) = \sum_{i=0}^{n} (t-\alpha)^i v_i$. Then

$$(t-\alpha)y_p'(t) + u(t) = \beta y_p(t)$$

By substituting (7), we get the following equality

$$\sum_{i=1}^{n} i(t-\alpha)^i v_i + \sum_{i=0}^{n} (t-\alpha)^i u_i = \beta \sum_{i=0}^{n} (t-\alpha)^i v_i.$$

It is easy to deduce that

$$m v_m + u_m = \beta v_m, \quad m = 0, 1, ..., n$$

that is

$$\beta v_m \ominus m v_m = u_m, \quad m = 0, 1, ..., n.$$

If we assume that $\beta > n$ then the last equality is equal to $(\beta - m)v_m = u_m$, that means that

$$v_m = \frac{1}{\beta - m} u_m, \quad m = 0, 1, ..., n.$$

We thus, obtain y_p as the following function

$$y_p(t) = \sum_{i=0}^{n} \frac{(t-\alpha)^i}{\beta - i} u_i, \qquad (9)$$

which is obtained, uniquely.

Now, we consider the equation (8) on interval $I = [a,b]$, with fuzzy initial condition $y(a) = y_0$. For the sake of instituting the initial condition, we addition a complementary function $y_h(t)$ and represent the solution function as

$$y(t) = y_p(t) + y_h(t).$$

Whereas the polynomial $y_p(t)$ is unique solution of equation (8), then the function $y_h(t)$ should be satisfied the following homogeneous equation

$$(t-\alpha)y_h'(t) = \beta y_h(t).$$

It is easy to check that the last equation has solution as follows

$$y_h(t) = \begin{cases} -(\alpha-t)^\beta v; & a \le t < \alpha < b,\ or\ \alpha \ge b, \\ (t-\alpha)^\beta v; & a < \alpha < t \le b,\ or\ \alpha \le a, \end{cases} \tag{10}$$

where v is an arbitrary fuzzy number. Now, we obtain the fuzzy number v based on initial condition $y(a) = y_0$, i.e. $y_p(a) + y_h(a) = y_0$. By (10) we get

$$y_h(a) = \begin{cases} -(\alpha-a)^\beta c; & \alpha > a, \\ (a-\alpha)^\beta c; & \alpha < a, \end{cases}$$

where c is an arbitrary fuzzy number. This equality can be written as

$$y_h(a) = |a-\alpha|^\beta u$$

with $u = \begin{cases} -c; & \alpha > a, \\ c; & \alpha < a. \end{cases}$

Considering (9), the initial condition leads to the following equality

$$\sum_{i=0}^{n} \frac{(a-\alpha)i}{\beta - i} u_i + |a-\alpha|^\beta u = y_0,$$

that gives

$$u = |a-\alpha|^{-\beta} \Big\{ y_0 \ominus \Big(\sum_{i=0}^{n} \frac{(a-\alpha)^i}{\beta - i} u_i \Big) \Big\}, \tag{11}$$

provided that the H-difference exists.

Therefore, the solution function of equation (1) is found as follows

$$y(t) = \sum_{i=0}^{n} \frac{(t-\alpha)^i}{\beta - i} u_i + |t-\alpha|^\beta u,$$

with v' given as (11).
The G-differentiability properties the function $y(t)$ simply, can be produced by Theorem 3.1 and Remark 3.3. In fact, we have proved the following result.

Theorem 4.1. Consider the initial value problem

$$\begin{cases} (t-\alpha)y'(t) + \sum_{i=0}^{n}(t-\alpha)^i u_i = \beta y(t), \\ y(a) = y_0, \quad a \le t \le b,\ \ t \ne \alpha. \end{cases}$$

Where α and β are real numbers and y_0 is a fuzzy number. If $\beta > n$ and the H-difference

$$y_0 \ominus \sum_{i=0}^{n} \frac{(a-\alpha)^i}{\beta - i} u_i \tag{12}$$

exists, then the equation has solution as

$$y(t) = \sum_{i=0}^{n} \frac{(t-\alpha)^i}{\beta - i} u_i + |t-\alpha|^\beta u, \tag{13}$$

where $u = |a-\alpha|^{-\beta} \Big\{ y_0 \ominus \Big(\sum_{i=0}^{n} \frac{(a-\alpha)^i}{\beta-i} u_i \Big) \Big\}$. Further, $y(t)$ is a (i)-differentiable function on (a,b), for $\alpha \le a$ and it is a (ii)-differentiable function on (a,b), for $\alpha \ge b$ and has a switching point at $t = \alpha$, for $\alpha \in (a,b)$.

5 Numerical examples

In order to the practical application and observe the behavior of solution function, we solve two examples.

Example 5.1. Consider fuzzy differential equation

$$\begin{cases} (t-1)y'(t) + u_0 + (t-1)u_1 = \frac{3}{2} y(t), \\ y(0) = y_0, \quad t \ge 0,\ \ t \ne 1. \end{cases} \tag{14}$$

Where $[u_0]^r = \frac{1}{4}[r, 2-r]$, $[u_1]^r = \frac{1}{4}[1+r, 3-r]$ and $[y_0]^r = [r, 2-r]$, $0 \le r \le 1$. For these values, the H-difference (12) exist, because

$$\Big[y_0 \ominus (\frac{2}{3} u_0 - 2u_1) \Big]^r = \frac{1}{6}[9-2r, 13-2r].$$

We thus, obtain of (13), the following

$$y(t) = \frac{2}{3} u_0 + 2(t-1)u_1 + |t-1|^{\frac{3}{2}} u, \quad t > 0,$$

which represents two criteria as solution for equation (14), one is (i)-differentiable for $t > 1$, with the following r-cuts

$$\begin{aligned} [y_1(t)]^r = \Big[&\frac{1}{6} r + \frac{1}{2}(1+r)(t-1) + \frac{1}{6}(9+2r)(t-1)^{\frac{3}{2}}, \\ &\frac{1}{3} - \frac{1}{6} r + \frac{1}{2}(3-r)(t-1) + \frac{1}{6}(13-2r)(t-1)^{\frac{3}{2}} \Big] \end{aligned}$$

and other is (ii)-differentiable for $t < 1$, with the following r-cuts

$$\begin{aligned} [y_2(t)]^r = \Big[&\frac{1}{6} r + \frac{1}{2}(3-r)(t-1) + \frac{1}{6}(9+2r)(1-t)^{\frac{3}{2}}, \\ &\frac{1}{3} - \frac{1}{6} r + \frac{1}{2}(1+r)(t-1) + \frac{1}{6}(13-2r)(1-t)^{\frac{3}{2}} \Big]. \end{aligned}$$

Also, $y(t)$ has a switching point at $t = 1$, by Theorem 4.1.

The graphical representation of the solution function $y(t)$, mean functions $y_1(t)$ and $y_2(t)$, for three r-cuts $r = 0$, $r = 0.5$ and $r = 1$, can be seen in Fig. 1.

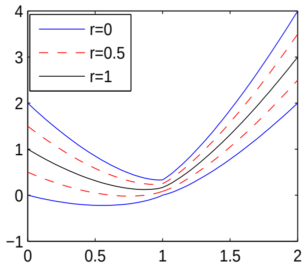

Fig. 1 The solution of example 5.1.

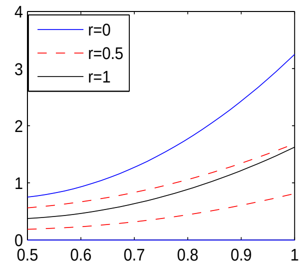

Fig. 2 The solution of example 5.2.

Example 5.2. Let us consider a fuzzy differential equation in different form as follows

$$\begin{cases} (t-\frac{1}{2})y'(t)+e^t a = 2y(t), \\ y(0)=y_0, \quad 0\le t\le 1, \;\; t\ne \frac{1}{2}. \end{cases} \tag{15}$$

Where $[a]^r=[\frac{1}{2}r, 1-\frac{1}{2}r]$, and $[y_0]^r=[-\frac{1}{2}+\frac{13}{8}, \frac{11}{4}-\frac{13}{8}r]$, $0\le r\le 1$.
Since $t\in[0,1]$, then a linear approximation can be a suitable alternative for factor $e^t a$. Therefore, we write

$$e^t a \cong (1+t)a = \Big((t-\frac{1}{2})+\frac{3}{2}\Big)a$$

that is $(t-\frac{1}{2})a+\frac{3}{2}a$, by assuming $t>\frac{1}{2}$. We thus, solve the following approximate equation instead of equation (15)

$$\begin{cases} (t-\frac{1}{2})y'(t)+u_0+(t-\frac{1}{2})u_1 = 2y(t), \\ y(0)=y_0, \quad \frac{1}{2}<t\le 1. \end{cases} \tag{16}$$

Where $u_0=\frac{3}{2}a$ and $u_1=a$.
The H-difference (12) exist, because

$$\Big[y_0\ominus(\frac{1}{2}u_0-\frac{1}{2}u_1)\Big]^r = \Big[y_0\ominus(\frac{3}{4}a-\frac{1}{2}a)\Big]^r$$
$$= [r, 2-r].$$

Then $u=4[r,2-r]$ and we obtain the solution function of (16) as follows, which is (i)-differentiable on interval $(\frac{1}{2},1)$, by Theorem 4.1.

$$y(t)=\frac{1}{2}u_0+(t-\frac{1}{2})u_1+(t-\frac{1}{2})^2u.$$

The other words

$$[y(t)]^r = \Big[\frac{3}{8}r+\frac{1}{2}r(t-\frac{1}{2})+4r(t-\frac{1}{2})^2,$$
$$\frac{3}{4}-\frac{3}{8}r+(1-\frac{1}{2}r)(t-\frac{1}{2})+4(2-r)(t-\frac{1}{2})^2\Big].$$

The graphical representation of solution, for $r=0$, $r=0.5$ and $r=1$ of r-cuts, can be seen in Fig. 2.

6 Conclusion

In this work, we obtain the solution function representation for a class of first-order fuzzy Cauchy-Euler differential equations, under generalized differentiability concept. Against the previous methods, proposed on fuzzy differential equations, we saw that the solution function can be obtained by studying the proper spaces of generalized differentiable fuzzy-valued functions. In other words, in our method, it is not necessary to turned the problem into a system of ordinary differential equations.

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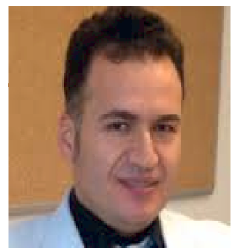

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