Mathematical Sciences Letters
An International Journal

http://dx.doi.org/10.18576/msl/050302

Fixed Points of Generalized Multivalued Contractive Mappings in Metric Spaces

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Received: 6 Feb. 2015, Revised: 12 Mar. 2016, Accepted: 16 Mar. 2016
Published online: 1 Sep. 2016

Abstract: In this paper, we establish some fixed point results for multivalued mappings satisfying a new generalize contractive conditions, involving some well-known contractive condition of rational type. The results of our paper generalized some theorems in the literature from single valued mappings to the multivalued mappings.

Keywords: Fixed point, rational type contraction, multivalued mapping.

1 Introduction and preliminaries

Fixed point theory for contraction mappings first studied by Banach in [3]. He proved that every contraction defined on a complete metric space has a unique fixed point. Since then the Banach contraction has been extended and generalized in many ways; see, for instance [1,4,5,9,11,12,13,14]. Among all these, an interesting generalization was given by Nadler [10]. He extended the Banach contraction principle from the single-valued mapping to the multivalued. Nadler proved the following theorem.

Theorem 1.*Let (X,d) be a complete metric space and let $T : X \to CB(X)$ be a multivalued mapping satisfying*

$$H(Tx,Ty) \leq kd(x,y),$$

for all $x,y \in X$, where k is a constant such that $k \in [0,1)$ and $CB(X)$ denotes the family of non-empty closed and bounded subset of X. Then T has a fixed point, i.e. there exists $x \in X$ such that $x \in T$.

On the other hand, Jaggi [7] and Dass, Gupta [6] have introduced the concept of contraction of rational type and they proved the existence of fixed point for this kind of mappings in complete metric spaces.
In this work, motivated and inspired by the above results, we defined a new contractive condition for multivalued mappings, involving some well-known contractions of rational type. We proved the existence of fixed point for this kind of mappings. Furthermore, it is shown that our results improve and extend those of A. Amini-Harandi [2] from single valued mappings to the multivalued mappings. To set up our main result in the next section, we need the following notations and definitions.

Let (X,d) be a metric space. We denote by $CB(X)$ the family of nonempty closed bounded subsets of X. Let $A,B \in CB(X)$, we will use the following notations:

$$D(x,B) = \inf\{d(x,y) : y \in B\}, \quad \forall x \in X,$$

$$H(A,B) = \max\{\sup_{x\in A} D(x,B), \sup_{y\in B} D(y,A)\}.$$

Then we called H the Hausdorff metric induced by d.

Definition 1.*The mapping $T : X \to CB(X)$ is continuous whenever $H(Tx_n,Tx) \to 0$ for all sequence $\{x_n\}$ in X with $x_n \to x$.*

Definition 2.*Let $T : X \to CB(X)$ be a multivalued mapping. A point $x \in X$ is said to be a fixed point of T if $x \in Tx$.*

2 Fixed Point Theorems

Throughout this work, we denote Λ the family of functions $\lambda(u_1,u_2,u_3,u_4,u_5) : \Re_+^5 \to \Re_+$, such that λ is nondecreasing in u_2,u_3,u_4,u_5 and $\lambda(u,u,v,u+v,0) \leq v$ for each $u,v \in \Re_+$, where $\Re_+ = [0,\infty)$.

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Theorem 2.*Let (X,d) be a complete metric space. Let $T : X \to CB(X)$ be a mapping satisfying*

$$H(Tx,Ty) \leq \alpha(d(x,y))N(x,y) + \beta(d(x,y))M(x,y),$$

for all $x,y \in X$, where

$$M(x,y) = Max\{d(x,y), D(x,Tx), D(y,Ty), \frac{1}{2}[D(y,Tx)+D(x,Ty)]\},$$

$$N(x,y) = \lambda(d(x,y), D(x,Tx), D(y,Ty), D(x,Ty), D(y,Tx)),$$

for $\lambda \in \Lambda$ and $\alpha, \beta : [0,\infty) \to [0,\infty)$ are mappings such that $\alpha(t)+\beta(t) < 1$ and $\limsup_{s\to t^+} \frac{\beta(s)}{1-\alpha(s)} < 1$, for all $t \in [0,\infty)$. Assume T is continuous or λ is continuous at $(0,0,u,u,0)$ for each $u \geq 0$. Then T has a fixed point.

*Proof.*Define a function β' from $[0,\infty)$ into $[0,1)$ by $\beta'(t) = \frac{\beta(t)+1-\alpha(t)}{2}$. Then we have the following;
1) $\beta(t) < \beta'(t)$, for all t,
2) $\limsup_{s\to t^+} \frac{\beta'(s)}{1-\alpha(s)} < 1$, for all $t \in [0,\infty)$.
Let $x_0 \in X$ and $x_1 \in Tx_0$. If $x_1 = x_0$ then x_0 is a fixed point of T, otherwise $x_1 \neq x_0$. Then we have

$$\begin{aligned} D(x_1,Tx_1) &\leq H(Tx_0,Tx_1) \\ &\leq \alpha(d(x_0,x_1))N(x_0,x_1) + \beta(d(x_0,x_1))M(x_0,x_1) \\ &< \alpha(d(x_0,x_1))N(x_0,x_1) + \beta'(d(x_0,x_1))M(x_0,x_1). \end{aligned}$$

Thus, there exist $x_2 \in Tx_1$ such that

$$d(x_1,x_2) \leq \alpha(d(x_0,x_1))N(x_0,x_1) + \beta'(d(x_0,x_1))M(x_0,x_1).$$

Now if $x_1 = x_2$, then x_1 is a fixed point of T, otherwise, $x_1 \neq x_2$, and we have

$$\begin{aligned} D(x_2,Tx_2) &\leq H(Tx_1,Tx_2) \\ &\leq \alpha(d(x_1,x_2))N(x_1,x_2) + \beta(d(x_1,x_2))M(x_1,x_2) \\ &< \alpha(d(x_1,x_2))N(x_1,x_2) + \beta'(d(x_1,x_2))M(x_1,x_2). \end{aligned}$$

Thus there exist $x_3 \in Tx_2$ such that

$$d(x_2,x_3) \leq \alpha(d(x_1,x_2))N(x_1,x_2) + \beta'(d(x_1,x_2))M(x_1,x_2).$$

By induction, we can find in this way a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$, $x_{n+1} \neq x_n$ and

$$\begin{aligned} d(x_n,x_{n+1}) &\leq \alpha(d(x_n,x_{n-1}))N(x_n,x_{n-1}) \\ &+\beta'(d(x_n,x_{n-1}))M(x_n,x_{n-1}). \end{aligned} \quad (1)$$

Since

$$\begin{aligned} N(x_{n-1},x_n) &= \lambda(d(x_{n-1},x_n), D(x_{n-1},Tx_{n-1}), D(x_n,Tx_n), D(x_{n-1},Tx_n), D(x_n,Tx_{n-1})) \\ &\leq \lambda(d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), d(x_{n-1},x_{n+1}), d(x_n,x_n)) \\ &\leq \lambda(d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), d(x_{n-1},x_n)+d(x_n,x_{n+1}), 0) \\ &= d(x_n,x_{n+1}), \end{aligned}$$

and

$$\begin{aligned} M(x_n,x_{n-1}) &= Max\{d(x_n,x_{n-1}), D(x_n,Tx_n), D(x_{n-1},Tx_{n-1}), \frac{D(x_{n-1},Tx_n)+D(x_n,Tx_{n-1})}{2}\} \\ &\leq Max\{d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n), \frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2}\} \\ &\leq Max\{d(x_n,x_{n-1}), d(x_n,x_{n+1}), d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\} \\ &= Max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}. \end{aligned}$$

Thus by using (1) we have

$$\begin{aligned} d(x_n,x_{n+1}) &\leq \alpha(d(x_n,x_{n-1}))d(x_n,x_{n+1}) \\ &+\beta'(d(x_n,x_{n-1}))Max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}. \end{aligned}$$

Therefore

$$(1-\alpha(d(x_n,x_{n+1})))d(x_n,x_{n+1}) \leq \beta'(d(x_n,x_{n-1}))Max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}. \quad (2)$$

If there exists n, such that $Max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_n,x_{n+1})$, then by (2), we have $1 \leq (\alpha+\beta')d(x_n,x_{n+1})$, which is a contradiction, Thus $Max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\} = d(x_{n-1},x_n)$, for each n. Thus from (2), we have

$$(1-\alpha(d(x_n,x_{n+1})))d(x_n,x_{n+1}) \leq \beta'(d(x_n,x_{n-1}))d(x_{n-1},x_n). \quad (3)$$

Let $\gamma(t) = \frac{\beta'(t)}{1-\alpha(t)}$ for each $t \in \Re_+$, then by (2), $\limsup_{t\to s^+} \gamma(t) < 1$. Thus by (3) we have

$$d(x_n,x_{n+1}) \leq \gamma(d(x_n,x_{n-1}))d(x_{n-1},x_n). \quad (4)$$

for all $n \in N$. Therefore $\{d(x_n,x_{n+1}\}$ is a non-increasing sequence, so $\lim_{n\to\infty} d(x_n,x_{n+1}) = r \geq 0$. Assume that $r > 0$. Then from (4) we have

$$\limsup_{s\to r^+} \gamma(s) \geq \limsup_{n\to\infty} \gamma(d(x_n,x_{n+1})) \geq \limsup_{n\to\infty} \frac{d(x_n,x_{n+1})}{d(x_{n-1},x_n)} = 1.$$

Which is contradiction, therefore

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

Let $\limsup_{s\to 0^+} \gamma(s) < r < 1$. Since $\limsup_{n\to\infty} \gamma(d(x_{n-1},x_n)) \leq \limsup_{t\to 0^+} \gamma(t) < 1$, there exists $N > 0$ such that $\gamma(d(x_{n-1},x_n) \leq r$, for $n \geq N$. Then from (4) we have $d(x_n,x_{n+1}) \leq rd(x_{n-1},x_n)$ for $n \geq N$. Hence $\Sigma_{n=1}^{\infty} d(x_n,x_{n+1}) < \infty$. This shows that $\{x_n\}$ is a Cauchy sequence. Since (X,d) is a complete metric space, then $\{x_n\}$ converges to some point $x^* \in X$. Now if T is continuous, then

$$D(x^*,Tx^*) = \limsup_{n\to\infty} D(x_{n+1},Tx^*) \le \limsup_{n\to\infty} H(Tx_n,Tx^*) = 0,$$

and so $x^* \in Tx^*$, i.e, x^* is a fixed point of T. Otherwise

$$\begin{aligned} D(x^*,Tx^*) &\le d(x^*,x_{n+1}) + D(x_{n+1},Tx^*) \\ &\le d(x^*,x_{n+1}) + H(Tx_n,Tx^*) \\ &\le d(x^*,x_{n+1}) + \alpha(d(x_n,x^*))N(x_n,x^*) \\ &+ \beta(d(x_n,x^*))M(x_n,x^*). \end{aligned}$$

Since

$$\begin{aligned} N(x_n,x^*) &= \lambda(d(x_n,x^*),D(x_n,Tx_n),D(x^*,Tx^*) \\ &,D(x_n,Tx^*),D(x^*,Tx_n)) \\ &\le \lambda(d(x_n,x^*),d(x_n,x_{n+1}),D(x^*,Tx^*) \\ &,D(x_n,Tx^*),d(x^*,x_{n+1})). \end{aligned}$$

Letting $n\to\infty$, then we get

$$\begin{aligned} \lim_{n\to\infty} N(x_n,x^*) &\le \lambda(0,0,D(x^*,Tx^*),D(x^*,Tx^*),0)) \\ &\le D(x^*,Tx^*). \end{aligned}$$

Also

$$\begin{aligned} D(x^*,Tx^*) &\le M(x_n,x^*) \\ &= Max\{d(x_n,x^*),D(x_n,Tx_n),D(x^*,Tx^*) \\ &,\frac{D(x_n,Tx^*)+D(x^*,Tx_n)}{2}\} \\ &\le Max\{d(x_n,x^*),d(x_n,x_{n+1}),D(x^*,Tx^*) \\ &,\frac{D(x_n,Tx^*)+d(x^*,x_{n+1})}{2}\}. \end{aligned}$$

Letting $n\to\infty$, then we get $\lim_{n\to\infty} M(x_n,x^*) = D(x^*,Tx^*)$.
Therefore we have

$$\begin{aligned} D(x^*,Tx^*) &\le \limsup_{n\to\infty} \alpha(d(x_n,x^*)D(x^*,Tx^*) \\ &+ \limsup_{n\to\infty} \beta(d(x_n,x^*))D(x^*,Tx^*) \\ &= \limsup_{n\to\infty}(\alpha(d(x_n,x^*)+\beta(d(x_n,x^*))D(x^*,Tx^*) \\ &\le \limsup_{t\to 0^+}(\alpha(t)+\beta(t))D(x^*,Tx^*). \end{aligned}$$

Since $\limsup_{t\to 0^+}(\alpha(t)+\beta(t)) < 1$, and so $D(x^*,Tx^*) = 0$, Thus $x^* \in Tx^*$. Therefore x^* is a fixed point of T.

Corollary 1.*Let (X,d) be a complete metric space. Let $T : X \to CB(X)$ be a continuous mapping satisfying*

$$H(Tx,Ty) \le \alpha(d(x,y))\frac{D(x,Tx)D(y,Ty)}{d(x,y)} + \beta(d(x,y))M(x,y),$$

for all $x,y \in X$, $\alpha,\beta : [0,\infty) \to [0,\infty)$ are mappings such that $\alpha(t)+\beta(t) < 1$ and $\limsup_{s\to t^+} \frac{\beta(s)}{1-\alpha(s)} < 1$, for all $t \in [0,\infty)$. Then T has a fixed point.

*Proof.*Define the mapping

$$\lambda(u_1,u_2,u_3,u_4,u_5) = \begin{cases} \frac{u_2u_3}{u_1} & \text{if } u_1 > 0, \\ 0 & \text{if } u_1 = 0, \end{cases}$$

by using theorem 2, T has a fixed point.

Corollary 2.*Let (X,d) be a complete metric space. Let $T : X \to CB(X)$ be a mapping satisfying*

$$\begin{aligned} H(Tx,Ty) \le \alpha(d(x,y))&\frac{D(y,Ty)(1+D(x,Tx))}{1+d(x,y)} \\ &+\beta(d(x,y))M(x,y), \end{aligned}$$

for all $x,y \in X$, $\alpha,\beta : [0,\infty) \to [0,\infty)$ are mappings such that $\alpha(t)+\beta(t) < 1$ and $\limsup_{s\to t^+} \frac{\beta(s)}{1-\alpha(s)} < 1$, for all $t \in [0,\infty)$. Then T has a fixed point.

*Proof.*Define the mapping

$$\lambda(u_1,u_2,u_3,u_4,u_5) = \frac{u_3(1+u_2)}{1+u_1}.$$

Then by using theorem 2, T has a fixed point.

Corollary 3.*Let (X,d) be a complete metric space. Let $T : X \to CB(X)$ be a mapping satisfying*

$$\begin{aligned} H(Tx,Ty) \le \alpha(d(x,y))&\frac{D(y,Ty)(1+D(x,Tx))(1+D(y,Tx))}{1+d(x,y)} \\ &+\beta(d(x,y))M(x,y), \end{aligned}$$

for all $x,y \in X$, $\alpha,\beta : [0,\infty) \to [0,\infty)$ are mappings such that $\alpha(t)+\beta(t) < 1$ and $\limsup_{s\to t^+} \frac{\beta(t)}{1-\alpha(t)} < 1$, for all $t \in [0,\infty)$. Then T has a fixed point.

*Proof.*Define the mapping

$$\lambda(u_1,u_2,u_3,u_4,u_5) = \frac{u_3(1+u_2)(1+u_5)}{1+u_1}.$$

Then by using Theorem 2, T has a fixed point.

*Example 1.*Let (X,d) be a metric space, where $X = \{1,2,3\}$, $d(1,2) = d(1,3) = 1$, $d(2,3) = 2$. Let $T : X \to CB(X)$ be given by

$$Tx = \begin{cases} \{1,3\} & \text{if } x \in \{1,3\}, \\ \{3\} & \text{if } x = 2, \end{cases}$$

It is obvious that (X,d) is a complete metric space. Moreover, since (X,d) is a discrete space then T is a continuous mapping. Now It is easy to show that if $\alpha = 0$ and $\beta = \frac{1}{2}$, we have

$$H(Tx,Ty) \le \alpha\frac{D(y,Ty)(1+D(x,Tx))}{1+d(x,y)} + \beta M(x,y),$$

for all $x,y \in X$. Then by Theorem 2, T has a fixed point.

3 Conclusion

Recently many results appeared in the literature giving the problems related to the fixed point for multivalued maps. In this paper we obtained the results for existence of the fixed points of multivalued maps that satisfying a new generalize contractive conditions. As a consequence we obtained some fixed point for multivalued contraction of rational type. We presented some examples to show the validity of established results.

Acknowledgement

I would like to thank the anonymous referees for the comments and suggestions on the manuscript.

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