

Numerical Solution of Some Fractional Partial Differential Equations using Collocation Finite Element Method

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Abstract: In this work, our aim is to obtain a numerical solution to some fractional differential equations. In the solution process, we have used fractional derivatives in Caputo sense. The fundamental characteristics of the present method is the fact that it converts complex problems into those requiring the solution of algebraic ones, which is obviously more easy for computational processing. The obtained approximate values show the accuracy and suitability of the present scheme for applying a wide range of fractional partial differential equations. Finally, the error norms L_2 and L_∞ are computed and found to be sufficiently small.

Keywords: Fractional differential equation, Finite element method, Collocation method, Cubic B-spline.

1 Introduction

Fractional differential equations rather than ordinary and partial ones more accurately describe physical phenomena having memory and hereditary characteristics thanks to memory effects of fractional derivatives. Because of these important characteristics, fractional differential equations have become more important in many fields of science in recent years. Thus, an urgent need for reliable and accurate methods for dealing with fractional partial differential equations has increasingly arisen. This study of fractional calculus has become more suitable for the formulation of natural phenomena. This is due to the fact that fractional differential equations rather than integer order differential equations can better model natural physics process and dynamic system processes. Moreover, having the memory effects, fractional differential equations can more suitably describe natural processes having memory and hereditary characteristics. However, in general, derivation of the exact solutions of several fractional differential equations is not so easy. Thus, obtaining some reliable and effective methods for solving fractional differential equations has become increasingly important recently.

In recent years, it has increasingly become evident that most of the phenomena in diverse fields of science such as engineering, chemistry, physics and many others can be accurately described by mathematical models in fractional calculus, namely, the area of integrals and derivatives of non-integer order [1]. The idea of differentiation and integration to equations with non integer order has its origin in early history. To be more precise, this idea has its roots almost as early as the those of the classical calculus were known [2]. Several studies have hinted that derivatives and integrals of non integer order describe the characteristics of several materials more accurately. It has become obvious that fractional order schemes are better than those using integer order ones in terms of accuracy. The ever increasing number of fractional derivative procedures in many areas of engineering and science obviously shows the fact that there has been a huge need for more accurate models of every-day objects. Thus, the fractional calculus can be seen one of possible approaches to more suitable mathematical modeling of every-day objects and procedures. Even though there are a few analytical techniques [3] for handling the fractional equations, just as in integer order partial differential equations, in several situations the auxiliary conditions are in such a way that the sole viable choice can be to apply approximation schemes. Even though

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there are lots of studies on the subject in recent years [4,5,6,7,8], this area of numerical mathematics is still not developed and understood as well as its integer counterpart [9]. Though there have been a lot of techniques used in solving fractional partial differential equations such as those used in and there in [10,11,12,13,14,15], we have lots of works to do in this nascent area of mathematics.

In this present paper, our goal is to apply the collocation finite element scheme to fractional partial differential equations with cubic B spline base functions. For this purpose, we will deal with one dimensional fractional anomalous diffusion equations having inhomogeneous source term, without losing its generality, given as follows,

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t) = f(x,t), \quad 0 \leq x \leq L, \quad t > 0, \quad (1)$$

having the auxiliary conditions

$$u(0,t) = g_0(t), \quad U(L,t) = g_1(t) \quad (2)$$

and

$$u(x,0) = G(x) \quad (3)$$

where $0 < \alpha < 1$ and $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes Caputo form fractional derivative having α order described by [16]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(n+1-\alpha)} \int_0^t (t-\tau)^{n-\alpha} \frac{\partial^{n+1} u(x,\tau)}{\partial \tau^{n+1}} d\tau, \quad n < \alpha \leq n+1. \quad (4)$$

In order to get finite element scheme to solve the fractional problem given by Eqs. (1)-(3), in the solution process of the current problem, we discretize the Caputo derivative using L1 formula [2]

$$\frac{\partial^\alpha f}{\partial t^\alpha} \Big|_{t_m} = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{m-1} b_k^\alpha [f(t_{m-k}) - f(t_{m-1-k})] + O(\Delta t)$$

where

$$b_{k,\alpha} = (k+1)^{1-\alpha} - k^{1-\alpha}.$$

2 Cubic B-spline Finite Element Collocation Solutions

Before solving Eq. (1) having both the initial condition (3) and the boundary conditions (2) with the aid of collocation method, let us firstly describe cubic B-spline basis functions. Assume the solution interval $[a, b]$ is divided in N finite elements of uniform equal lengths with nodal points x_i , $i = 0, 1, 2, \dots, N$ in such a way that $x_0 < x_1 \dots < x_N$ with $h = (x_i - x_{i-1})$. The cubic B-spline functions $\phi_i(x)$, $(i = -1(1)N + 1)$, at nodal points x_i are described in the solution interval $[a, b]$ by [17]

$$\phi_i(x) = \frac{1}{h^3} \begin{cases} (x - x_{i-2})^3, & x_{i-2} \leq x \leq x_{i-1}, \\ h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x_{i-1} \leq x \leq x_i, \\ h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x_i \leq x \leq x_{i+1}, \\ (x_{i+2} - x)^3, & x_{i+1} \leq x \leq x_{i+2}, \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

The set of splines $\{\phi_{-1}(x), \dots, \phi_{N+1}(x)\}$ constitutes a basis for those functions given in the solution domain $[a, b]$. So, an approximate solution $U_N(x, t)$ may be defined using the cubic B-splines as test functions:

$$U_N(x, t) = \sum_{i=-1}^{N+1} \delta_i(t) \phi_i(x) \quad (6)$$

where $\delta_i(t)$'s are time dependent variables which are going to be found using the supplementary and cubic B-spline collocation constraints. Since every cubic B spline extends over four consecutive elements, every element $[x_i, x_{i+1}]$ is spanned with four consecutive cubic B-splines. In the present problem, these elements are described on $[x_i, x_{i+1}]$ and the elements knots x_i, x_{i+1} . If we the nodal values U_i , U_i' and U_i'' defined in terms of the time dependent variable $\delta_i(t)$ by:

$$\begin{aligned} U_i &= U(x_i) = \delta_{i-1}(t) + 4\delta_i(t) + \delta_{i+1}(t), \\ U_i' &= U'(x_i) = \frac{3}{h}(-\delta_{i-1}(t) + \delta_{i+1}(t)), \\ U_i'' &= U''(x_i) = \frac{6}{h^2}(\delta_{i-1}(t) - 2\delta_i(t) + \delta_{i+1}(t)) \end{aligned} \quad (7)$$

and the variation of $U_N(x, t)$ on a typical element $[x_i, x_{i+1}]$ is given by

$$U_N(x, t) = \sum_{j=i-1}^{i+2} \delta_j(t) \phi_j(x). \tag{8}$$

If we replace the global approximation (6) and its required derivatives (7) in Eq. (1), we easily result in a set of the α -th order fractional ordinary differential equations given as follows:

$$(\dot{\delta}_{i-1} + 4\dot{\delta}_i + \dot{\delta}_{i+1}) - \frac{6}{h^2}(\delta_{i-1} - 2\delta_i + \delta_{i+1}) + (\delta_{i-1} + 4\delta_i + \delta_{i+1}) = f(x_i, t) \tag{9}$$

where dot denotes α -th fractional derivative in terms of time. When time dependent parameters $\delta_i(t)$'s and the time fractional derivatives $\dot{\delta}_i(t)$ given in Eq. (9) are discretized using the Crank-Nicolson formulae and $L1$ formula, respectively:

$$\delta_i = \frac{1}{2}(\delta^n + \delta^{n+1}), \tag{10}$$

and

$$\dot{\delta} = \frac{d^\alpha \delta}{dt^\alpha} = \frac{(\Delta t)^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} [(k+1)^{1-\alpha} - k^{1-\alpha}] [\delta^{n-k} - \delta^{n-k-1}],$$

we get a iterative scheme between consecutive times which are interrelating unknown time dependent parameters $\delta_i^{n+1}(t)$ for $i = 0, \dots, N$,

$$\begin{aligned} & (1 - (6 - h^2)\gamma) \delta_{i-1}^{n+1} + (1 + 4(3 + h^2)\gamma) \delta_i^{n+1} + (1 - (6 - h^2)\gamma) \delta_{i+1}^{n+1} = (1 + (6 - h^2)\gamma) \delta_{i-1}^n \\ & + (1 - 4(3 + h^2)\gamma) \delta_i^n + (1 + (6 - h^2)\gamma) \delta_{i+1}^n + 2h^2\gamma f(x_i, t_n) \\ & - \sum_{k=1}^n [(k+1)^{1-\alpha} - k^{1-\alpha}] [(\delta_{i-1}^{n-k+1} - \delta_{i-1}^{n-k}) + 4(\delta_i^{n-k+1} - \delta_i^{n-k}) + (\delta_{i+1}^{n-k+1} - \delta_{i+1}^{n-k})] \end{aligned} \tag{11}$$

where

$$\gamma = \frac{(\Delta t)^\alpha \Gamma(2-\alpha)}{2h^2}.$$

The newly obtained system (11) consists of $N + 1$ linear equations yet includes $N + 3$ unknown parameters $(\delta_{-1}, \dots, \delta_{N+1})^T$. Thus, to be able to obtain a unique solution to this system, we should find two additional constraints. These are obtained from the boundary conditions and then are used to get rid of δ_{N+1} and δ_{-1} in this system.

2.1 Statement of Initial Condition

To be able to begin, the iterative process, we firstly need the initial vector

$$\mathbf{d}^0 = (\delta_0^0, \delta_1^0, \delta_2^0, \dots, \delta_{N-2}^0, \delta_{N-1}^0, \delta_N^0)^T.$$

It can be easily computed using the boundary and initial conditions. So, the general approximation (6) can be particularly stated for the starting point of iteration as follows

$$U_N(x, 0) = \sum_{i=-1}^{N+1} \delta_i^0(t) \phi_i(x) \tag{12}$$

where the δ_i^0 's are time dependent unknown parameters. We impose the initial numerical approximation $U_N(x, 0)$ satisfy the following requirements:

$$\begin{aligned} U_N(x, 0) &= u(x_i, 0), & i &= 0, 1, \dots, N \\ (U_N)_{xx}(0, 0) &= G''(0), & (U_N)_{xx}(L, 0) &= G''(L). \end{aligned} \tag{13}$$

then we result in the matrix equation

$$W\mathbf{d}^0 = \mathbf{b} \tag{14}$$

where

$$W = \begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{bmatrix}$$

$$\mathbf{d}^0 = (\delta_0^0, \delta_1^0, \delta_2^0, \dots, \delta_{N-2}^0, \delta_{N-1}^0, \delta_N^0)^T$$

and

$$\mathbf{b} = (u(x_0, 0) - \frac{h^2}{6}G''(0), u(x_1, 0), u(x_2, 0), \dots, u(x_{N-2}, 0), u(x_{N-1}, 0), u(x_N, 0) - \frac{h^2}{6}G''(L))^T.$$

3 Numerical examples and results

Here, we are going to provide two numerical examples to show the application of collocation finite element scheme using cubic B-spline basis functions to deal with the following two fractional partial differential equations. To show how accurate the results acquired with our the present method, the error norm L_2

$$L_2 = \|U^{analytical} - U_N\|_2 = \sqrt{h \sum_{j=0}^N |U_j^{analytical} - (U_N)_j|^2} \quad (15)$$

and the error norm L_∞

$$L_\infty = \|U^{analytical} - U_N\|_\infty = \max_j |U_j^{analytical} - (U_N)_j|. \quad (16)$$

are calculated.

Example 3.1: Firstly, we will consider the following problem

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} - \frac{\partial^2 u(x, t)}{\partial x^2} + u(x, t) = \Gamma(2 + \alpha)e^{xt}, \quad 0 \leq x \leq 1 \quad t > 0, \quad (17)$$

having auxiliary conditions

$$u(0, t) = t^{1+\alpha}, \quad u(1, t) = e t^{1+\alpha} \quad (18)$$

and

$$u(x, 0) = 0. \quad (19)$$

The problem has the following analytical solution [18]

$$u(x, t) = e^x t^{1+\alpha}. \quad (20)$$

Numerical results for the Eq. (17) with the auxiliary conditions (18) and (19) are acquired with collocation scheme with cubic B-spline basis functions.

We have compared the exact solution and numerical solutions for our problem using values of $\alpha = 0.20$, $\alpha = 0.50$, $\alpha = 0.75$ and $\alpha = 0.90$ and tabulated them in Table 1. We can obviously see in this table that the exact and approximate solutions acquired by the scheme are in harmony with respect to each other.

We have also illustrated the approximate values for $\alpha = 0.50$, $\Delta t = 0.0001$ and $t_f = 0.1$ and for various divisions of the solution region in Table 2. In Table 2, we can observe that when the division number increases, the acquired approximate values get more precise. This conclusion can be drawn from the decreasing values of the error norms L_2 and L_∞ .

In Figure 1, the graphs of numerical solutions acquired for $\alpha = 0.50$, $\Delta t = 0.0001$ and $N = 40$ at various times have been illustrated.

Example 3.2: Secondly, we will deal with the following problem

$$\frac{\partial^\alpha u(x, t)}{\partial t^\alpha} - \frac{\partial^2 u(x, t)}{\partial x^2} - u(x, t) = \frac{2}{\Gamma(3 - \alpha)} t^{2-\alpha} \sin x, \quad 0 \leq x \leq \pi, \quad t > 0, \quad (21)$$

Table 1: A comparison of the analytical and the approximate solutions of the problem using $N = 40$, $\Delta t = 0.0001$ and $t_f = 0.1$ at various values of α with the error norms L_2 and L_∞ .

x	$\alpha = 0.20$		$\alpha = 0.50$		$\alpha = 0.75$		$\alpha = 0.90$	
	Numerical	Exact	Numerical	Exact	Numerical	Exact	Numerical	Exact
0.0	0.063087	0.063096	0.031616	0.031623	0.017778	0.017783	0.012586	0.012589
0.1	0.069729	0.069732	0.034945	0.034949	0.019651	0.019653	0.013914	0.013913
0.2	0.077063	0.077065	0.038621	0.038624	0.021718	0.021720	0.015379	0.015377
0.3	0.085168	0.085170	0.042683	0.042686	0.024003	0.024004	0.016997	0.016994
0.4	0.094125	0.094128	0.047172	0.047176	0.026528	0.026529	0.018785	0.018781
0.5	0.104024	0.104027	0.052133	0.052137	0.029317	0.029319	0.020761	0.020756
0.6	0.114964	0.114968	0.057615	0.057620	0.032400	0.032402	0.022943	0.022939
0.7	0.127055	0.127059	0.063675	0.063680	0.035807	0.035810	0.025355	0.025352
0.8	0.140418	0.140422	0.070371	0.070378	0.039572	0.039576	0.028020	0.028018
0.9	0.155186	0.155190	0.077771	0.077779	0.043732	0.043739	0.030964	0.030965
1.0	0.171489	0.171512	0.085940	0.085960	0.048324	0.048339	0.034213	0.034221
$L_2 \times 10^3$	0.005014		0.006235		0.004091		0.003304	
$L_\infty \times 10^3$	0.023228		0.019343		0.014279		0.008490	

Table 2: A comparison of the analytical and approximate solutions of the problem with $\alpha = 0.5$, $\Delta t = 0.0001$ and $t_f = 0.1$ for various values of N with the error norms L_2 and L_∞ .

x	N=10	N=20	N=40	N=80	N=100	Exact
0.0	0.031509	0.031594	0.031616	0.031621	0.031622	0.031623
0.1	0.034890	0.034934	0.034945	0.034948	0.034948	0.034949
0.2	0.038566	0.038610	0.038621	0.038623	0.038624	0.038624
0.3	0.042627	0.042671	0.042683	0.042685	0.042686	0.042686
0.4	0.047111	0.047160	0.047172	0.0471765	0.047175	0.047176
0.5	0.052065	0.052119	0.052133	0.052136	0.052137	0.052137
0.6	0.057538	0.057600	0.057615	0.057619	0.057620	0.057620
0.7	0.063584	0.063657	0.063675	0.063679	0.063680	0.063680
0.8	0.070264	0.070349	0.070371	0.070376	0.070378	0.070378
0.9	0.077642	0.077745	0.077771	0.077777	0.077778	0.077779
1.0	0.085650	0.085882	0.085940	0.085955	0.085957	0.085960
$L_2 \times 10^3$	0.127728	0.027530	0.006235	0.001435	0.000887	
$L_\infty \times 10^3$	0.309488	0.077372	0.019343	0.004836	0.003095	

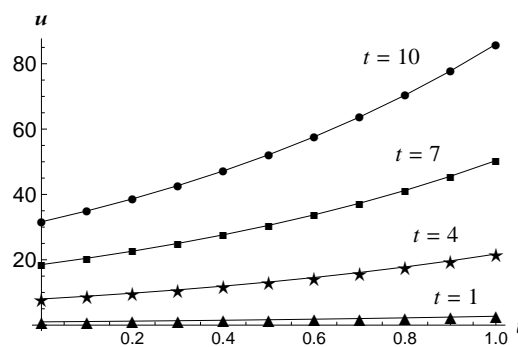


Fig. 1: A comparison of the analytical (line) and approximate solutions for $\alpha = 0.50$, $\Delta t = 0.0001$ and $N = 40$ at $t = 1$ (triangle), $t = 4$ (star), $t = 7$ (square) and $t = 10$ (circle).

having the following boundary conditions

$$u(0,t) = u(\pi,t) = 0 \tag{22}$$

and initial condition

$$u(x, 0) = 0. \quad (23)$$

This problem has the following analytical solution

$$u(x, t) = t^2 \sin x. \quad (24)$$

Numerical results for the Eq. (21) with the boundary conditions (22) and the initial condition (23) are acquired using collocation finite element scheme with cubic B-spline basis functions.

A comparison of the analytical and the newly obtained numerical solutions for values $\alpha = 0.20$, $\alpha = 0.50$, $\alpha = 0.75$ and $\alpha = 0.90$ has been given in Table 3. It is evident in this table that both the exact and approximate solutions acquired using the present method are in harmony with each other.

Table 3: A comparison of the analytical and approximate solutions of the problem with $N = 40$, $\Delta t = 0.0001$ and $t_f = 0.1$ for various values of α with the error norms L_2 and L_∞ .

x	$\alpha = 0.20$	$\alpha = 0.50$	$\alpha = 0.75$	$\alpha = 0.90$	Exact
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.314159	0.003089	0.003090	0.003090	0.003092	0.003090
0.628319	0.005876	0.005877	0.005878	0.005880	0.005878
0.942478	0.008088	0.008090	0.008091	0.008094	0.008090
1.256637	0.009508	0.009510	0.009511	0.009515	0.009511
1.570796	0.009997	0.009999	0.010001	0.010004	0.010000
1.884956	0.009508	0.009510	0.009511	0.009515	0.009511
2.199115	0.008088	0.008090	0.008091	0.008094	0.008090
2.513274	0.005876	0.005877	0.005878	0.005880	0.005878
2.827433	0.003089	0.003090	0.003090	0.003092	0.003090
3.141593	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.003338	0.001017	0.001194	0.005420	
$L_\infty \times 10^3$	0.002664	0.000812	0.000953	0.004324	

In Table 4, the approximate values for $\alpha = 0.50$, $\Delta t = 0.0001$ and $t_f = 0.1$ and for various division numbers of the solution region have been tabulated. Table 4 obviously illustrates that when division number is increased, the acquired approximate values get more precise. This conclusion can be drawn by looking at the decreasing values of the error norms L_2 and L_∞ .

Table 4: A comparison of the analytical approximate solutions of the problem with $\alpha = 0.5$, $\Delta t = 0.0001$ and $t_f = 0.1$ for various values of N with the error norms L_2 and L_∞ .

x	N=10	N=20	N=40	N=80	N=100	Exact
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.314159	0.003085	0.003089	0.003090	0.003090	0.003090	0.003090
0.628319	0.005869	0.005876	0.005877	0.005878	0.005878	0.005878
0.942478	0.008078	0.008087	0.008090	0.008090	0.008090	0.008090
1.256637	0.009496	0.009507	0.009510	0.009510	0.009511	0.009511
1.570796	0.009985	0.009996	0.009999	0.010000	0.010000	0.010000
1.884956	0.009496	0.009507	0.009510	0.009510	0.009511	0.009511
2.199115	0.008078	0.008087	0.008090	0.008090	0.008090	0.008090
2.513274	0.005869	0.005876	0.005877	0.005878	0.005878	0.005878
2.827433	0.003085	0.003089	0.003090	0.003090	0.003090	0.003090
3.141593	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.019422	0.004693	0.001017	0.000099	0.000011	
$L_\infty \times 10^3$	0.015496	0.003745	0.000812	0.000079	0.000009	

In Figure 2, we have illustrated the graphical views of approximate values acquired for $\alpha = 0.50$, $\Delta t = 0.0001$ and $N = 40$ at different times.

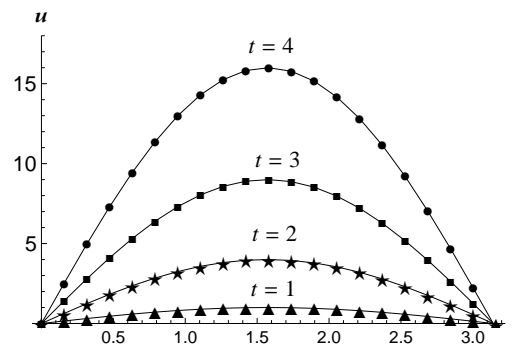


Fig. 2: A comparison of the analytical (lines) and approximate solutions for $\alpha = 0.50$, $\Delta t = 0.0001$ and $N = 40$ at $t = 1$ (triangles), $t = 2$ (stars), $t = 3$ (squares) and $t = 4$ (circles).

4 Perspective

In the present paper, we have presented collocation finite element scheme using cubic B-spline basis functions to acquire approximate solutions for fractional partial differential equations. The fractional derivatives are used in Caputo form. The fundamental characteristics of the present method is that it changes a fractional differential problem into algebraic solvable problem, which is obviously more suitable for numerical calculations. In conclusion, it can be said that collocation finite element method can be easily used in finding approximate solutions of many more similar equations.

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