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Finite Difference Methods for Fractional Gas Dynamics Equation

Alaattin Esen[1], *Berat Karaagac*[1] *and Orkun Tasbozan*[2,*]

[1] Department of Mathematics, Inonu University, Malatya, Turkey
[2] Department of Mathematics, Mustafa Kemal University, Hatay, Turkey

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Abstract: In this article, the explicit and the Crank-Nicolson finite difference methods have been successfully applied to obtain approximate solutions of the the nonlinear time fractional gas dynamics equation. The time fractional derivative in the equation has been considered in the Caputo form. The $L1$ discretization formula has been applied to the equation. To test the accuracy of the proposed methods, the error norms L_2 and L_∞ have also been computed . The newly obtained solutions by the proposed method indicate the easy implementation and effectiveness of the approach used in the article.

Keywords: Finite difference method, Fractional gas dynamics equation, Caputo derivative

1 Introduction

Fractional calculus constitutes an important branch of applied mathematics and mainly deals with derivatives and integrals of non-integer orders. The application of differentiation and integration to non-integer orders has a long history, so it is not new by no means. In fact, interest in the subject dates back to the ideas of the classical calculus [1]. However, in the last few decades, many authors have pointed out that derivatives and integrals of non-integer order are very suitable for the description of many phenomena in the nature. It has been shown that new fractional-order models are more adequate than previously used integer-order models. The growing number of fractional derivative applications in various fields of science and engineering indicates that there is a significant demand for better mathematical models of real objects, and that the fractional calculus provides one possible approach on the way to more adequate mathematical modeling of real objects and processes. They are widely used to model problems in fluid mechanics, acoustics, biology, electromagnetism, diffusion, signal processing, and many other physical processes, look at [2] and references therein. There are a wide range of studies dealing with the obtaining both analytical and numerical solutions of fractional differential equations using numerous techniques and methods. Although there are few analytical methods such as found by [3,4,5,6] providing exact solutions of the fractional equations, the numerical methods are more common and the most appropriate and even sometimes the only way to handle most of the problems involving fractional equations. Thus effective, accurate and easily implemented numerical methods are of great importance. Though there have been many methods applied to solve fractional partial differential equations, there is still a long way to go in this field. There are several studies about fractional equations in the literature. Murillo and Yuste [7] have used an explicit difference for solving fractional diffusion and diffusion-wave equations in the Caputo form. Sweilam *et al.* [8] solved time-fractional diffusion equation by using Crank-Nicolson finite difference method. Monami and Odibat [9] have implemented relatively new analytical techniques, the variational iteration method and the Adomian decomposition method, for solving linear fractional partial differential equations arising in fluid mechanics. In this paper, we will use finite difference methods to obtain the numerical solutions of the fractional gas dynamics equation by using the $L1$ discretizaton formula of the fractional derivative as used by [7].The equations of gas dynamics are mathematical expressions based on the physical laws of conservation namely, the laws of conservation of mass,

* Corresponding author e-mail: orkun.tasbozan@inonu.edu.tr

conservation of momentum, conservation of energy etc [10]. The various forms of gas dynamics equations have been solved by many authors. Among others, Liu [11] has taken some partial differential equations related to gas dynamics and mechanics into consideration and solved them numerically and Rasulov and Karaguler [12] have applied difference scheme to solve some gas dynamics problems.

In this study, we will consider the homogenous nonlinear time-fractional gas dynamics equation as a model is given by

$$D_t^{\gamma} u + uu_x - u(1-u) = 0 \qquad (1)$$

where

$$D_t^{\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-\tau)^{-\gamma} f'(\tau)d\tau \quad 0 < \gamma \leq 1$$

is the fractional derivative in the Caputo's sense [3,13]. In this paper, for fractional gas dynamics equation, we will take the boundary conditions of the model problem (1) given in the interval $0 \leq x \leq 1$ as

$$u(0,t) = E_{\gamma}(t^{\gamma}), \quad u(1,t) = e^{-1}E_{\gamma}(t^{\gamma})$$

and the initial condition as

$$u(x,0) = e^{-x}.$$

The exact solution of the problem is given as [10]

$$u(x,t) = e^{-x}E_{\gamma}(t^{\gamma})$$

where E_{γ} is the Mittag-Leffler function [13].

In our numerical solutions, to obtain a finite difference schemes for solving the fractional gas dynamics equation as [7] used in explicit finite difference method, we will also discretize the Caputo derivative by means of the so-called $L1$ formula [1]

$$D_t^{\gamma} f(t)|_{t_m} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{m-1} b_k^{\gamma} [f(t_{m-k}) - f(t_{m-1-k})] + O(\Delta t) \qquad (2)$$

where

$$b_k^{\gamma} = (k+1)^{1-\gamma} - k^{1-\gamma}.$$

2 The Finite Difference Schemes

Let's suppose that the solution domain of the problem $0 \leq x \leq 1$ is divided into intervals having equal length Δx in the x direction and having equal time intervals Δt in time t such that $x_j = jh$, $j = 0(1)N$ and $t_m = m\Delta t$, $m = 0(1)M$ and u_j^m will denote $u(x_j, t_m)$ throughout the article.

2.1 The Explicit Finite Difference Scheme

Using Eq.(2) in Eq.(1) and applying the the following discretize for the terms uu_x and $u(1-u)$, respectively:

$$uu_x \simeq u_j^m \left(\frac{u_{j+1}^m - u_{j-1}^m}{2\Delta x} \right)$$

and

$$u(1-u) \simeq u_j^m (1 - u_j^m),$$

we can easily obtain the following system of algebraic equations

$$u_j^{m+1} = u_j^m - Su_j^m (u_{j+1}^m - u_{j-1}^m + 2\Delta x u_j^m - 2\Delta x) - \sum_{k=1}^{m} b_k^{\gamma} \left[u_j^{m+1-k} - u_j^{m-k} \right],$$

$$j = 1(1)N-1, m = 0(1)M$$

where

$$S = \Gamma(2-\gamma) \frac{(\Delta t)^{\gamma}}{2\Delta x}.$$

2.2 The Crank-Nicolson Finite Difference Scheme

Using Eq.(2) in Eq.(1) and applying the the following discretize for the terms uu_x and $u(1-u)$, respectively:

$$uu_x \cong \frac{u_j^m}{2} \left(\frac{u_{j+1}^{m+1} - u_{j-1}^{m+1}}{2\Delta x} + \frac{u_{j+1}^m - u_{j-1}^m}{2\Delta x} \right)$$

and

$$u(1-u) \cong \frac{u_j^{m+1} + u_j^m}{2} (1 - u_j^m),$$

we can easily obtain the following system of algebraic equations

$$(-Su_j^m)u_{j-1}^{m+1} + (1 - 2\Delta x S + 2\Delta x S u_j^m)u_j^{m+1} + (Su_j^m)u_{j+1}^{m+1} = (Su_j^m)u_{j-1}^m + (1 + 2\Delta x S - 2\Delta x S u_j^m)u_j^m + (-Su_j^m)u_{j+1}^m - \sum_{k=1}^{m} b_k^{\gamma} \left[u_j^{m+1-k} - u_j^{m-k} \right],$$

$$j = 1(1)N-1, m = 0(1)M$$

where

$$S = \Gamma(2-\gamma) \frac{(\Delta t)^{\gamma}}{4\Delta x}.$$

3 Numerical examples and results

Numerical results for the fractional gas dynamics problem are obtained by the explicit and the Crank-Nicolson finite difference methods. The accuracy of the methods are measured by the error norm L_2

$$L_2 = \left\|u^{exact} - u_N\right\|_2 \simeq \sqrt{h \sum_{j=0}^{N} \left|u_j^{exact} - (u_N)_j\right|^2}$$

and the error norm L_∞

$$L_\infty = \left\|u^{exact} - u_N\right\|_\infty \simeq \max_j \left|u_j^{exact} - (u_N)_j\right|.$$

The comparison of the analytical solutions and numerical solutions obtained by the explicit and the Crank-Nicolson difference methods for fractional gas dynamics equation for values of $\gamma = 0.25$, $\gamma = 0.50$ and $\gamma = 0.75$ is given in Table 1-2, respectively. As it is clearly seen from the both of the tables, the analytical and numerical solutions obtained by the present schemes are in good agreement with each other. In Table 3, we demonstrate the numerical results by the Crank-Nicolson finite difference method for $\gamma = 0.5$, $\Delta t = 0.001$ and $t_f = 1.0$. and for different number of divisions of the region. Table 3 clearly show that as the number of division increases, the obtained numerical results become more accurate. We see these from the decreasing values of the error norms L_2 and L_∞.

Table 1: The comparison of the exact solutions with the numerical solutions of the fractional gas dynamics problem by the explicit finite difference method with $\Delta x = 0.025$, $\Delta t = 0.0001$ and $t_f = 0.1$ for different values of γ and the error norms

x	$\gamma = 0.50$		$\gamma = 0.75$	
	Numerical	Exact	Numerical	Exact
0.0	1.486763	1.486763	1.219661	1.219661
0.1	1.345271	1.345279	1.103587	1.103595
0.2	1.217237	1.217259	0.998544	0.998574
0.3	1.101382	1.101421	0.903485	0.903547
0.4	0.996550	0.996607	0.817493	0.817563
0.5	0.901695	0.901768	0.739696	0.739762
0.6	0.815872	0.815953	0.669303	0.669364
0.7	0.738226	0.738305	0.605610	0.605666
0.8	0.667984	0.668046	0.547979	0.548029
0.9	0.604440	0.604473	0.495847	0.495877
1.0	0.546950	0.546950	0.448688	0.448688
$L_2 \times 10^3$	0.101129		0.052767	
$L_\infty \times 10^3$	0.236551		0.083227	

In Table 4-5, we show the error norms L_2 and L_∞ for $\Delta x = 0.025$ and $\Delta t = 0.0001$ for different values of t_f.

The comparison of the results of the finite difference methods, the other studies and exact solution for $\gamma = 1$ and $t = 0.1$ is given in Table 6. It shows the finite difference methods is in good agrement at almost all points x.

Figure 1 shows the graphs of the exact (denoted by lines) solutions and the numerical solutions by the Crank-Nicolson finite difference method for $\Delta x = 0.025$ and $\Delta t = 0.0001$ at $t = 0.1$ (stars), $t = 0.5$ (squares) and $t = 1.0$ (triangles) for $\gamma = 1.0$.

Table 2: The comparison of the exact solutions with the numerical solutions of the fractional gas dynamics problem by the Crank-Nicolson finite difference method with $\Delta x = 0.025$, $\Delta t = 0.0001$ and $t_f = 0.1$ for different values of γ and the error norms L_2 and L_∞.

x	$\gamma = 0.25$		$\gamma = 0.50$		$\gamma = 0.75$	
	Numerical	Exact	Numerical	Exact	Numerical	Exact
0.0	2.364040	2.364040	1.486763	1.486763	1.219661	1.219661
0.1	2.139085	2.139072	1.345281	1.345279	1.103594	1.103595
0.2	1.935535	1.935512	1.217257	1.217259	0.998557	0.998574
0.3	1.751353	1.751324	1.101411	1.101421	0.903502	0.903547
0.4	1.584696	1.584663	0.996585	0.996607	0.817509	0.817563
0.5	1.433896	1.433863	0.901734	0.901768	0.739710	0.739762
0.6	1.297444	1.297412	0.815912	0.815953	0.669316	0.669364
0.7	1.173975	1.173947	0.738263	0.738305	0.605621	0.605666
0.8	1.062253	1.062232	0.668013	0.668046	0.547989	0.548029
0.9	0.961159	0.961147	0.604456	0.604473	0.495855	0.495877
1.0	0.869682	0.869682	0.546950	0.546950	0.448688	0.448688
$L_2 \times 10^3$	0.065720	0.064386		0.040855		
$L_\infty \times 10^3$	0.133187	0.161156		0.067735		

Table 3: The comparison of the exact solutions with the numerical solutions of the gas dynamics problem with by the Crank-Nicolson finite difference method $\gamma = 0.5$, $\Delta t = 0.0001$ and $t_f = 1.0$ for different values of Δx and the error norms L_2 and L_∞.

x	$\Delta x = 0.1$	$\Delta x = 0.05$	$\Delta x = 0.025$	Exact
0.0	5.008980	5.008980	5.008980	5.008980
0.1	4.534760	4.532429	4.532333	4.532313
0.2	4.101863	4.101205	4.101040	4.101006
0.3	3.714275	3.710996	3.710787	3.710744
0.4	3.358817	3.357899	3.357669	3.357620
0.5	3.042397	3.038383	3.038150	3.038100
0.6	2.750114	2.749251	2.749034	2.748987
0.7	2.492261	2.487612	2.487428	2.487386
0.8	2.251395	2.250849	2.250712	2.250680
0.9	2.041870	2.036593	2.036518	2.036499
1.0	1.842701	1.842701	1.842701	1.842701
$L_2 \times 10^3$	3.057607	0.388016	0.304294	
$L_\infty \times 10^3$	5.370697	0.650771	0.571857	

Table 4: The comparison of the exact solutions with the numerical solutions of the fractional gas dynamics problem by the explicit finite difference method method with $\Delta x = 0.025$ and $\Delta t = 0.0001$ for different values of t_f and the error norms L_2 and L_∞.

t	$\gamma = 0.50$		$\gamma = 0.75$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.005	0.312097	0.424766	0.084208	0.120952
0.01	0.234807	0.311849	0.074993	0.106386
0.05	0.129177	0.242196	0.058112	0.079220
0.1	0.101129	0.236551	0.052767	0.083227

Table 5: The comparison of the exact solutions with the numerical solutions of the fractional gas dynamics problem by the Crank-Nicolson finite difference method method with $\Delta x = 0.025$ and $\Delta t = 0.0001$ for different values of t_f and the error norms L_2 and L_∞.

t	$\gamma = 0.25$		$\gamma = 0.50$		$\gamma = 0.75$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.005	0.718327	1.547650	0.278203	0.378029	0.081599	0.117052
0.01	0.449973	1.030954	0.200572	0.266704	0.071278	0.100966
0.05	0.143310	0.313828	0.093259	0.183434	0.049698	0.067631
0.1	0.065720	0.133187	0.064386	0.161156	0.040855	0.067735

Table 6: The results obtained by the finite difference methods in comparison with the other studies and exact solution at $\gamma = 1.0$, $\Delta t = 0.0001$,$\Delta x = 0.025$ and $t_f = 0.1$ for different values of x.

x	HPSTM [2]	ADM [2]	Explicit	Crank-Nicolson	Exact
0.1	0.9999431595	0.9999431595	1.0000048609	1.0000093850	1.0000000000
0.3	0.8186842160	0.8186842160	0.8187335710	0.8187376655	0.8187307531
0.5	0.6702819447	0.6702819447	0.6703212919	0.6703246440	0.6703200460
0.7	0.5487804413	0.5487804413	0.5488119545	0.5488146988	0.5488116361
0.9	0.4493037263	0.4493037263	0.4493287612	0.4493310007	0.4493289641
L_2			0.0000022940	0.0000054172	
L_∞			0.0000048609	0.0000095657	

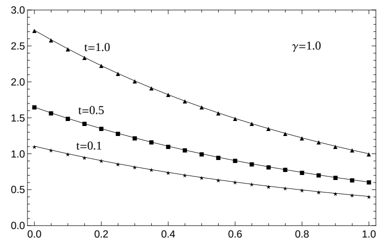

Fig. 1: The comparison of the exact(lines) and numerical solutions by the Crank-Nicolson finite difference method for $\gamma = 1.0$, $\Delta x = 0.025$ and $\Delta t = 0.0001$ at $t = 0.1$ (stars), $t = 0.5$ (squares), and $t = 1.0$ (triangles).

4 Conclusion

In the present study, a finite difference methods have been successfully used to obtain the numerical solutions of fractional gas dynamics equation. In these equations, the fractional derivative is considered of the Caputo form. In this study, the fractional derivative appearing in the fractional gas equation is approximated by means of the so-called $L1$ formulae. One can easily conclude from the presented results that the applied method is a highly good one to obtain numerical solutions of this kind fractional partial differential equations.

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Alaatin ESEN received his diploma in mathematics from the Inonu University in 1994. He has completed his M.Sc. and Ph.D. degrees in applied mathematics. He is currently studying about the numerical solutions of a wide range of partial differential equations. He has many research papers published in various national and international journals. He has given talks and conferences. His research interests include finite difference methods, finite element methods, computational methods and algorithms.

Berat KARAAGAC graduated from the department of Mathematics of the Inonu University. She has completed his M.Sc. degree in applied mathematics. She is currently studying his Ph.D. thesis. Her main interest areas include numerical solutions, fractional differential equations, FEM etc.

Orkun TASBOZAN graduated from the department of Mathematics of the Afyon Kocatepe University. He has completed his M.Sc. degree and Ph.D. thesis in applied mathematics. His main interest areas include analytical solutions, fractional differential equations, HAM, FEM etc.