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Quantum Jensen-Shannon Divergence Between Quantum Ensembles

Zhaoqi Wu[1], *Shifang Zhang*[2] *and Chuanxi Zhu*[1]

[1] Department of Mathematics, Nanchang University, Nanchang 330031, P. R. China
[2] School of Mathematics and Computer Science, Fujian Normal University, Fuzhou 350007, P. R. China

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Abstract: In this paper, we first define quantum Jensen-Shannon divergence (QJSD) between quantum states in infinite-dimensional case and discuss its properties. Then, using the probabilistic coupling technique, we further propose the notion of quantum Jensen-Shannon divergence (QJSD) between quantum ensembles. Some fundamental properties of this quantity are also discussed.

Keywords: quantum Jensen-Shannon divergence, quantum ensemble, distinguishability measure.

1. Introduction

Quantum information theory has become a significant branch during the last few years, and the study on quantum entanglement and other related problems has attracted much attention by many scholars ([1-6]). On the other hand, entropy is an important quantity to characterize both classical and quantum information ([7-9]). It is well known that distance measures play a central role in quantum computation and quantum information, which is closely related to quantum entanglement. Recently, Majtey et.al.([10]) introduced the concept of Quantum Jensen-Shannon divergence (QJSD) for quantum states in finite-dimensional Hilbert space, which is a modification of the notion of quantum relative entropy. Many properties of QJSD are discussed ([10]) and the metric property of QJSD is studied ([11]). In fact, Rao ([12]) and Lin ([13]) have introduced Jensen-shannon divergence (JSD) as a symmetrized version of the Kullback-Leibler divergence independently in classical case and this quantity has been recently applied to many problems arising in statistics and physics ([14-16]). In the framework of information theory, the JSD can be related to mutual information ([17]). The Fisher divergence (FD) and Jensen-Shannon divergence (JSD) are compared for quantitative measures of the discrepancies between two arbitrary D-dimensioanl distribution functions, the FD being the local character and the JSD of global one ([18]). Sachlas and Papaioannou investigated the properties of the Jensen's difference in the case of non-probability vectors, which appears in actuarial graduation ([19]).

Density matrices can be thought of as generalizations of classical probability distributions. However, in many scenarios, one often deals with an even more general concept, which is a hybrid between the quantum and classical cases. This is the concept of a probabilistic ensemble of quantum states. Oreshkov and Calsamiglia defined two distinguishability measures between quantum ensembles called Kantorovich distance and Kantorovich fidelity respectively and discussed their properties ([20]). Recently, Luo et.al.([21]) defined the quantum relative entropy between quantum ensembles.

Up to now, the major attention in quantum information theory has been paid to finite-dimensional systems. However, an important class of Gaussian channels (see, e.g., [22-23]) act in infinite-dimensional Hilbert space. In 2006, Holevo and Shirokov studied χ-capacity of infinite-dimensional quantum channels ([24]). In 2008, they developed an approximation approach to infinite-dimensional quantum channels based on a detailed investigation of continuity properties of entropic characteristics of quantum channels and operations ([25]). Recently, mutual information and coherent information for infinite-dimensional quantum channels are established and discussed ([26]).

In this paper, we first define QJSD between quantum states under the general framework of infinite-dimensional separable Hilbert space, then we further extend this quan-

* Corresponding author: e-mail: wuzhaoqi_conquer@163.com

tity to the one between quantum ensembles and discuss its properties. We first recall the concept of quantum ensemble.

A quantum ensemble $\{p_i, \rho_i\}$ is a family of distinct quantum states $\{\rho_i\}$ together with a probability distribution $\{p_i\}$ on the states (i.e., $p_i \geq 0, \sum_i p_i = 1$). This notion has several natural interpretations. For example, it may be interpreted as the final outcome of a general quantum measurement. In this case, the quantum ensemble arises from an original quantum state ρ and a general quantum measurement $M = \{M_i\}$ as

$$p_i = trM_i(\rho), \rho_i = \frac{1}{p_i}M_i(\rho).$$

We now give the definition of Quantum Jensen-Shannon divergence in the context of infinite-dimensional separable complex Hilbert space.

Definition 1.1. Let H be a infinite-dimensional separable complex Hilbert space, and ρ and σ two quantum states on H. Then

$$JS(\rho \parallel \sigma) = \frac{1}{2}[D(\rho\|\frac{\rho+\sigma}{2}) + D(\sigma\|\frac{\rho+\sigma}{2})] \quad (1)$$

is called the quantum Jensen-Shannon divergence between quantum states (QJSD), where $D(\rho \parallel \sigma)$ is the quantum relative entropy between ρ and σ.

2. Quantum Jensen-Shannon divergence in infinite dimensional case

In this section, we first establish the properties of QJSD in infinite-dimensional case by approximation of finite dimensional ones through projection operators.

Lemma 2.1([27]). Let H be a separable Hilbert space, $\{P_n\}$ be a nondecreasing sequence of projectors converging to the identity operator I in the strong operator topology, and $A, B \in B(H)$ be two arbitrary positive trace class operators. Then the sequences $\{S(P_nAP_n\}$ and $\{D(P_nAP_n\|P_nBP_n)\}$ are nondecreasing, and

$$S(A) = \lim_{n\to\infty} S(P_nAP_n),$$

$$D(A\|B) = \lim_{n\to\infty} D(P_nAP_n\|P_nBP_n)\},$$

where $S(A) = -trAlogA$ is the von Neumann entropy of A, and $D(A\|B) = tr(AlogA - AlogB + B - A)$ is the quantum relative entropy between A and B.

For finite dimensional case, the properties in the next theorem have been discussed in [1]. It is natural to generalize these properties in infinite dimensional case, so we have the following theorem.

Theorem 2.1. Let H, H_1 and H_2 be separable complex Hilbert spaces.

(1) If ρ and σ are two states on H, then $0 \leq JS(\rho\|\sigma) \leq 1$. $JS(\rho\|\sigma) = 0$ iff $\rho = \sigma$, $JS(\rho\|\sigma) = 1$ iff ρ and σ have support on orthogonal vector spaces.

(2) $JS(\rho\|\sigma) = JS(\sigma\|\rho)$;

(3) JS is invariant under unitary transformations, that is, if ρ and σ are states on H, and U is the unitary operator on H, then

$$JS(U\rho U^\dagger\|U\sigma U^\dagger) = JS(\rho\|\sigma).$$

(4) (Restricted additivity) If ρ_1 and σ_1 are two states on H_1, and ρ_2 is a state on H_2, then

$$JS(\rho_1 \otimes \rho_2\|\sigma_1 \otimes \rho_2) = JS(\rho_1\|\sigma_1).$$

(5) (Joint convexity) If ρ_j and σ_j are states on H, $\lambda_i > 0$, $j = 1, 2, \cdots, n$, and $\sum_j \lambda_j = 1$, then

$$JS(\sum_j \lambda_j\rho_j\| \sum_j \lambda_j\sigma_j) \leq \sum_j \lambda_j JS(\rho_j\|\sigma_j).$$

(6) (Monotonicity) Let Φ be a trace-preserving completely positive map of $T(H)$ into itself. Then for any states ρ and σ on H,

$$JS(\Phi(\rho)||\Phi(\sigma)) \leq JS(\rho||\sigma).$$

(7) For any states ρ and σ on $H_1 \otimes H_2$,

$$JS(\rho_1||\sigma_1) \leq JS(\rho||\sigma),$$

where ρ_1 and σ_1 are the partial traces of ρ and σ on H_1, respectively.

Proof. It is easy to verify that (1)-(3) holds.

(4) Take increasing finite dimensional projector sequence $\{P_n^A\}$ and $\{P_n^B\}$ in H_A and H_B respectively, which converge to I_A and I_B in strong operator topology respectively. Let $\rho_A^n = \mu_n^{-1}P_n^A\rho_AP_n^A, \rho_A^n = \mu_n^{-1}P_n^A\rho_AP_n^A$, where $\mu_n = tr(P_n^A\rho_A), \nu_n = tr(P_n^A\sigma_A)$. Then

$$\frac{\rho_A^n + \sigma_A^n}{2} = P_n^A\frac{\rho_A/\mu_n + \sigma_A/\nu_n}{2}P_n^A,$$

By Lemma 2.1, we have

$$\lim_{n\to\infty} D(\rho_A^n\|\frac{\rho_A^n + \sigma_A^n}{2}) = \lim_{n\to\infty} D(P_n^n\rho_AP_n^A\|P_n^A\rho_AP_n^A)$$

$$= D(\rho_A\|\frac{\rho_A + \sigma_A}{2}),$$

and

$$\lim_{n\to\infty} D(\sigma_A^n\|\frac{\rho_A^n + \sigma_A^n}{2}) = \lim_{n\to\infty} D(P_n^n\sigma_AP_n^A\|P_n^A\rho_AP_n^A)$$

$$= D(\sigma_A\|\frac{\rho_A + \sigma_A}{2}).$$

Then by the definition of QJSD, we obtain

$$\lim_{n\to\infty} JS(\rho_A^n\|\sigma_A^n) = JS(\rho_A\|\sigma_A). \quad (2)$$

Let $\rho_B^k = \xi_k^{-1}P_k^B\rho_BP_k^B$, where $\xi_k = tr(P_k^B\rho_B)$. Then

$$\rho_A^n \otimes \rho_B^k = \mu_n^{-1}\xi_k^{-1}(P_n^A \otimes P_k^B)(\rho_A \otimes \rho_B)(P_n^A \otimes P_k^B),$$

$$\sigma_A^n \otimes \rho_B^k = \nu_n^{-1}\xi_k^{-1}(P_n^A \otimes P_k^B)(\sigma_A \otimes \rho_B)(P_n^A \otimes P_k^B),$$

and

$$\frac{\rho_A^n \otimes \rho_B^k + \sigma_A^n \otimes \rho_B^k}{2}$$

$$= (P_n^A \otimes P_k^B)(\frac{\mu_n^{-1}\xi_k^{-1}\rho_A \otimes \rho_B + \nu_n^{-1}\xi_k^{-1}\sigma_A \otimes \rho_B}{2})(P_n^A \otimes P_k^B).$$

Using Lemma 2.1 again, we have

$$\lim_{n,k\to\infty} D(\rho_A^n \otimes \rho_B^k \| \frac{\rho_A^n \otimes \rho_B^k + \sigma_A^n \otimes \rho_B^k}{2})$$

$$= D(\rho_A \otimes \rho_B \| \frac{\rho_A \otimes \rho_B + \sigma_A \otimes \rho_B}{2}),$$

and

$$\lim_{n,k\to\infty} D(\sigma_A^n \otimes \rho_B^k \| \frac{\rho_A^n \otimes \rho_B^k + \sigma_A^n \otimes \rho_B^k}{2})$$

$$= D(\sigma_A \otimes \rho_B \| \frac{\rho_A \otimes \rho_B + \sigma_A \otimes \rho_B}{2}),$$

Then by the definition of QJSD, we obtain

$$\lim_{n,k\to\infty} JS(\rho_A^n \otimes \rho_B^k \| \sigma_A^n \otimes \rho_B^k) = JS(\rho_A \otimes \rho_B \| \sigma_A \otimes \rho_B). \quad (3)$$

From [10], we know that the restricted additivity holds for finite-dimensional case, that is,

$$JS(\rho_A^n \otimes \rho_B^k \| \sigma_A^n \otimes \rho_B^k) = JS(\rho_A^n \| \sigma_A^n).$$

Taking the limit on both sides of the above equality and by (2) and (3), the conclusion follows immediately.

(5) We prove that for $\lambda \in [0,1]$, we have

$$JS(\lambda\rho_1 + (1-\lambda)\rho_2 \| \lambda\sigma_1 + (1-\lambda)\sigma_2)$$

$$\leq \lambda JS(\rho_1\|\sigma_1) + (1-\lambda)JS(\rho_2\|\sigma_2).$$

Take increasing finite dimensional projector sequence $\{P_n\}$ in H, which converges to I in the strong operator topology. Put $\rho = \lambda\rho_1 + (1-\lambda)\rho_2, \sigma = \lambda\sigma_1 + (1-\lambda)\sigma_2$.

Let

$$\rho_n = \frac{P_n\rho P_n}{tr(P_n\rho)} = \frac{\lambda P_n\rho_1 P_n + (1-\lambda)P_n\rho_2 P_n}{\lambda tr(P_n\rho_1) + (1-\lambda)tr(P_n\rho_2)}$$

$$= \frac{\alpha_1^n\rho_1^n + \alpha_2^n\rho_2^n}{\alpha_1^n + \alpha_2^n},$$

where $\alpha_1^n = \lambda tr(P_n\rho_1), \rho_1^n = \lambda\frac{P_n\rho_1 P_n}{\alpha_1^n}$ and $\alpha_2^n = \lambda tr(P_n\rho_2)$, $\rho_2^n = (1-\lambda)\frac{P_n\rho_2 P_n}{\alpha_2^n}$.

Also, let

$$\sigma_n = \frac{P_n\sigma P_n}{tr(P_n\sigma)} = \frac{\lambda P_n\sigma_1 P_n + (1-\lambda)P_n\sigma_2 P_n}{\lambda tr(P_n\sigma_1) + (1-\lambda)tr(P_n\sigma_2)}$$

$$= \frac{\beta_1^n\sigma_1^n + \beta_2^n\sigma_2^n}{\beta_1^n + \beta_2^n},$$

where $\beta_1^n = \lambda tr(P_n\sigma_1), \sigma_1^n = \lambda\frac{P_n\sigma_1 P_n}{\beta_1^n}$ and $\beta_2^n = \lambda tr(P_n\sigma_2)$, $\sigma_2^n = (1-\lambda)\frac{P_n\sigma_2 P_n}{\beta_2^n}$.

Denote

$$\alpha_n = \frac{\alpha_1^n}{\alpha_1^n + \alpha_2^n}, \beta_n = \frac{\beta_1^n}{\beta_1^n + \beta_2^n}.$$

Noting that as $n \to \infty$, $\alpha_1^n, \beta_1^n \to \lambda$, $\alpha_2^n, \beta_2^n \to 1-\lambda$, we have $\alpha_n, \beta_n \to \lambda$, as $n \to \infty$. By the joint convexity of $(\rho,\sigma) \mapsto JS(\rho\|\sigma)$ for finite-dimensional case, we have

$$JS(\alpha_n\rho_1^n + (1-\alpha_n)\rho_2^n \| \alpha_n\sigma_1^n + (1-\alpha_n)\sigma_2^n) \quad (4)$$

$$\leq \alpha_n JS(\rho_1^n\|\sigma_1^n) + (1-\alpha_n)JS(\rho_2^n\|\sigma_2^n).$$

Similar to the proof of (4) (Restricted additivity), we can easily obtain that

$$\lim_{n\to\infty} JS(\rho_1^n\|\sigma_1^n) = JS(\rho_1\|\sigma_1),$$

$$\lim_{n\to\infty} JS(\rho_2^n\|\sigma_2^n) = JS(\rho_2\|\sigma_2),$$

and

$$\lim_{n\to\infty} JS(\rho_n\|\sigma_n) = JS(\rho\|\sigma).$$

Note that

$$\lim_{n\to\infty} JS(\rho_n\|\sigma_n) = \lim_{n\to\infty} JS(\alpha_n\rho_1^n + (1-\alpha_n)\rho_2^n \| \beta_n\sigma_1^n + (1-\beta_n)\sigma_2^n)$$

$$= \lim_{n\to\infty} JS(\alpha_n\rho_1^n + (1-\alpha_n)\rho_2^n \| \alpha_n\frac{\beta_n}{\alpha_n}\sigma_1^n + (1-\alpha_n)\frac{(1-\beta_n)}{1-\alpha_n}\sigma_2^n)$$

$$= \lim_{n\to\infty} JS(\alpha_n\rho_1^n + (1-\alpha_n)\rho_2^n \| \alpha_n\sigma_1^n + (1-\alpha_n)\sigma_2^n).$$

Taking the limit on both sides of (4), the conclusion follows immediately.

(6)-(7) can by proved by the monotonicity of the relative entropy and monotonicity of the relative entropy with respect to taking the partial trace.

3. Quantum Jensen-Shannon divergence between quantum ensembles

Using the probabilistic coupling technique ([21]), we now define QJSD between quantum ensembles as follows:

$$D(\mathcal{E}\|\mathcal{F}) := \inf_c \sum_{ij} c_{ij} JS(\rho_i\|\sigma_j). \quad (5)$$

Some fundamental properties of QJSD between quantum ensembles of which many are natural heredities of the properties of the QJSD between quantum states are summarized in the following theorem.

Theorem 3.1. Let H_1, H_2 be two separable complex Hilbert spaces and $\mathcal{E} = \{p_i, \rho_i\}$, $\mathcal{F} = \{q_j, \sigma_j\}$ be any two quantum ensembles on H_1, $\mathcal{G} = \{r_k, \tau_k\}$ be any quantum ensemble on H_2.

(1) $0 \leq JS(\mathcal{E}\|\mathcal{F}) \leq 1$, $JS(\mathcal{E}\|\mathcal{F}) = 0$ iff $\mathcal{E}$ and $\mathcal{F}$ are identical and $JS(\mathcal{E}\|\mathcal{F}) = 1$ iff the supports of $\mathcal{E}$ and $\mathcal{F}$ are orthogonal sets of states.

(2) (Symmetry) $JS(\mathcal{E}\|\mathcal{F}) = JS(\mathcal{F}\|\mathcal{E})$.

(3) (Joint convexity) $JS(p\mathcal{E}_1 + (1-p)\mathcal{E}_2\|p\mathcal{F}_1 + (1-p)\mathcal{F}_2) \leq pJS(\mathcal{E}_1\|\mathcal{F}_1) + (1-p)JS(\mathcal{E}_2\|\mathcal{F}_2), \forall p \in [0,1]$.

(4) (Monotonicity under CPTP maps) $JS(\mathcal{E}\|\mathcal{F})$ is monotone under any trace-preseving quantum operation M in the sense that

$$JS(M(\mathcal{E})\|M(\mathcal{F})) \leq JS(\mathcal{E}\|\mathcal{F}).$$

In particular, $JS(\mathcal{E}\|\mathcal{F})$ is unitarily invariant in the sense that for any unitary operator U,

$$JS(U\mathcal{E}U^*\|U\mathcal{F}U^*) \leq JS(\mathcal{E}\|\mathcal{F}).$$

(5) (Monotonicity under averaging) Let $\bar{\rho} = \sum_i p_i\rho_i$ and $\bar{\sigma} = \sum_j q_j\sigma_j$ be the averages of the ensembles $\mathcal{E}$ and $\mathcal{F}$ respectively, then

$$JS(\bar{\rho}\|\bar{\sigma}) \leq JS(\mathcal{E}\|\mathcal{F}).$$

(6) (Stability) If we define the tensor product of two ensembles as the ensembles $\{p_i r_k, \rho_i \otimes \tau_k\}$ which we will denote by $\mathcal{E} \otimes \mathcal{G}$ for short, then

$$JS(\mathcal{E} \otimes \mathcal{G}\|\mathcal{F} \otimes \mathcal{G}) = JS(\mathcal{E}\|\mathcal{F}).$$

Proof. (1) From the definition (5), it is obvious that $JS(\mathcal{E}\|\mathcal{F}) \geq 0$ and $JS(\mathcal{E}\|\mathcal{F}) = 0$ iff $\mathcal{E} = \mathcal{F}$. The proof for the other part is similar to the proof of property 2 (Normalization) in [20], so we omit it here.

(2) The symmetry follows from definition (5) and the symmetry of QJSD between quantum states.

(3) Let c_{ij}^1 and c_{ij}^2 be two joint probability distributions which achieve the minimum in (5) for the pais of distributions $(\mathcal{E}_1, \mathcal{F}_1)$ and $(\mathcal{E}_2, \mathcal{F}_2)$, respectively. Then we can see that

$$c_{ij}^{12} = pc_{ij}^1 + (1-p)c_{ij}^2$$

is a joint probability distribution with marginals $p\mathcal{E}_1 + (1-p)\mathcal{E}_2$ and $p\mathcal{F}_1 + (1-p)\mathcal{F}_2$. Thus we have

$$\begin{aligned} JS(p\mathcal{E}_1 + (1-p)\mathcal{E}_2\|p\mathcal{F}_1 + (1-p)\mathcal{F}_2) \\ \leq \sum_{ij} c_{ij}^{12} JS(\rho_i\|\sigma_j) \\ = p\sum_{ij} c_{ij}^1 JS(\rho_i\|\sigma_j) + (1-p)\sum_{ij} c_{ij}^2 JS(\rho_i\|\sigma_j) \\ = pJS(\mathcal{E}_1\|\mathcal{F}_1) + (1-p)JS(\mathcal{E}_2\|\mathcal{F}_2). \end{aligned}$$

(4) By the monotonicity of the conventional QJSD (Theorem 2.1(6)), we have

$$JS(M(\rho_i)\|M(\sigma_j)) \leq JS(\rho_i\|\sigma_j),$$

which implies that

$$c_{ij}JS(M(\rho_i)\|M(\sigma_j)) \leq c_{ij}JS(\rho_i\|\sigma_j).$$

By taking the sum, we have

$$\sum_{ij} c_{ij}JS(M(\rho_i)\|M(\sigma_j)) \leq \sum_{ij} c_{ij}JS(\rho_i\|\sigma_j).$$

Consequently, from the definition (5), we have

$$JS(M(\mathcal{E})\|M(\mathcal{F})) \leq JS(\mathcal{E}\|\mathcal{F}).$$

(5) Note that $\rho := \sum_i p_i\rho_i$ and $\sigma := \sum_j q_j\sigma_j$ can be rewritten as

$$\rho = \sum_{ij} c_{ij}\rho_i, \sigma = \sum_{ij} c_{ij}\sigma_j,$$

where $c = \{c_{ij}\}$ is any coupling for $p = \{p_i\}$ and $q = \{q_j\}$. From the joint convexity of the conventional QJSD (Theorem 2.1(5)), we have

$$JS(\sum_{ij} c_{ij}\rho_i\|\sum_{ij} c_{ij}\sigma_j) \leq \sum_{ij} c_{ij}JS(\rho_i\|\sigma_j),$$

that is,

$$JS(\rho\|\sigma) \leq \sum_{ij} c_{ij}JS(\rho_i\|\sigma_j).$$

By taking the infimum over the coupling c, we obtain the desired result.

(6) Let $JS(\mathcal{E} \otimes \mathcal{G}\|\mathcal{F} \otimes \mathcal{G}) = \sum_{ijkk'} c_{ijkk'}JS(\rho_i \otimes \tau_k\|\sigma_j \otimes \tau_{k'})$, where

$$\sum_{jk'} c_{ijkk'} = p_i r_k, \sum_{ik} c_{ijkk'} = q_j r_{k'}.$$

By the monotonicity of QJSD under partial trace operation, it follows that

$$JS(\mathcal{E} \otimes \mathcal{G}\|\mathcal{F} \otimes \mathcal{G}) \geq \sum_{ij} c'_{ij}JS(\rho_i\|\sigma_j),$$

where $c'_{ij} = \sum_{kk'} c_{ijkk'}$ with marginals p_i and q_j. Therefore, we have

$$JS(\mathcal{E} \otimes \mathcal{G}\|\mathcal{F} \otimes \mathcal{G}) \geq JS(\mathcal{E}\|\mathcal{F}). \tag{6}$$

Taking $c_{ijkk'} = c_{ij}r_k\delta_{kk'}$, where c_{ij} is a joint distribution which attains the minimum in the definition of $JS(\mathcal{E}\|\mathcal{F})$, and using the stability of QJSD between quantum states (Theorem 2.1 (4)), the equality in (6) is attained.

The QJSD between quantum ensembles reduces to three limiting cases which is listed in the following theorem:

Theorem 3.2. Let H be a separable complex Hilbert space, $\mathcal{E}$ and $\mathcal{F}$ be any two quantum ensembles on H.

(1) (Two singleton ensembles) If $p_i = \delta_{ik}$, $q_j = \delta_{jk}$, i.e., the ensembles $\mathcal{E}$ and $\mathcal{F}$ degenerate to sets of single quantum state, $\mathcal{E} = \{1, \rho\}, \mathcal{F} = \{1, \sigma\}$, then $JS(\mathcal{E}\|\mathcal{F})$ reduces to the conventional QJSD between quantum states:

$$JS(\mathcal{E}\|\mathcal{F}) = JS(\rho\|\sigma).$$

(2) (One singleton ensemble) If the ensemble $\mathcal{F}$ consists of only one state σ, i.e., $q_j = \delta_{jk}$, then $JS(\mathcal{E}\|\mathcal{F})$ reduces to the average QJSD between a state drawn from the ensemble $\mathcal{E}$ and the state σ:

$$JS(\mathcal{E}\|\mathcal{F}) = \sum_i p_i JS(\rho_i\|\sigma).$$

(3) (Classical distribution) If the ensemble consists of only perfectly distinguishable states, i.e., $JS(\rho_i\|\sigma_j) = 1 - \delta_{ij}$, then $JS(\mathcal{E}\|\mathcal{F})$ reduces to the Kolmogorov distance between the classical probability distributions $\{p_i\}$ and $\{q_j\}$:

$$JS(\mathcal{E}\|\mathcal{F}) = \frac{1}{2}\sum_i |p_i - q_i|. \tag{7}$$

Proof. (1) The only joint probability distribution with marginals $\{p_i\}$ and $\{q_j\}$ in this case is $c_{ij} = \delta_{jk}\delta_{ik}$, and the conclusion follows.

(2) The only joint probability distribution with marginals $\{p_i\}$ and $\{q_j\}$ in this case is $c_{ij} = \delta_{jk}p_i$, and the conclusion follows.

(3) Noting that $\sum\limits_{ij,i\neq j} c_{ij} + \sum\limits_i c_{ii} = 1$, we can write the right hand side of (5) as

$$\inf_c \sum_{ij,i\neq j} c_{ij}1 + \sum_i c_{ii}0 = \inf_c (1 - \sum_i c_{ii}). \tag{8}$$

The minimum in (8) is achieved when $\sum\limits_i c_{ii}$ is maximal, which in turn is achieved when each of the terms c_{ii} is maximal. Since the maximum value of c_{ii} is $\min\{p_i, q_j\}$, we obtain

$$JS(\mathcal{E}\|\mathcal{F}) = 1 - \sum_i \min\{p_i, q_i\} = \frac{1}{2}\sum_i |p_i - q_i|.$$

This completes the proof.

Remark 3.1. Note that limiting case III corresponds to the probability distributions over a set of orthogonal states. Consider another limit corresponding to the classical probability distribution, i.e., when $\{p_i\}$ and $\{q_j\}$ are spectra of ρ and σ(which means that ρ and σ commute). Then the QJSD reduces to $JS(\{p_i\}, \{q_i\})$, which is different from (7).

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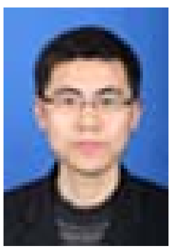

Zhaoqi Wu received his Ph.D. degree in mathematics from Zhejiang University, P. R. China. His research focuses on quantum information theory, especially on quantum measurement, quantum information measure and quantum operation. His research interests also includes nonlinear functional analysis, especially fixed point theory for nonlinear operators in probabilistic metric spaces. He is currently working as a lecturer in the Department of Mathematics, Nanchang University.

Shifang Zhang received his Ph.D. degree in mathematics from Zhejiang University, P. R. China. His research focuses on quantum information theory, especially on the mathematical foundations of quantum measurement and quantum operation. His research interests also includes operator theory, especially the theory of operator matrices and its spectra. He is currently working as a lecturer in School of Mathematics and Computer Science, Fujian Normal University.

Chuanxi Zhu received his Ph.D. degree in mathematics from Xi'an Jiaotong University, P. R. China. His research mainly focuses on nonlinear functional analysis, especially fixed point theory in probabilistic metric space and random functional analysis. His research interests also includes decision analysis and its applications in management science. He is now the dean of College of Science of Nanchang University and the department chair of Department of Mathematics. He published a large number of papers appeared in international peer-reviewed journals of mathematics.