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# Application of Fractional Calculus to a Class of Multivalent $\beta$-Uniformly Convex Functions

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In this paper we introduce a class of multivalent functions which is $\beta$-uniformly convex in the unit disc. Characterization property exhibited and relation with other fractional calculus operators are given. Connections with the popular classes like $\beta$-uniformly convex and parabolic convex functions are pointed out. Results on modified Hadamard product, extreme points, growth and distortion theorems, class preserving integral operators, region of $p$-valency and radius of $\beta$-uniform convexity are also derived.

**Keywords:** Multivalent, convex, $\beta$-uniformly convex, fractional calculus operator, region of $p$-valency, radius of $\beta$-uniform convexity.

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## 1 Introduction and Preliminaries

Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (n, p \in I\!N), \tag{1.1}$$

which are analytic in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $S(p)$ denote the class of functions defined by (1.1) which are analytic and multivalent in $U$. Consider the subclass $T(p)$ of $S(p)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \quad (a_k \geq 0, n, p \in I\!N). \tag{1.2}$$

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A function $f(z) \in S(p)$ is said to be multivalently starlike of order $s, 0 \leq s < p$ in $U$, if

$$Re\left\{z\frac{f'(z)}{f(z)}\right\} > s \tag{1.3}$$

and multivalently convex of order $s, 0 \leq s < p$ in $U$, if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > s. \tag{1.4}$$

A function $f(z) \in S(p)$ is said to be uniformly convex in $U$, if $f(z)$ is convex in $U$ and has the property that every circular are $\gamma$, contained in $U$ with center $\xi$ in $U$, arc $f(\gamma)$ is convex with respect to $f(\xi)$.

This definition of uniformly convex functions was given by A. W. Goodman [4] in 1991.

The class of uniformly convex functions is denoted by $UCV$. We have the characterization: $f \in UCV$, if and only if

$$Re\left\{1 + z\frac{f''(z)}{f'(z)}\right\} \geq \left|1 + z\frac{f''(z)}{f'(z)} - p\right|. \tag{1.5}$$

We can further generalize the class $UCV$ by introducing a parameter $\alpha, -p \leq \alpha < p$.

$f \in UCV(\alpha)$ if and only if

$$Re\left\{1 + z\frac{f''(z)}{f'(z)} - \alpha\right\} \geq \left|1 + z\frac{f''(z)}{f'(z)} - p\right|. \tag{1.6}$$

Further, let $0 \leq \beta < \infty$. Then the function $f \in S(p)$ is said to be $\beta$-uniformly convex in $U$, if the image of every circular arc $\gamma$ contained in $U$, with center $\xi$ in $U$, where $|\xi| \leq \beta$, is convex. For fixed $\beta$, the class of all $\beta$-uniformly convex functions is denoted by $\beta - UCV$. Notice that, $0 - UCV = CV$, set of all convex functions and $1 - UCV = UCV$ as defined in (1.5).

$0 - UCV(\alpha) = CV(\alpha)$, set of all convex functions of order $\alpha, -p \leq \alpha < p, 1 - UCV(\alpha) = UCV(\alpha)$ as defined in (1.6) as before. We again note that $f \in \beta - UCV(\alpha)$, if and only if

$$Re\left\{1 + z\frac{f''(z)}{f'(z)} - \alpha\right\} \geq \beta\left|1 + z\frac{zf''(z)}{f'(z)} - p\right|. \tag{1.7}$$

The class $\beta - UCV$ was introduced by S. Kanas *et al.* [5], where its geometric properties and connections with convex domains were considered. S. Kanas and H. M. Srivastava [6] studied this class in detail. Later on, A. Gangadharan *et al.* [3] used linear operators to find the connections between the class $\beta - UCV$ and the different subclasses of the class of analytic and univalent functions defined in the unit disc.

Let the function $f(z)$ and $g(z)$ defined by

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \tag{1.8}$$

and

$$g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k \tag{1.9}$$

belong to $K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$ and $K(\mu, \gamma, \eta, a, b, c, \xi, \beta)$, respectively. Then the modified Hadamard product of $f$ and $g$ is defined by

$$(f * g)(z) = z^p - \sum_{k=p+n}^{\infty} a_k b_k z^k. \tag{1.10}$$

The incomplete beta function $\phi_p(a, c; z)$ is defined by

$$\phi_p(a, c; z) = z^p + \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} z^k \tag{1.11}$$

for $a \in I\!R$ and $c \in I\!R \setminus \overline{z}_0$ where $\overline{z}_0 = \{0, -1, -2, \ldots\}$, $z \in U$. $(a)_k$ is the Pochhammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & , \quad k = 0 \\ a(a+1)\cdots(a+k-1) & , \quad k \in I\!N \end{cases} .$$

Next, we consider the Carlson-Shaffer operator [1] defined by

$$L_p(a, c)f(z) = \phi_p(a, c; z) * f(z), \quad \text{for} \quad f \in S(p) = z^p + \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}}{(c)_{k-p}} \ a_k z^k. \tag{1.12}$$

The Gaussian hypergeometric function denoted by ${}_2F_1(a, b; c; z)$ and is defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \ \ z \in U \tag{1.13}$$

and $a + b < c$.

Now, using the convolution theorem we can define the Hohlov operator $F_p(a, b; c) : T(p) \to T(p)$ by the following relation:

$$F_p(a, b; c)(f(z)) = z^p \ {}_2F_1(a, b; c; z) * f(z) = z^p - \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}(b)_{k-p}}{(c)_{k-p}(k-p)!} a_k z^k, \tag{1.14}$$

$a, b \in I\!R$ and $c \in I\!R \setminus \overline{z}_0$, where $\overline{z}_0 = \{0, -1, -2, \ldots\}$, $z \in U$. Notice that, Hohlov operator reduces to Carlson-Shaffer operator if $b = 1$. Also for $a = m + 1, b = c = 1$, we get the famous Ruscheweyh derivative operator of order $m$. We can write

$$F_p(a, b; c)f(z) = z^p - \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}(b)_{k-p}}{(c)_{k-p}(1)_{k-p}} a_k z^k. \tag{1.15}$$

**Definition 1.1.** Let $\mu > 0$ and $\gamma, \eta \in I\!R$. Then the generalized fractional integral operator $I_{0,z}^{\mu,\gamma,\eta}$ of a function $f(z)$ is defined by

$$I_{0,}^{\mu,\gamma,\eta} f(z) = \frac{z^{-\mu-\gamma}}{\Gamma(\mu)} \int_0^z (z-t)^{\mu-1} f(t)\, {}_2F_1\Big(\mu+\gamma, -\eta; \mu; 1-\frac{t}{z}\Big) dt, \tag{1.16}$$

where $f(z)$ is analytic in a simply-connected region of the $z$-plane containing the origin, with order

$$f(z) = 0(|z|^r), \ \ z \to 0, \tag{1.17}$$

where $r > \max\{0, \mu-\eta\} - 1$ and the multiplicity of $(z-t)^{\mu-1}$ is removed by requiring $\log(z-t)$ to be real, when $(z-t) > 0$ and is well defined in the unit disc.

**Definition 1.2.** Let $0 \le \mu < 1$ and $\gamma, \eta \in I\!R$. Then the generalized fractional derivative operator $J_{0,z}^{\mu,\gamma,\eta}$ of a function $f(z)$ is defined by

$$J_{0,z}^{\mu,\gamma,\eta} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{dz}\Big\{ z^{\mu-\gamma} \int_0^z (z-t)^{-\mu} f(t)\, {}_2F_1\big(\gamma-\mu, 1-\eta; 1-\mu; 1-\frac{t}{z}\big) dt \Big\}, \tag{1.18}$$

where the function is analytic in the simply-connected region of $z$-plane containing the origin, with the order as given in (1.17) and multiplicity of $(z-t)^{-\mu}$ is removed by requiring $\log(z-t)$ to be real when $(z-t) > 0$. Notice that, we have the following relationships with the fractional integral and derivative operators of order $\mu$.

$$I_{0,z}^{\mu,-\mu,\eta} f(z) = D_{0,z}^{-\mu} f(z) \quad (\mu > 0),$$

$$J_{0,z}^{\mu,\mu,\eta} f(z) = D_{0,z}^{\mu} f(z) \quad (0 \le \mu < 1).$$

Consider the fractional operator $U_{0,z}^{\mu,\gamma,\eta}$ defined in terms of $J_{0,z}^{\mu,\gamma,\eta}$ as follows:

$$U_{0,z}^{\mu,\gamma,\eta} f(z) = \begin{cases} \dfrac{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p)\Gamma(1+p+\eta-\gamma)} z^{\gamma} J_{0,z}^{\mu,\gamma,\eta}(f(z), & 0 \le \mu < 1 \\ \dfrac{\Gamma(1+p-\gamma)\Gamma(1+p+\eta-\mu)}{\Gamma(1+p)\Gamma(1+p+\eta-\gamma)} z^{\gamma} I_{0,z}^{-\mu,\gamma,\eta} f(z), & -\infty < \mu < 0 \end{cases}. \tag{1.19}$$

Let

$$\begin{aligned} Lf(z) &= M_{0,z}^{\mu,\gamma,\eta,a,b,c} f(z) \\ &= F_p(a,b;c;z) * U_{0,z}^{\mu,\gamma,\eta} f(z) \\ &= z^p + \sum_{k=p+n}^{\infty} \frac{(a)_{k-p}(b)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1)_{k-p}(1+p+\eta-\mu)_{k-p}(1+p-\gamma)_{k-p}} a_k z^k \end{aligned} \tag{1.20}$$

for $a, b \in I\!R, c \in I\!R \setminus \overline{z}_0, \overline{z}_0 = \{0, -1, -2, \ldots\}, \ -\infty < \mu < 1, -\infty < \gamma < 1, \eta \in I\!R^+$, $-p \le \alpha < p, \beta \ge 0$ and $f \in S(p)$.

For convenience, we will write $Lf$ as follows:

$$Lf(z) = z^p + \sum_{k=p+n}^{\infty} g(k)a_k z^k, \tag{1.21}$$

where

$$g(k) = \frac{(a)_{k-p}(b)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1)_{k-p}(1+p+\eta-\mu)_{k-p}(1+p-\gamma)_{k-p}}. \tag{1.22}$$

Let $K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$ denote the class of function $f \in S(p)$ satisfying

$$Re\left\{1 + \frac{z(Lf)''}{(Lf)'} - \alpha\right\} \geq \beta\left|1 + \frac{z(Lf)''}{(Lf)'} - p\right|, \tag{1.23}$$

where $(a, b \in I\!R, c \in I\!R \setminus \overline{z}_0, \overline{z}_0 = \{0, -1, -2, \ldots\}, -\infty < \mu < 1, -\infty < \gamma < 1, \eta \in I\!R^+$, and $-p \leq \alpha < p, \beta \geq 0, z \in U)$.

It is very interesting to notice that the class $K(\mu, \gamma, \eta, a, b, c)$ reduces to the class of convex, $\beta$-uniformly convex parabolic convex functions for suitable choice of the parameters $a, b, c, \mu, \gamma, \eta, \alpha$ and $\beta$. For instance,

1. If $a = c, b = 1, \mu = \gamma = 0$ the class reduces to $\beta - UCV(\alpha)$.
2. If $a = c, b = 1, \mu = \gamma = 0, \alpha = 2\rho - 1, (0 \leq \rho < 1)$ the class reduces to parabolic convex of order $\rho$.

Other interesting classes studied by different authors can be derived from $K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$.

## 2 Some Results for the Class $K(\mu, \gamma, \eta, a, b, c)$

**Theorem 2.1.** *A function $f \in T(p)$ is in the class $K(\mu, \gamma, \eta, a, b, c)$ if and only if*

$$\sum_{k=p+n}^{\infty} k[k(1+\beta) - (\alpha + p\beta)]g(k)a_k \leq p(p-\alpha). \tag{2.1}$$

*The result is sharp for the function*

$$f(z) = z^p - \frac{p(p-\alpha)}{k[k(1+\beta) - (\alpha + p\beta)]g(k)} z^{p+n}, \quad n \in I\!N. \tag{2.2}$$

*Proof.* Assume that $f \in K(\mu, \gamma, \eta, a, b, c)$ and $z$ is real. Then we have from (1.23)

$$\frac{p^2 - \sum_{k=p+n}^{\infty} k^2 g(k)a_k z^{k-p}}{p - \sum_{k=p+n}^{\infty} kg(k)a_k z^{k-p}} - \alpha \geq \beta\left|\frac{\sum_{k=p+n}^{\infty}(k-p)g(k)a_k z^{k-p}}{p - \sum_{k=p+n}^{\infty} kg(k)a_k z^{k-p}}\right|.$$

Allowing $z \to 1$ along the real axis, we obtain the desired inequality (2.1).

Conversely, let us assume that (2.1) holds, then we show that

$$\beta\left|1+\frac{z(Lf)''}{(Lf)'}-p\right|-Re\left\{1+\frac{z(Lf)''}{(Lf)'}-p\right\}\leq p-\alpha.$$

Notice that

$$\begin{aligned}\beta\left|1+z\frac{(Lf)''}{(Lf)'}-p\right|-Re\left\{1+\frac{z(Lf)''}{(Lf)'}-p\right\}&\leq(1+\beta)\left|1+\frac{z(Lf)''}{(Lf)'}-p\right|\\&\leq\frac{(1+\beta)\sum_{k=p+n}^{\infty}(k-p)g(k)a_k}{p-\sum_{k=p+n}^{\infty}kg(k)a_k}.\end{aligned}$$

This expression is bounded above by $(p-\alpha)$ if

$$\sum_{k=p+n}^{\infty}k[k(1+\beta)-(\alpha+p\beta)]g(k)a_k\leq p(p-\alpha).$$ □

**Corollary 2.1.** *Let the function $f(z)$ defined by (1.2) be in the class $K(\mu,\gamma,\eta,a,b,c,\alpha,\beta)$. Then*

$$a_k\leq\frac{p(p-\alpha)}{k[k(1+\beta)-(\alpha+p\beta)]g(k)},\quad(k\geq p+n,n\in I\!N)$$

*with equality for the function $f(z)$ given by (2.2).*

**Theorem 2.2.** *Let the function $f$ and $g$ be in the class $K(\mu,\gamma,\eta,a,b,c)$. Then for $\lambda\in[0,1]$, the function*

$$h(z)=(1-\lambda)f(z)+\lambda g(z)=z^p-\sum_{k=p+n}^{\infty}d_kz^k$$

*is in the class $K(\mu,\gamma,\eta,a,b,c)$.*

*Proof.* Since $f$ and $g$ are in the class $K(\mu,\gamma,\eta,a,b,c)$, they satisfy the inequality (2.1). Thus, the function $h(z)$ defined by

$$h(z)=(1-\lambda)f(z)+\lambda g(z)=z^p-\sum_{k=p+n}^{\infty}[(1-\lambda)a_k+\lambda b_k]z^k$$

is also in the class $K(\mu,\gamma,\eta,a,b,c)$. This immediately follows by setting $d_k=(1-\lambda)a_k+\lambda b_k>0$. Therefore, $K(\mu,\gamma,\eta,a,b,c)$ is a convex set. □

**Theorem 2.3.** *Let $f(z)$ and $g(z)$ defined by (1.8) and (1.9) be in the class $K(\mu,\gamma,\eta,a,b,c)$. Then the function $h(z)$ defined by*

$$h(z)=z^p-\sum_{k=p+n}^{\infty}(a_k^2+b_k^2)z^k$$

*is in the class $K(\mu,\gamma,\eta,a,b,c,\theta,\beta)$, where*

$$\theta=p-\frac{2p(1+\beta)(p-\alpha)^2}{(1+p)(1+p+\beta-\alpha)^2g(p+1)-2p(p-\alpha)^2}.$$

*Proof.* In view of Theorem 2.1 it is sufficient to show that

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta)-(\theta+p\beta)]}{p(p-\theta)} g(k)(a_k^2+b_k^2) \leq 1. \tag{2.3}$$

Notice that $f(z)$ and $g(z)$ belong to $K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$, thus

$$\sum_{k=p+n}^{\infty} \left\{ \frac{k[k(1+\beta)-(\alpha+p\beta)]g(k)}{p(p-\alpha)} \right\}^2 a_k^2 \leq \left[ \sum_{k=p+n}^{\infty} \frac{k[k(1+\beta)-(\alpha+p\beta)]g(k)a_k}{p(p-\alpha)} \right]^2 \leq 1, \tag{2.4}$$

$$\sum_{k=p+n}^{\infty} \left\{ \frac{k[k(1+\beta)-(\alpha+p\beta)]g(k)}{p(p-\alpha)} \right\}^2 b_k^2 \leq \left[ \sum_{k=p+n}^{\infty} \frac{k[k(1+\beta)-(\alpha+p\beta)]g(k)b_k}{p(p-\alpha)} \right]^2 \leq 1. \tag{2.5}$$

Adding (2.4) and (2.5), we get

$$\sum_{k=p+n}^{\infty} \frac{1}{2} \left\{ \frac{k[k(1+\beta)-(\alpha+p\beta)g(k)}{p(p-\alpha)} \right\}^2 (a_k^2+b_k^2) \leq 1. \tag{2.6}$$

Thus, (2.3) will hold if

$$\frac{k(1+\beta)-(\theta+p\beta)}{(p-\theta)} \leq \frac{1}{2} \frac{k[k(1+\beta)-(\alpha+p\beta)]^2 g(k)}{p(p-\alpha)^2}.$$

That is, if

$$\theta \leq p - \frac{2p(1+\beta)(k-p)(p-\alpha)^2}{k[k(1+\beta)-(\alpha+p\beta)]^2 g(k) - 2p(p-\alpha)^2}. \tag{2.7}$$

Notice that, $\theta$ can be further improved by using the fact that $g(k)$ is a non-increasing function of $k$, for $k \geq p+n, n \in I\!N$. Thus, $g(p+n) \leq g(p+1)$ for $n \in I\!N$ and

$$g(p+1) = \frac{ab(1+p)(1+p+\eta-\gamma)}{c(1+p+\eta-\mu)(1+p-\gamma)}. \tag{2.8}$$

Therefore,

$$\theta = p - \frac{2p(1+\beta)(p-\alpha)^2}{(1+p)(1+p+\beta-\alpha)^2 g(p+1) - 2p(p-\alpha)^2},$$

where $g(p+1)$ is given by (2.8). □

Next, we give another inclusion property of the class.

**Theorem 2.4.** *Let $f_j(z)$ defined by*

$$f_j(z) = z^p - \sum_{k=p+n}^{\infty} a_{k,j} z^k, \quad j = 1, 2, \ldots, \ell$$

*belong to the class $K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$. Then the function*

$$h(z) = \frac{1}{\ell} \sum_{j=1}^{\ell} f_j(z)$$

*is also in the class $K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$.*

*Proof.* Since $f_j(z) \in K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$, in view of Theorem 2.1, we have

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta) - (\alpha + p\beta)]g(k)}{p(p-\alpha)} a_{k,j} \leq 1. \tag{2.9}$$

Now,

$$h(z) = \frac{1}{\ell} \sum_{j=1}^{\ell} f_j(z) = z^p - \frac{1}{\ell} \sum_{j=1}^{\ell} \sum_{k=p+n}^{\infty} a_{k,j} z^k = z^p - \sum_{k=p+n}^{\infty} e_k z^k,$$

where

$$e_k = \frac{1}{\ell} \sum_{j=1}^{\ell} a_{k,j}.$$

Notice that

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta) - (\alpha + p\beta)]g(k)}{p(p-\alpha)} \frac{1}{\ell} \sum_{j=1}^{\ell} a_{k,j} \leq 1$$

using (2.9). Thus, $h(z) \in K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$. □

## 3 Connections with Other Fractional Calculus Operators

**Theorem 3.1.** *Let*

$$\frac{ab(1+p)(1+p+\eta-\gamma)}{c(1+p+\eta-\mu)(1+p-\gamma)} \leq 1 \tag{3.1}$$

*for $a, b \in I\!R, c \in I\!R \setminus \overline{z}_0, \overline{z}_0 = \{0, -1, -2, \ldots\}, -\infty < \mu < 1, -\infty < \gamma < 1, \eta \in I\!R^+$, $-p \leq \alpha < p, \beta \geq 0$. Also let the function $f(z)$ defined by (1.2) satisfy*

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta) - (\alpha + p\beta)]g(k)}{p(p-\alpha)} a_k \leq \frac{c(1+p+\eta-\mu)(1+p-\gamma)}{ab(1+p)(1+p+\eta-\gamma)}. \tag{3.2}$$

*Then $Lf(z) \in K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$ where $g(k)$ is given by (1.22).*

*Proof.* We have,

$$Lf(z) = z^p - \sum_{k=p+n}^{\infty} g(k) a_k z^k, \tag{3.3}$$

where

$$g(k) = \frac{(a)_{k-p}(b)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1)_{k-p}(1+p+\eta-\mu)_{k-p}(1+p-\gamma)_{k-p}}.$$

Under the hypothesis of the theorem, we observe that the function $g(k)$ is a non-increasing function, that is, $g(p+n) \leq g(p+1), n \in I\!N$. Thus,

$$0 < g(p+n) \leq g(p+1) = \frac{ab(1+p)(1+p+\eta-\gamma)}{c(1+p+\eta-\mu)(1+p-\gamma)}. \tag{3.4}$$

In view of (3.2) and (3.4), we now have

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta)-(\alpha+p\beta)]g^2(k)}{p(p-\alpha)} a_k \leq g(p+1),$$
$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta)-(\alpha+p\beta)]g(k)}{p(p-\alpha)} \leq 1.$$

Therefore, by Theorem 2.1, we conclude that

$$Lf(z) \in K(\mu,\gamma,\eta,a,b,c,\alpha,\beta). \qquad \square$$

**Remark 3.1.** The equality in (3.2) is attained for the function

$$f(z) = z^p - \frac{cp(p-\alpha)(1+p+\eta-\mu)(1+p-\gamma)}{ab(1+p+\beta-\alpha)(1+p)^2(1+p+\eta-\gamma)} z^{p+1}. \tag{3.5}$$

**Corollary 3.1.** *Let $\mu,\gamma,\eta$ be such that $\mu \geq 0, \gamma < 1+p$, and*

$$\max\{\mu,\gamma\} - (1+p) < \eta \leq \frac{\mu(\gamma-(2+p))}{\gamma}. \tag{3.6}$$

*Also let the function $f(z)$ by (1.2) satisfy*

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta)-(\alpha+p\beta)]g(k)}{p(p-\alpha)} a_k \leq \frac{(1+p+\eta-\mu)(1+p-\gamma)}{(1+p)(1+p+\eta-\gamma)} \tag{3.7}$$

*for $-p \leq \alpha < p, \beta \geq 0$. Then*

$$Lf(z) = J_{0,z}^{\mu,\gamma,\eta} f(z) \in \beta - UCV(\alpha).$$

*Proof.* The corollary follows from Theorem 3.1 by setting $a = c, b = 1$. $\square$

**Corollary 3.2.** *Let $\mu,\gamma,\eta \in I\!R$ such that $\mu \geq 0, \gamma < 1+p$, and*

$$\max\{\mu,\gamma\} - (1+p) < \eta \leq \frac{\mu(\gamma-(2+p))}{\gamma}.$$

*Also let the function $f(z)$ defined by (1.2) satisfy*

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta)-(\alpha+p\beta)]g(k)}{p(p-\alpha)} a_k \leq \frac{(1+p-\mu)}{(1+p)}$$

*for $-p \leq \alpha < p, \beta \geq 0$. Then*

$$Lf(z) = D_{0,z}^{\mu} f(z) \in \beta - UCV(\alpha).$$

*Proof.* The corollary follows from Theorem 3.1 by setting $a = c, b = 1, \mu = \gamma$. □

**Corollary 3.3.** *Let $\mu, \gamma, \eta \in I\!R$ such that $\mu \geq 0, \gamma < 1 + p$, and*

$$\max\{\mu, \gamma\} - (1 + p) < \eta \leq \frac{\mu(\gamma - (2 + p))}{\gamma}.$$

*Also, let the function $f(z)$ defined by (1.2) satisfy*

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta) - (\alpha + p\beta)]g(k)}{p(p-\alpha)} a_k \leq \frac{c}{ab}.$$

*Then $Lf(z) = F_p(a, b; c)f(z) \in \beta - UCV(\alpha)$.*

*Proof.* Corollary follows from Theorem 3.1 by setting $\mu = \gamma = 0$. □

**Corollary 3.4.** *Let the hypothesis of Corollary 3.3 be true and*

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta) - (\alpha + p\beta)]g(k)}{p(p-\alpha)} a_k \leq \frac{c}{a}.$$

*then*

$$Lf(z) = L_p(a, c)f(z) \in \beta - UCV(\alpha).$$

*Proof.* The corollary follows from Theorem 3.1 by setting $\mu = \gamma = 0, b = 1$. □

## 4 Results on Modified Hadamard Product

**Theorem 4.1.** *Let the function $f(z)$ and $g(z)$ defined by*

$$f(z) = z^p - \sum_{k=p+n}^{\infty} a_k z^k \tag{4.1}$$

*and*

$$g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k \tag{4.2}$$

*belong to $K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$ and $K(\mu, \gamma, \eta, a, b, c, \xi, \beta)$, respectively. Also assume that*

$$\frac{ab(1+p)(1+p+\eta-\gamma)}{c(1+p+\eta-\mu)(1+p-\gamma)} \leq 1.$$

*Then $(f * g)(z) \in K(\mu, \gamma, \eta, a, b, c, \delta, \beta)$, where*

$$\delta = p - \frac{p(1+\beta)(p-\alpha)(p-\xi)}{k(1+p+\beta-\alpha)(1+p+\beta-\xi)g(p+1) - p(p-\alpha)(p-\xi)}, \tag{4.3}$$

*and the result is sharp for*

$$f(z) = z^p - \frac{p(p-\alpha)}{(p+1)(1+p+\beta-\alpha)g(p+1)} z^{p+1},$$

$$g(z) = z^p - \frac{p(p-\xi)}{(p+1)(1+p+\beta-\xi)g(p+1)} z^{p+1}.$$

*Proof.* To prove the theorem it is sufficient to show that

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta)-(\delta+p\beta)]}{p(p-\delta)} g(k) a_k b_k \le 1, \tag{4.4}$$

where $g(p+1)$ is defined by (3.4).

Since $f(z) \in K(\mu,\gamma,\eta,a,b,c,\alpha,\beta)$ and $g(z) \in K(\mu,\gamma,\eta,a,b,c,\xi,\beta)$ , we have

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta)-(\alpha+p\beta)]g(k)}{p(p-\alpha)} a_k \le 1, \tag{4.5}$$

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta)-(\xi+p\beta)]g(k)}{p(p-\xi)} b_k \le 1. \tag{4.6}$$

Applying Cauchy-Schwarz inequality to (4.5) and (4.6), we get

$$\sum_{k=p+n}^{\infty} \frac{k\sqrt{[k(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\xi+p\beta)]}g(k)}{p\sqrt{(p-\alpha)(p-\xi)}} \sqrt{a_k b_k} \le 1. \tag{4.7}$$

In view of (4.4) it suffices to show that

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta)-(\delta+p\beta)]g(k)}{p(p-\delta)} a_k b_k$$
$$\le \sum_{k=p+n}^{\infty} \frac{k\sqrt{[k(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\xi+p\beta)]}g(k)}{p\sqrt{(p-\alpha)(p-\xi)}} \sqrt{a_k b_k}.$$

Or equivalently, for $k \ge p+1$.

$$\sqrt{a_k b_k} \le \frac{\sqrt{k[(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\xi+p\beta)}}{\sqrt{(p-\alpha)(p-\xi)}} \frac{(p-\delta)}{[k(1+\beta)-(\delta+p\beta)]}. \tag{4.8}$$

In view of (4.7) and (4.8) it is enough to show that

$$\frac{p\sqrt{(p-\alpha)(p-\xi)}}{k\sqrt{[k(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\xi+p\beta)]}g(k)}$$
$$\le \frac{\sqrt{[k(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\xi+p\beta)}(p-\delta)}{\sqrt{(p-\alpha)(p-\xi)}[k(1+\beta)-(\delta+p\beta)]},$$

which simplices to

$$\delta \leq p - \frac{p(k-p)(1+\beta)(p-\alpha)(p-\xi)}{k[k(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\xi+p\beta)]g(k)-p(p-\alpha)(p-\xi)}$$

with $g(k)$ given by (1.22). Using the fact that $g(k)$ is a decreasing function of $k$ $(k \geq p+1)$, we can choose $\delta$ such that

$$\delta = p - \frac{p(1+\beta)(p-\alpha)(p-\xi)}{k(1+p+\beta-\alpha)(1+p+\beta-\xi)g(p+1)-p(p-\alpha)(p-\xi)},$$

where

$$g(p+1) = \frac{ab(1+p)(1+p+\eta-\gamma)}{c(1+p+\eta-\mu)(1+p-\gamma)}.$$ □

**Theorem 4.2.** *Let the function $f(z)$ and $g(z)$ defined by (4.1) and (4.2) be in the class $K(\mu,\gamma,\eta,a,b,c,\alpha,\beta)$. Then $(f*g)(z) \in K(\mu,\gamma,\eta,a,b,c,\delta,\beta)$ where*

$$\delta = p - \frac{p(1+\beta)(p-\alpha)^2}{k(1+p+\beta-\alpha)^2 g(p+1)-p(p-\alpha)^2}$$

*for $g(p+1)$ given by (2.8).*

*Proof.* Substituting $\alpha = \beta$ in Theorem 4.1, the result follows. □

**Corollary 4.1.** *Let the function $f(z)$ defined by (1.2) be in the class $K(\mu,\gamma,\eta,a,b,c,\alpha,\beta)$ and let $g(z) = z^p - \sum_{k=p+n}^{\infty} b_k z^k$ for $|b_k| \leq 1$. Then $(f*g)(z) \in K(\mu,\gamma,\eta,a,b,c,\alpha,\beta)$.*

## 5 Extreme Points of the Class $K(\mu,\gamma,\eta,a,b,c,\alpha,\beta)$

**Theorem 5.1.** *Let $f(z)_p = z^p$ and*

$$f_k(z) = z^k - \frac{p(p-\alpha)}{k[k(1+\beta)-(\alpha+p\beta)]}g(k)z^k, \quad (k \geq p+1).$$

*Then $f(z) \in K(\mu,\gamma,\eta,a,b,c,\alpha,\beta)$ if and only if $f(z)$ can be expressed in the form*

$$f(z) = \sum_{k=p}^{\infty} \lambda_k f_k(z), \tag{5.1}$$

*where $\lambda_k \geq 0$ and $\sum_{k=p}^{\infty} \lambda_k = 1$.*

*Proof.* Let $f(z)$ be expressible in the form

$$f(z) = \sum_{k=p}^{\infty} \lambda_k f_k(z) = z^k - \sum_{k=p+1}^{\infty} \frac{p(p-\alpha)}{k[k(1+\beta)-(\alpha+p\beta)]g(k)}\lambda_k z^k.$$

But

$$\sum_{k=p+1}^{\infty} \frac{p(p-\alpha)\lambda_k}{k[k(1+\beta)-(\alpha+p\beta)]g(k)} \frac{k[k(1+\beta)-(\alpha+p\beta)]g(k)}{p(p-\alpha)} = \sum_{k=p+1}^{\infty} \lambda_k = 1-\lambda_p \leq 1.$$

Therefore, $f(z) \in K(\mu,\gamma,\eta,a,b,c,\alpha,\beta)$. Conversely, suppose that $f(z) \in K(\mu,\gamma,\eta,a,b,c,\alpha,\beta)$. Setting

$$\lambda_k = \frac{k[k(1+\beta)-(\alpha+p\beta)]g(k)}{p(p-\alpha)} a_k \text{ and } \lambda_p = 1 - \sum_{k=p+1}^{\infty} \lambda_k$$

we see that $f(z)$ can be expressed in the form (5.1). □

**Corollary 5.1.** *The extreme points of the class $K(\mu,\gamma,\eta,a,b,c,\alpha,\beta)$ are $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{p(p-\alpha)}{k[k(1+\beta)-(\alpha+p\beta)]g(k)} z^k, \quad k \geq p+1.$$

## 6 Growth and Distortion Theorems

**Theorem 6.1.** *Let the function $f(z)$ defined by (1.2) be in the class $K(\mu,\gamma,\eta,a,b,c,\alpha,\beta)$. Then*

$$||Lf(z)| - |z|^p| \leq \frac{cp(p-\alpha)(1+p-\gamma)(1+p+\eta-\mu)}{ab(1+p)(1+p+\eta-\gamma)(1+p+\beta-\alpha)} |z|^{p+1}, \tag{6.1}$$

*and*

$$||(Lf(z))'| - p|z|^{p-1}| \leq \frac{cp(p-\alpha)(1+p-\gamma)(1+p+\eta-\mu)}{ab(1+p+\eta-\gamma)(1+p+\beta-\alpha)} |z|^p. \tag{6.2}$$

**Remark 6.1.** The result (6.1) and (6.2) are sharp for the extremal function $f(z)$ given by

$$f(z) = z^p - \frac{cp(p-\alpha)(1+p-\gamma)(1+p+\eta-\mu)}{ab(1+p)(1+p+\eta-\gamma)(1+p+\beta-\alpha)} z^{p+1}. \tag{6.3}$$

**Corollary 6.1.** *Let $Lf(z) \in K(\mu,\gamma,\eta,a,b,c,\alpha,\beta)$ then the disc $|z| < 1$ is mapped onto a domain that contains the disc*

$$|w| < 1 - \frac{cp(p-\alpha)(1+p-\gamma)(1+p+\eta-\mu)}{ab(1+p)(1+p+\eta-\mu)(1+p+\beta-\alpha)}.$$

*Also $(Lf(z))'$ maps the disc $|z| < 1$ onto a domain that contains the disc*

$$|w| < p - \frac{cp(p-\alpha)(1+p-\gamma)(1+p+\eta-\mu)}{ab(1+p+\eta-\gamma)(1+p+\beta-\alpha)}.$$

**Remark 6.2.** We can obtain the Growth and Distortion Theorems for $J_{0,}^{\mu,\gamma,\eta} f(z)$, $D_{0,z}^{\mu} f(z)$, $F_p(a,b;c)f(z)$ and $L_p(a,c)f(z)$ by accordingly initializing the parameters.

## 7 Class Preserving Integral Operators

The integral operator $F(z)$ defined by

$$F(z) = z^{p-1} \int_0^z \frac{f(t)}{t^p} dt \tag{7.1}$$

is class preserving. The Komatu integral operator [5] is defined by

$$H(z) = P_{c,p}^d f(z) = \frac{(c+p)^d}{\Gamma(d) z^c} \int_0^z t^{c-1} \Big( \log \frac{z}{t} \Big)^{d-1} f(t) dt, \quad d > 0,\ c > -p,\ z \in U \tag{7.2}$$

and the integral operator

$$I(z) = Q_{c,p}^d f(z) = \begin{pmatrix} d+c+p-1 \\ c+p-1 \end{pmatrix} \frac{d}{z^c} \int_0^z t^{c-1} \left(1 - \frac{t}{z}\right)^{d-1} f(t) dt, \tag{7.3}$$

$(d > 0, c > -p, z \in U)$, is also class preserving. We note that

$$H(z) = z^p - \sum_{k=p+n}^{\infty} \left( \frac{c+p}{c+k} \right)^d a_k z^k \tag{7.4}$$

and

$$I(z) = z^p - \sum_{k=p+n}^{\infty} \frac{\Gamma(c+k)\Gamma(d+c+p)}{\Gamma(d+c+k)\Gamma(c+p)} a_k z^k. \tag{7.5}$$

It can be easily proved that these are class preserving integral operators.

**Theorem 7.1.** *Let $d > 0, c > -p$ and $f(z)$ belong to the class $K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$. Then the function $H(z)$ defined by (7.2) is p-valent in the disc $|z| < R_1$, where*

$$R_1 = \inf_k \left\{ \frac{[k(1+\beta) - (\alpha + p\beta)] g(k) (c+k)^d}{(p-\alpha)(c+p)^d} \right\}^{1/(k-p)}. \tag{7.6}$$

*The result is sharp for the function $f(z)$ given by*

$$f(z) = z^p - \frac{(p-\alpha)(c+1)^d}{[k(1+\beta) - (\alpha + p\beta)] g(k) (c+k)^d} z^{p+n}, \ \ n \in N.$$

*Proof.* We show that

$$\left| \frac{H'(z)}{z^{p-1}} - p \right| \le p \ \text{ in } \ |z| < R_1, \tag{7.7}$$

where $R_1$ is given by (7.6).

In view of (7.4), we have

$$\left| \frac{H'(z)}{z^{p-1}} - p \right| = \left| - \sum_{k=p+n}^{\infty} k \left( \frac{c+p}{c+k} \right)^d a_k z^{k-p} \right| \le \sum_{k=p+n}^{\infty} k \left( \frac{c+p}{c+k} \right)^d a_k |z|^{k-p}.$$

The last inequality is bounded above by $p$ if

$$\sum_{k=p+n}^{\infty} \frac{k}{p} \left(\frac{c+1}{c+k}\right)^d a_k |z|^{k-p} \leq 1. \tag{7.8}$$

Also, $f(z) \in K(\gamma, \eta, , a, b, c, \alpha, \beta)$ and so

$$\sum_{k=p+n}^{\infty} \frac{k[k(1+\beta)-(\alpha+p\beta)]g(k)}{p(p-\alpha)} a_k \leq 1. \tag{7.9}$$

Thus, inequality (7.8) will hold if

$$\frac{k}{p}\left(\frac{c+p}{c+k}\right)^d |z|^{k-p} \leq \frac{k[k(1+\beta)-(\alpha+p\beta)]g(k)}{p(p-\alpha)}.$$

That is, if

$$|z| \leq \left\{\frac{[k(1+\beta)-(\alpha+p\beta)]g(k)(c+k)^d}{(p-\alpha)(c+p)^d}\right\}^{1/(k-p)} \quad \text{for } k \geq p+n, n \in I\!N.$$

The result follows by setting $|z| = R_1$. □

**Theorem 7.2.** *Let $d > 0, c > -p$ and $f(z)$ belong to the class $K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$. Then the function $I(z)$ defined by (7.3) is p-valent in the disc $|z| < R_2$, where*

$$R_2 = \inf_k \left\{\frac{[k(1+\beta)-(\alpha+p\beta)]\Gamma(c+d+k)\Gamma(c+p)g(k)}{(p-\alpha)\Gamma(c+k)\Gamma(d+c+p)}\right\}^{1/(k-p)}.$$

*The result is sharp for the function given by*

$$f(z) = z^p - \frac{(p-\alpha)\Gamma(c+k)\Gamma(d+c+p)}{[k(1+\beta)-(\alpha+p\beta)]\Gamma(c+d+k)\Gamma(c+p)g(k)} z^{p+n}, \quad n \in I\!N.$$

*Proof.* In view of the arguments similar to Theorem 7.1 and relation (7.5), we get

$$|z| = \left\{\frac{[k(1+\beta)-(\alpha+p\beta)]\Gamma(c+d+k)\Gamma(c+p)g(k)}{(p-\alpha)\Gamma(c+k)\Gamma(d+c+p)}\right\}^{1/(k-p)} \quad \text{for } k \geq p+n,\ n \in I\!N.$$

□

## 8 Radius of $\beta$-Uniform Convexity

**Theorem 8.1.** *Let the function $f(z)$ defined by (1.2) be in the class $K(\mu, \gamma, \eta, a, b, c, \alpha, \beta)$. Then $f(z)$ is $\beta$-uniformly convex in $|z| < R_3$ of order $\delta, 0 \leq \delta < p, 0 \leq \alpha + \delta < p$ where*

$$|z| < R_3 = \inf_k \left\{\frac{[k(1+\beta)-(\alpha+p\beta)]g(k)(p-\delta-\alpha)}{(p-\alpha)[\beta(k-p)-(p-\delta-\alpha)]}\right\}.$$

*the result is sharp for*

$$f(z) = z^p - \sum_{k=p+n}^{\infty} \frac{p(p-\alpha)}{k[k(1+\beta)-(\alpha+p\beta)]g(k)} z^k \ \text{ for some } k.$$

*Proof.* To prove the result it is sufficient to show that

$$\beta\left|1+\frac{zf''(z)}{f'(z)}-p\right|+\alpha \leq p-\delta. \tag{8.1}$$

Simplifying by fairly straight forward calculations and using Theorem 2.1, we get

$$|z|^{k-p} \leq \frac{[k(1+\beta)-(\alpha+p\beta)]g(k)(p-\delta-\alpha)}{(p-\alpha)[\beta(k-p)-(p-\delta-\alpha)]}. \tag{8.2}$$

Setting $|z| = R_3$ in (8.2) we get the desired result. □

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