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Characterization, Reliability and Information measures of Even-Power Weighted Generalized Gamma Distribution

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Abstract: The Weighted distributions arise when the observations generated from a stochastic process are not given equal chance of being recorded; instead they are recorded according to some weighted function. Since the widely using of the weighted distribution in many fields of real life such various areas including medicine, ecology, reliability, and so on , then we try to shed light and record our contribution in this field thru the research. In this paper, a new class of an Even Power Weighted Generalized Gamma Distribution is introduced. An Even-Power Weighted Generalized Gamma distribution is obtained by taking the weights as the variate values have been defined. The important statistical properties including hazard functions, reverse hazard functions, measures of central tendency and dispersion, moment generating function, characteristic function, Shannon's entropy and Fisher's information matrix of a new model has been derived and studied. Here, we also study EPWGG entropy estimation,Akaike and Bayesian information criterion. A likelihood ratio test for an Even Power Weighted Generalized Gamma Distribution is conducted. A simulation study has been done by using R-software.

Keywords: Generalized Gamma distribution, Even-Power Generalized Gamma distribution Shannon's entropy, Fisher's information matrix, Likelihood ratio test.

1 Introduction

The Generalized Gamma (GG) distribution presents a flexible family in the varieties of shapes and hazard functions for modeling duration. The study of life testing models begins with the estimation of the unknown parameters involved in the models. Amorose Stacy (1962) proposed a generalized Gamma model and studied its characteristics. Stacy and Mihram (1965) and Harter (1967) have derived maximum likelihood estimators of generalized Gamma model under different situations. Prantice (1974) has considered maximum likelihood estimators for generalized Gamma model by using the technique of reparametrization. Hwang T. et al (2006) introduced a new moment estimation of parameters of the generalized Gamma distribution using its characterization. In information theory, thus far a maximum entropy (ME) derivation of GG is found in Kapur (1989), where it is referred to as generalized Weibull distribution, and the entropy of GG has appeared in the context of flexible families of distributions. Some concepts of this family in information theory has introduced by Dadpay et al (2007).

The PDF of the Generalized Gamma Distribution is given by:

$$f(x;\lambda,\beta,k)=\frac{\lambda\beta}{\Gamma k}(\lambda x)^{k\beta-1}e^{-(\lambda x)^{\beta}} \quad , for\ x>0\, and\ \lambda,\beta,k>0 \tag{1.1}$$

Where $\Gamma(.)$ is the Gamma function, K and β are shape parameters, and λ is the scale parameter. The generalized Gamma family is flexible in that it includes several well-known models as subfamilies. The subfamilies of generalized Gamma thus far considered in the literature are exponential, Gamma and Weibull. The lognormal distribution is also obtained as a limiting distribution when $n\to\infty$.

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The CDF of the Generalized Gamma distribution is given by:

$$F(x;\lambda,\beta,k)=\frac{\gamma\left(k,(\lambda x)^{\beta}\right)}{\Gamma k}$$

The Structural properties of the Generalized Gamma distribution are given as:

$$E(X)=\frac{\Gamma\left(k+\frac{1}{\beta}\right)}{\lambda\Gamma k} \tag{1.2}$$

$$E(X^2)=\frac{\Gamma\left(k+\frac{2}{\beta}\right)}{\lambda^2\Gamma k} \tag{1.3}$$

$$V(X)=\frac{\Gamma\left(k+\frac{2}{\beta}\right)\Gamma(k)-\Gamma^2\left(k+\frac{1}{\beta}\right)}{\lambda^2\Gamma^2(k)} \tag{1.4}$$

$$C.V(X)=\frac{\left[\Gamma\left(k+\frac{2}{\beta}\right)\Gamma(k)-\Gamma^2\left(k+\frac{1}{\beta}\right)\right]^{\frac{1}{2}}}{\Gamma\left(k+\frac{1}{\beta}\right)} \tag{1.5}$$

2 Even-Power Weighted Generalized Gamma Distribution

Traditional enviromentric theory and practice have been occupied with randomization and replication. But in environmental and ecological work, observations also fall in the non-experimental, non-replicated and non-random categories. The problems of model specification and data interpretation then acquire special importance and great concern. The theory of weighted distributions provides a unifying approach for these problems. Weighted distributions take into account the method of ascertainment, by adjusting the probabilities of actual occurrence of events to arrive at a specification of the probabilities of those events as observed and recorded. Failure to make such adjustments can lead to incorrect conclusions. The concept of weighted distributions can be traced to the work of Fisher (1934), in connection with his studies on how methods of ascertainment can influence the form of distribution of recorded observations.

Later it was introduced and formulated in general terms by Rao (1965), in connection with modeling statistical data where the usual practice of using standard distributions for the purpose was not found to be appropriate. In Rao's paper (1965), he identified various situations that can be modeled by weighted distributions. These situations refer to instances where the recorded observations cannot be considered as a random sample from the original distributions. This may occur due to non-observability of some events or damage caused to the original observation resulting in a reduced value or adoption of a sampling procedure which gives unequal chances to the units in the original. The concept of weighted distributions can be traced to the study of the effect of methods of ascertainment upon estimation of frequencies by Fisher (1934). In extending the basic ideas of Fisher, Rao [(1934, 1965)] saw the need for a unifying concept and identified various sampling situations that can be modeled by what he called weighted distributions.

Weighted distributions may occur in clinical trials, reliability theory, and survival analysis and population studies, where a proper sampling frame is absent. In such situations, items are sampled at a rate proportional to their length, so that larger values of the quantity being measured are sampled with higher probabilities. Numerous works on various aspects of length-biased sampling are available in literature which include family size and sex ratio , wild life population and line transect sampling , analysis of family data, cell cycle analysis , efficacy of early screening for disease, aerial survey and visibility bias Patil and Rao (1978). Shaban and Boudrissa (2000) have shown that the length biased version of the Weibull distribution known as Weibull Length-biased (WLB) distribution is uni-modal throughout examining its shape, with other properties, Das and Roy (2011) discussed the length-biased Weighted Generalized Rayleigh distribution with its properties, also they have develop the length-biased from of the weighted Weibull distribution. Patil and Ord (1976) introduced the concept of size-biased sampling and weighted distributions by identifying some of the situations where the underlying models retain their form.

Other contributions have been made by (Oluyede and George (2000), Ghitany and Al-Mutairi (2008), Oluyede and Terbeche (2007), Mir et al(2013a,b), Reshi et al (2013) , Reshi et al (2014a,b,c,d,e,f) and Ahmed et al (2013a,b).

2.1 Proposed Model

Suppose X is a non-negative random variable with its natural probability density function $f(x;\theta)$, where the natural parameter is θ. Suppose a realization x of X under $f(x;\theta)$ enters the investigator's record with probability proportional to $w(x;\beta)$, so that the weight function $w(x;\beta)$ is a non-negative function with the parameter β representing the recording mechanism. Clearly, the recorded x is not an observation on X, but on the random variable X_w, having a PDF

$$f_w^{2c}(x;\theta,\beta) = \frac{w(x;\beta)^{2c} f(x;\theta)}{\int_0^\infty \left(w(x)\right)^{2c} f(x)}; \; c = 1,2,3....... \tag{2.1}$$

Assuming that $E(X^{2c}) = \int_0^\infty \left(w(x)\right)^{2c} f(x)\, dx$ i,e if the moment of w(x) exists.

By taking weight w(x) = x^2 we obtain Even-power Weighted distribution (See Kareema et al (2013)). Where w is the normalizing factor obtained to make the total probability equal to unity by choosing $w = E[w(x,\beta)]$.The variable X_W is called weighted version of X, and its distribution is related to that of X and is called the weighted distribution with weight function w.For example, when $w(x;\beta) = x^2$, in that case, we get an Even Power Weighted Distribution. The distribution of X^* is called the Even Power Weighted Distribution with PDF

$$f_{W^2}{}^*(x;\theta) = \frac{x^2 f(x;\theta)}{\mu_2'} \tag{2.2}$$

An Even-Power Weighted Generalized Gamma distribution (EPWGGMD) is obtained by applying the weights x^{2c}, where c =1 to the Generalized Gamma distribution.

$$f_{W^2}(x;\lambda,\beta,k) = \frac{x^2 \dfrac{\lambda\beta}{\Gamma k}(\lambda x)^{k\beta-1} e^{-(\lambda x)^\beta}}{\dfrac{\Gamma\left(k+\dfrac{2}{\beta}\right)}{\lambda^2 \Gamma k}}$$

$$f_{W^2}(x;\lambda,\beta,k) = \frac{\beta\lambda^{k\beta+2}}{\Gamma\left(k+\dfrac{2}{\beta}\right)} x^{k\beta+1} e^{-(\lambda x)^\beta} \quad ;0 < x < \infty\,;\, \lambda,\beta > 0,\, k \ge 0 \tag{2.3}$$

Where $\Gamma(.)$ is the Gamma function, K and β are shape parameters, and λ is the scale parameter. The above equation (2.3) represents the probability density function of Even-Power Weighted Generalized Gamma Distribution

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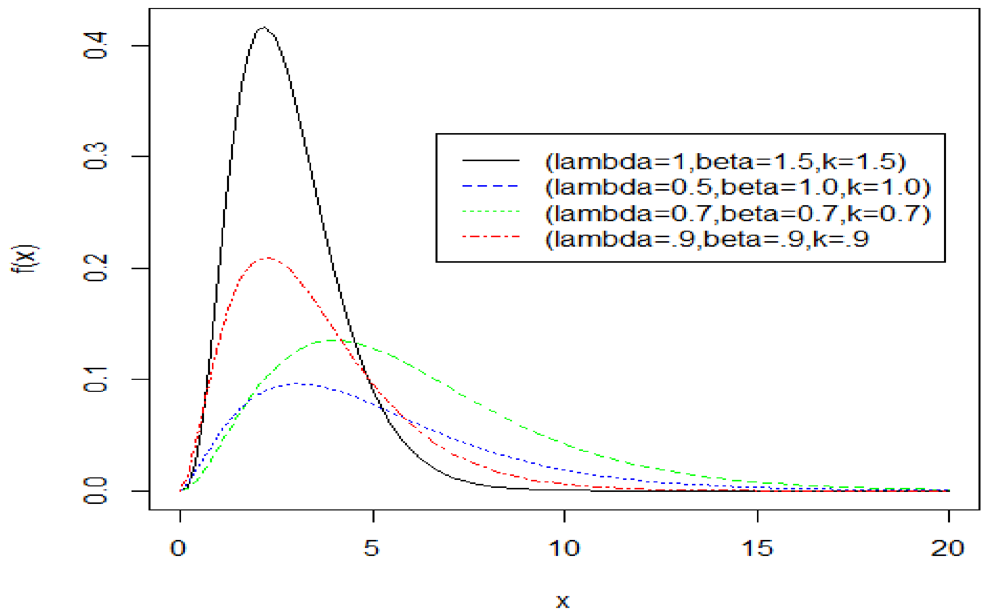

Figure 1: PDF plot for Even Power Weighted Generalized Gamma (lambda, beta, k)

The CDF of Even-Power Weighted Generalized Gamma Distribution is given as:

$$F_{w^2}(x;\lambda,\beta,k)=\frac{\gamma\left(\left(k+\frac{2}{\beta}\right),(\lambda x)^{\beta}\right)}{\Gamma\left(k+\frac{2}{\beta}\right)} \tag{2.4}$$

2.1.1 Reliability Measures

The hazard function and reverse hazard function of the Even Power Weighted Generalized Gamma Distribution are given as:

$$h_{w^2}(x;\lambda,\beta,k)=\frac{\lambda\beta(\lambda x)^{k\beta+1}e^{-(\lambda x)^{\beta}}}{\Gamma\left(k+\frac{2}{\beta}\right)-\gamma\left(\left(k+\frac{2}{\beta}\right),(\lambda x)^{\beta}\right)} \tag{2.5}$$

$$h_{rv}(x;\lambda,\beta,k)=\frac{\lambda\beta(\lambda x)^{k\beta+1}e^{-(\lambda x)^{\beta}}}{\gamma\left(\left(k+\frac{2}{\beta}\right),(\lambda x)^{\beta}\right)} \tag{2.6}$$

2.1.2 Special Cases

1. When $\beta=k=1$, then Even-Power Weighted Generalized Gamma distribution reduced to Even-Power Weighted classical exponential distribution and its probability distribution is given by

$$f(x;\lambda)=\frac{1}{2}\lambda^{3}x^{3}\ e^{-\lambda x};\lambda>0 \tag{2.7}$$

2. When $\beta=1$ then Even-Power Weighted Generalized Gamma distribution reduced to Even-Power classical Gamma distribution and its probability distribution is given by

$$f(x;\lambda,k+2)=\frac{\lambda^{k+2}}{\Gamma k+2}\,e^{-\lambda x}\,x^{k+1};\lambda>0,k\geq 0\;,0<x<\infty \tag{2.8}$$

3. When $k=0\,and\,\beta=1$ then Even-Power Weighted Generalized Gamma distribution reduced to Size-biased Weighted exponential distribution (see Mir et al (2013)) and its probability distribution is given by

$$f(x;\lambda)=\lambda^2 x\,e^{-\lambda x};\lambda>0 \tag{2.9}$$

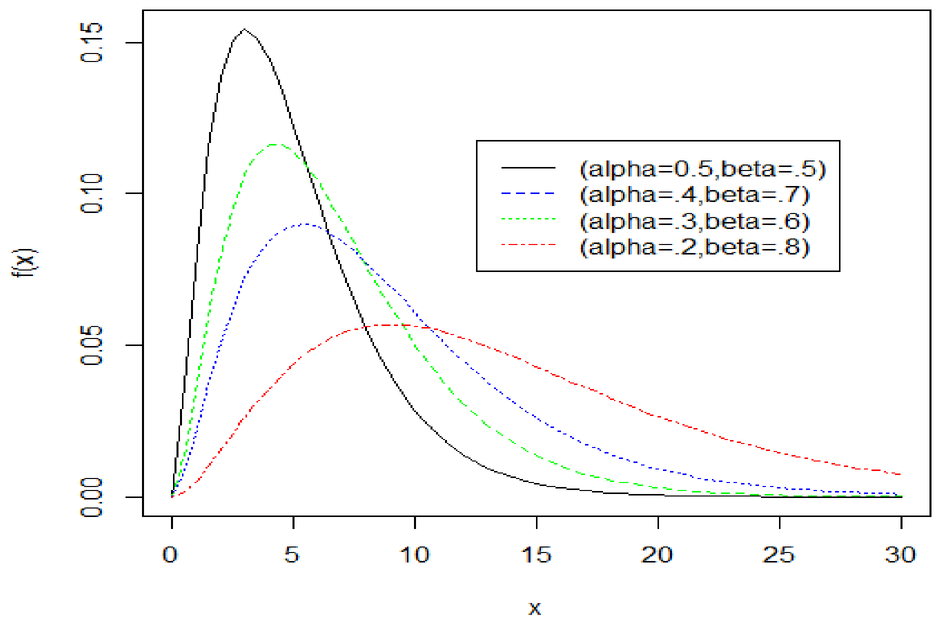

Figure 2: PDF plot for Even Power Classical Weighted Gamma Distribution (alpha, beta)

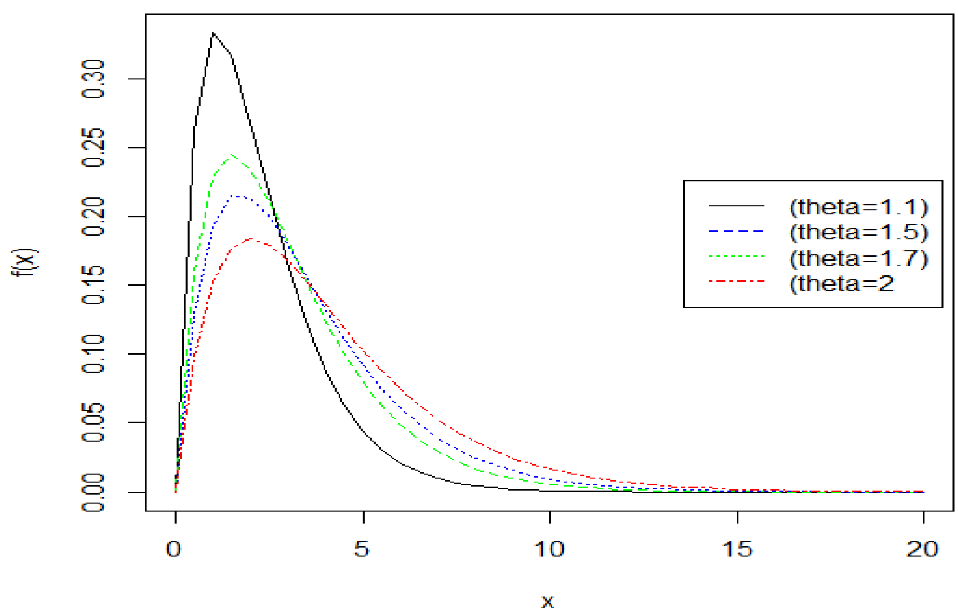

Figure 3: PDF plot for Size- biased Exponential Distribution (theta)

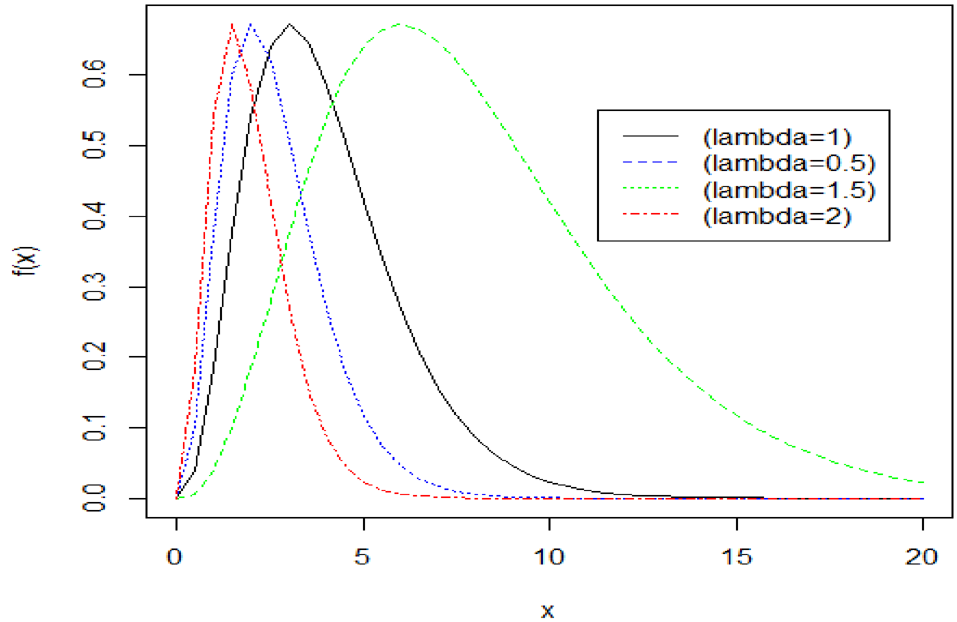

Figure 4: PDF plot for Even Power Weighted Exponential (lambda)

3 Structural properties of Even-Power Weighted Generalized Gamma Distribution.

In this section, we derive some important structural properties of Even-Power Weighted Generalized Gamma distribution.

3.1 Moments of Even-Power Weighted Generalized Gamma distribution

Using equation (2.3), the rth moments are obtained as:

$$E(X^r) = \int_0^\infty x^r \frac{\beta\lambda^{k\beta+2}}{\Gamma\left(k+\frac{2}{\beta}\right)} x^{k\beta+1} e^{-(\lambda x)^\beta} \, dx$$

$$E(X^r) = \frac{\beta\,\lambda^{k\beta+2}}{\Gamma\left(k+\frac{2}{\beta}\right)} \int_0^\infty x^{(k\beta+r+2)-1}\, e^{-(\lambda x)^\beta} dx \tag{3.1}$$

On solving the above equation (3.1), we get

$$E(X^m) = \frac{\Gamma\left(k+\frac{r+2}{\beta}\right)}{\lambda^r\,\Gamma\left(k+\frac{2}{\beta}\right)} \tag{3.2}$$

Using the equation (3.2), the mean and variance of the EPWGGMD are given as:

$$E(X) = \frac{\Gamma\left(k+\frac{3}{\beta}\right)}{\lambda\Gamma\left(k+\frac{2}{\beta}\right)} \tag{3.3}$$

$$V(X) = \frac{\left[\Gamma\left(k+\frac{4}{\beta}\right)\Gamma\left(k+\frac{2}{\beta}\right) - \Gamma^2\left(k+\frac{3}{\beta}\right)\right]}{\lambda^2\,\Gamma^2\left(k+\frac{2}{\beta}\right)} \tag{3.4}$$

The coefficient of variation of EPWGGMD is given by:

$$CV = \frac{\left[\Gamma\left(k+\frac{4}{\beta}\right)\Gamma\left(k+\frac{2}{\beta}\right) - \Gamma^2\left(k+\frac{3}{\beta}\right)\right]^{\frac{1}{2}}}{\Gamma\left(k+\frac{3}{\beta}\right)} \tag{3.5}$$

3.2 Moment generating function of Even Power Weighted Generalized Gamma Distribution:

The moment generating function of EPWGG distribution is obtained as:

$$E\left(e^{tx^\beta}\right) = \int_0^\infty e^{tx^\beta} f_{w^2}(x;\lambda,\beta,k)\,dx$$

$$E\left(e^{tx^\beta}\right) = \frac{\beta\lambda^{k\beta+2}}{\Gamma\left(k+\frac{2}{\beta}\right)} \int_0^\infty x^{k\beta+1} e^{-x^\beta\left(\lambda^\beta - t\right)}\,dx \tag{3.6}$$

On solving the above equation (3.6), we get

$$E\left(e^{tx^\beta}\right) = \left(\frac{\lambda^\beta}{\lambda^\beta - t}\right)^{k+\frac{2}{\beta}} \tag{3.7}$$

,

3.3 Characteristic function of Even Power Weighted Generalized Gamma Distribution:

The Characteristic of EPWGG distribution is obtained as:

$$\Phi_x(t) = \int_0^\infty e^{itx^\beta} f_{w^2}(x;\lambda,\beta,k)\,dx$$

$$\Phi_x(t) = \frac{\beta\lambda^{k\beta+2}}{\Gamma\left(k+\frac{2}{\beta}\right)} \int_0^\infty x^{k\beta+1} e^{-x^\beta\left(\lambda^\beta - it\right)}\,dx \tag{3.8}$$

On solving the above equation (3.8), we get

$$\Phi_x(t) = \left(\frac{\lambda^\beta}{\lambda^\beta - it}\right)^{k+\frac{2}{\beta}} \tag{3.9}$$

3.4 Shannon's Entropy of Even Power Weighted Generalized Gamma Distribution:

For deriving the Shannon's entropy of the Even-Power Weighted Generalized Gamma distribution, we need the two definitions that are more details of them can be found in Shannon (1948).

Shannon's entropy of the Even-Power Weighted Generalized Gamma distribution is obtained as:

$$H\left[f(x;\alpha,\beta,k)\right]=E\left[-\log\left\{f(x;\alpha,\beta,k\right\}\right]$$

$$H\left[f(x;\alpha,\beta,k)\right]=E\left[-\log\frac{\beta\lambda^{k\beta+2}}{\Gamma\left(k+\frac{2}{\beta}\right)}x^{k\beta+1}e^{-(\lambda x)^{\beta}}\right]$$

$$H\left(f\left(x;\lambda,k,\beta\right)\right)=-\log\left(\frac{\beta\lambda^{k\beta+2}}{\Gamma\left(k+\frac{2}{\beta}\right)}\right)-(k\beta+1)\,E\left(\log x\right)+\left(k+\frac{2}{\beta}\right) \quad (3.10)$$

Now, $E(\log(x))=\int_0^\infty \log x f(x;\alpha,\beta,k)dx$

$$E(\log(x))=\int_0^\infty \log x\frac{\beta\lambda^{k\beta+2}}{\Gamma\left(k+\frac{2}{\beta}\right)}x^{k\beta+1}e^{-(\lambda x)^{\beta}}dx$$

Let $(\lambda x)^{\beta}=t\Rightarrow x^{\beta}=\frac{t}{\lambda^{\beta}}\Rightarrow x=\left(\frac{t}{\lambda^{\beta}}\right)^{\frac{1}{\beta}} \qquad dx=\frac{dt}{\lambda\beta t^{1-\frac{1}{\beta}}}$

$$E(\log(x))=\frac{\beta\left(\lambda^{\beta}\right)^{k+\frac{2}{\beta}}}{\Gamma\left(k+\frac{2}{\beta}\right)}\int_0^\infty \log\left(\frac{t}{\lambda^{\beta}}\right)^{\frac{1}{\beta}}\left(\frac{t}{\lambda^{\beta}}\right)^{k+\frac{1}{\beta}}e^{-t}\frac{dt}{\lambda\beta t^{1-\frac{1}{\beta}}}$$

$$E(\log(x))=\frac{\Psi\left(k+\frac{1}{\beta}\right)-\log\lambda^{\beta}}{\beta} \quad (3.11)$$

Substitute the value of equation (3.11) in equation (3.10), we have

$$H\left(f\left(x;\lambda,k,\beta\right)\right)=-\log\left(\frac{\beta\left(\lambda^{\beta}\right)^{k+\frac{2}{\beta}}}{\Gamma\left(k+\frac{2}{\beta}\right)}\right)-\left(k+\frac{1}{\beta}\right)\left\{\frac{\Psi\left(k+\frac{1}{\beta}\right)-\log\lambda^{\beta}}{\beta}\right\}+\left(k+\frac{2}{\beta}\right) \quad (3.12)$$

3.5 Entropy estimation

Consider the PDF of Even-Power Weighted Generalized Gamma distribution (2.3)

$$f_{W^2}(x;\lambda,\beta,k)=\frac{\beta\lambda^{k\beta+2}}{\Gamma\left(k+\frac{2}{\beta}\right)}x^{k\beta+1}e^{-(\lambda x)^{\beta}} \quad ;0<x<\infty\,;\,\lambda,\beta>0,\,k\geq 0$$

$$\log L^{*}(X;\lambda,\beta,k) = n(2+k\beta)\log\lambda+n\log\beta-n\log\Gamma\left(k+\frac{2}{\beta}\right)+(k\beta+1)\sum_{i=1}^{n}\log x_i-\lambda^{\beta}\sum_{i=1}^{n}x_i^{\beta} \tag{3.13}$$

$$l(X;\lambda,\beta,k) = n\left((2+k\beta)\log\lambda+\log\beta-\log\Gamma\left(k+\frac{2}{\beta}\right)\right)+(k\beta+1)\sum_{i=1}^{n}\log x_i-\lambda^{\beta}\sum_{i=1}^{n}x_i^{\beta} \tag{3.14}$$

$$\frac{l(x;\lambda,\beta,k)}{n} = (2+k\beta)\log\lambda+\log\beta-\log\Gamma\left(k+\frac{2}{\beta}\right)+(k\beta+1)\overline{\log x}-\lambda^{\beta}\,\overline{x^{\beta}} \tag{3.15}$$

The Shannon's entropy of Size-biased Generalized Gamma Distribution is given as:

$$\hat{H}\,(EPWGGMD)=-\left[(2+\hat{k}\hat{\beta})\log\hat{\lambda}+\log\hat{\beta}-\log\Gamma\left(\hat{k}+\frac{2}{\hat{\beta}}\right)+\left(\hat{k}\hat{\beta}+1\right)\overline{\log x}-\hat{\lambda}^{\hat{\beta}}\,\overline{x^{\hat{\beta}}}\right] \tag{3.16}$$

From equation (3.15) and (3.16), we can write

$$\hat{H}(EPWGGMD)=-\frac{l(x;\hat{\lambda},\hat{\beta},\hat{k})}{n} \tag{3.17}$$

3.6 Akaike and Bayesian information criterion

In order to introducing of an approach for model selection, we remember Akaike and Bayesian information criterion based on entropy estimation. Akaike's information criterion, developed by HirotsuguAkaike (1974) under the name of "an information criterion" (AIC) in 1971 and proposed in Akaike (1973) is a measure of the goodness of fit of an estimated statistical model. The AIC is not a test of the model in the sense of hypothesis testing; rather it is a test between models - a tool for model selection. Given a data set, several competing models may be ranked according to their AIC, with the one having the lowest AIC being the best. From the AIC value one may infer that e.g. the top three models are in a tie and the rest are far worse, but it would be arbitrary to assign a value above which a given model is "rejected". In the general case, the AIC is

$$AIC=2K-2\log L(\hat{\theta})$$

Where K is the number of parameters in the statistical model and L is the maximized value of the likelihood function for the estimated model.

The Bayesian information criterion (BIC) or Schwarz Criterion is a criterion for model selection among a class of parametric models with different numbers of parameters. Choosing a model to optimize BIC is a form of regularization. It is very closely related to AIC. In BIC, the penalty for additional parameters is stronger than that of the AIC.

The formula for the BIC is

$$BIC=K\log n-2\log L(\hat{\theta})$$

The AIC and BIC methodology attempts to find the model that best explains the data with a minimum of their values, from (3.17) we have

$$l(x;\hat{\lambda},\hat{\beta},\hat{k})=-n\hat{H}(EPWGGMD)$$

Then for SBGG family we have

$$AIC=2K+2n\hat{H}(EPWGGMD) \tag{3.18}$$

And, $$BIC=K\log n+2n\hat{H}(EPWGGMD) \tag{3.19}$$

3.7 Likelihood Ratio Test for Even-Power Weighted Generalized Gamma distribution.

Let x_1, x_2 … x_n be random samples can be drawn from Generalized Gamma distribution or Even-Power Weighted Generalized Gamma distribution. We test the hypothesis

$$H_0 : f(x) = f(x;\lambda,\beta,k) againest\ H_1 : f(x) = f_{W^2}(x;\lambda,\beta,k) .$$

To test whether the random sample of size n comes from the Generalized Gamma distribution or Even-Power Weighted Generalized Gamma distribution, then the following test statistic is used.

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left[\frac{f_{W^2}(x;\lambda,\beta,k))}{f(x;\lambda,\beta,k))}\right]$$

$$\Delta = \frac{L_1}{L_0} = \prod_{i=1}^{n}\left[\frac{\lambda^2\Gamma(k)}{\Gamma\left(k+\frac{2}{\beta}\right)}x^2\right].$$

$$\Delta = \left[\frac{\lambda^2\Gamma(k)}{\Gamma\left(k+\frac{2}{\beta}\right)}\right]^n \prod_{i=1}^{n} x_i^{\ 2} \tag{3.20}$$

We reject the null hypothesis.

$$\left[\frac{\lambda^2\Gamma(k)}{\Gamma\left(k+\frac{2}{\beta}\right)}\right]^n \prod_{i=1}^{n} x_i^{\ 2} \ > k \tag{3.21}$$

Equalivalently, we rejected the null hypothesis where

$$\Delta^* = \prod_{i=1}^{n} x_i^{\ 2} > k^*, where k^* = k\left[\frac{\lambda^2\Gamma(k)}{\Gamma\left(k+\frac{2}{\beta}\right)}\right]^n > 0$$

For a large sample size of n, $2\log\Delta$ is distributed as a Chi-square distribution with one degree of freedom. Thus, the p-value is obtained from the Chi-square distribution. Also, we can reject the reject the null hypothesis, when probability value s given by:

$P(\Delta^* > \lambda^*), Where\ \lambda^* = \prod_{i=1}^{n} x_i$ is less than a specified level of significance, where $\prod_{i=1}^{n} x_i$ is the observed value of the test statistic.

3.8 Fisher's information matrix of Even Power Weighted Generalized Gamma Distribution

The Even-Power Weighted Generalized gamma distribution has a probability density function of the form:

$$f_{W^2}(x;\lambda,\beta,k)=\frac{\beta\lambda^{k\beta+2}}{\Gamma\left(k+\frac{2}{\beta}\right)}x^{k\beta+1}e^{-(\lambda x)^{\beta}} \tag{3.22}$$

Applying log on both sides in equation, we have

$$\log f_{w^2}(x;\lambda,\beta,k)=\log\beta+(k\beta+2)\log\lambda-\log\Gamma\left(k+\frac{2}{\beta}\right)+(k\beta+1)\log x-\lambda^{\beta}x^{\beta} \tag{3.23}$$

Differentiating equation (3.22) partially with respect to λ,β and k we get

$$\frac{\partial\log f_{w^2}(x;\lambda,\beta,k)}{\partial\lambda}=\frac{(k\beta+2)-\beta(\lambda x)^{\beta}}{\lambda}$$

$$\frac{\partial\log f_{w^2}(x;\lambda,\beta,k)}{\partial\beta}=\frac{\psi\left(k+\frac{2}{\beta}\right)}{\beta^2}+\frac{1}{\beta}+k\log\lambda x-(\lambda x)^{\beta}\log(\lambda x)$$

$$\frac{\partial\log f_{w^2}(x;\lambda,\beta,k)}{\partial k}=\beta\log\lambda x-\psi\left(k+\frac{2}{\beta}\right)$$

Differentiating again the above equation partially with respect to λ,β and k we have

$$\frac{\partial^2\log f_{w^2}(x;\lambda,\beta,k)}{\partial\lambda^2}=\frac{-(k\beta+2)-\beta(\beta-1)(\lambda x)^{\beta}}{\lambda^2}$$

$$\frac{\partial^2\log f_{w^2}(x;\lambda,\beta,k)}{\partial\beta\partial\lambda}=\frac{k-\left[(\lambda x)^{\beta}-\beta(\lambda x)^{\beta}\log(\lambda x)\right]}{\lambda}$$

$$\frac{\partial^2\log f_{w^2}(x;\lambda,\beta,k)}{\partial k\partial\lambda}=\frac{\beta}{\lambda}$$

$$\frac{\partial^2\log f_{w^2}(x;\lambda,\beta,k)}{\partial\beta^2}=-\left(\frac{1}{\beta^2}+\frac{\psi'\left(k+\frac{2}{\beta}\right)}{\beta^4}+\frac{2\psi\left(k+\frac{2}{\beta}\right)}{\beta^3}+(\lambda x)^{\beta}\log(\lambda x)^2\right)$$

$$\frac{\partial^2\log f_{w^2}(x;\lambda,\beta,k)}{\partial\lambda\partial\beta}=\frac{k-(\lambda x)^{\beta}\left[(\beta\log\lambda+1)\right]-\beta(\lambda x)^{\beta}\log x}{\lambda}$$

$$\frac{\partial^2\log f_{w^2}(x;\lambda,\beta,k)}{\partial k\partial\beta}=\log\lambda+\log x+\psi'\left(k+\frac{2}{\beta}\right)$$

$$\frac{\partial^2 \log f_{w^2}(x;\lambda,\beta,k)}{\partial k^2} = -\psi'\left(k+\frac{2}{\beta}\right)$$

$$\frac{\partial^2 \log f_{w^2}(x;\lambda,\beta,k)}{\partial\lambda\partial k} = \frac{\beta}{\lambda}$$

$$\frac{\partial^2 \log f_{w^2}(x;\lambda,\beta,k)}{\partial\beta\partial k} = \log\lambda + \log x + \frac{\psi'\left(k+\frac{2}{\beta}\right)}{\beta^2}$$

Taking expectations on both sides of the above equations, we get

$$-E\left(\frac{\partial^2 \log f_{w^2}(x;\lambda,\beta,k)}{\partial\lambda^2}\right) = \frac{(k\beta+2)+\beta(\beta-1)(\lambda)^\beta E(x)^\beta}{\lambda^2}$$

$$-E\left(\frac{\partial^2 \log f_{w^2}(x;\lambda,\beta,k)}{\partial\beta\partial\lambda}\right) = \frac{-k-\lambda^\beta\left[(\beta\log\lambda-1)E(x)^\beta+\beta E\left(x^\beta\log x\right)\right]}{\lambda}$$

$$-E\left(\frac{\partial^2 \log f_{w^2}(x;\lambda,\beta,k)}{\partial k\partial\lambda}\right) = -\frac{\beta}{\lambda}$$

$$-E\left(\frac{\partial^2 \log f_{w^2}(x;\lambda,\beta,k)}{\partial\lambda\partial\beta}\right) = \frac{-k-\lambda^\beta\left[(\beta\log\lambda+1)E(x)^\beta+\beta\lambda^\beta E\left(x^\beta\log x\right)\right]}{\lambda}$$

$$-E\left(\frac{\partial^2 \log f_{w^2}(x;\lambda,\beta,k)}{\partial\beta^2}\right) = \left(\frac{1}{\beta^2}+\frac{\psi'\left(k+\frac{2}{\beta}\right)}{\beta^2}+\frac{2\psi\left(k+\frac{2}{\beta}\right)}{\beta^3}+\lambda^\beta E\left[x^\beta(\log x)^2\right]+\lambda^\beta(\log\lambda)^2E(x)^\beta+2\lambda^\beta\log\lambda E\left(x^\beta\log x\right)\right)$$

$$-E\left(\frac{\partial^2 \log f_{w^2}(x;\lambda,\beta,k)}{\partial k\partial\beta}\right) = -\log\lambda - E(\log x) - \frac{\psi'\left(k+\frac{2}{\beta}\right)}{\beta^2}$$

$$-E\left(\frac{\partial^2 \log f_{w^2}(x;\lambda,\beta,k)}{\partial\lambda\partial k}\right) = -\frac{\beta}{\lambda}$$

$$-E\left(\frac{\partial^2 \log f_{w^2}(x;\lambda,\beta,k)}{\partial\beta\partial k}\right) = -\log\lambda - E(\log x) - \frac{\Psi'\left(k+\frac{2}{\beta}\right)}{\beta^2}$$

$$-E\left(\frac{\partial^2 \log f_{w^2}(x;\lambda,\beta,k)}{\partial k^2}\right) = \psi'\left(k+\frac{2}{\beta}\right)$$

We know that, $E\left(x^\beta\log x\right) = \int_0^\infty x^\beta \log x f_{w^2}(x;\lambda,\beta,k)dx$

$$E\left(x^{\beta}\log x\right)=\frac{\Gamma'\left(k+\frac{2}{\beta}+1\right)-\beta\log\lambda\Gamma\left(k+\frac{2}{\beta}+1\right)}{\beta\lambda^{\beta}\Gamma\left(k+\frac{2}{\beta}\right)} \tag{3.23}$$

Also, $E\left(x^{\beta}(\log x)^{2}\right)=\int_{0}^{\infty}x^{\beta}(\log x)^{2}f_{w^2}(x;\lambda,\beta,k)dx$

$$E\left(x^{\beta}(\log x)^{2}\right)=\frac{\Gamma''\left(k+\frac{2}{\beta}+1\right)-2\beta\log\lambda\Gamma'\left(k+\frac{2}{\beta}+1\right)+\beta^{2}(\log\lambda)^{2}\Gamma\left(k+\frac{2}{\beta}+1\right)}{\beta^{2}\lambda^{\beta}\Gamma\left(k+\frac{2}{\beta}\right)} \tag{3.24}$$

Also, $E(X^{\beta})=\dfrac{\left(k+\frac{2}{\beta}\right)}{\lambda^{\beta}}$ (3.25)

Substitute the values of equations (3.23), (3.24) and (3.25) in the above entries of a Fisher information matrix, we get

$$I(1,1)=\frac{(k\beta+2)+\beta(\beta-1)\left(k+\frac{2}{\beta}\right)}{\lambda^{2}}$$

$$I(1,2)=\frac{-k\Gamma\left(k+\frac{2}{\beta}\right)+\left[\left(\beta\log\lambda-\beta^{2}\log\lambda+1\right)\Gamma\left(k+\frac{2}{\beta}+1\right)+\beta\Gamma'\left(k+\frac{2}{\beta}+1\right)\right]}{\lambda\Gamma\left(k+\frac{2}{\beta}\right)}$$

$$I(1,3)=-\frac{\beta}{\lambda}$$

$$I(2,1)=\frac{-k\Gamma\left(k+\frac{2}{\beta}\right)-(1+\beta\log\lambda)\Gamma\left(k+\frac{2}{\beta}+1\right)-\left[\beta\log\lambda\Gamma\left(k+\frac{2}{\beta}+1\right)-\left(\Gamma'\left(k+\frac{2}{\beta}+1\right)\right)\right]}{\lambda\Gamma\left(k+\frac{2}{\beta}\right)}$$

$$I(2,2)=\frac{\left(\Gamma''\left(k+\frac{2}{\beta}+1\right)-2\beta\log\lambda\Gamma'\left(k+\frac{2}{\beta}+1\right)+\beta^{2}(\log\lambda)^{2}\Gamma\left(k+\frac{2}{\beta}+1\right)\right)}{\beta^{2}\Gamma\left(k+\frac{2}{\beta}\right)}+2\log\lambda\left\{\frac{\left(\Gamma'\left(k+\frac{2}{\beta}+1\right)-\beta\log\lambda\Gamma\left(k+\frac{2}{\beta}+1\right)\right)}{\beta\Gamma\left(k+\frac{2}{\beta}\right)}\right\}$$

$$I(2,3)=-\log\lambda-\left[\frac{\psi\left(k+\frac{2}{\beta}\right)-\beta\log\lambda}{\beta}\right]-\psi'\left(k+\frac{2}{\beta}\right)$$

$$I(3,1)=-\frac{\beta}{\lambda}$$

$$I(3,2) = -\log\lambda - \left[\frac{\Psi\left(k+\frac{2}{\beta}\right) - \beta\log\lambda}{\beta}\right] - \frac{\psi'\left(k+\frac{2}{\beta}\right)}{\beta^2}$$

$$I(3,3) = \psi'\left(k+\frac{2}{\beta}\right)$$

3.9 Mode of Even-Power Weighted Generalized Gamma distribution:

The Even-Power Weighted generalized gamma distribution has a probability density function of the form:

$$f_{W^2}(x;\lambda,\beta,k) = \frac{\beta\lambda^{k\beta+2}}{\Gamma\left(k+\frac{2}{\beta}\right)} x^{k\beta+1} e^{-(\lambda x)^\beta} \quad ;0<x<\infty \,;\, \lambda,\beta>0,\, k\geq 0$$

In order to discuss monotonicity of an Even-Power Weighted Generalized Gamma distribution. We take the logarithm of its PDF:

$$\log f_{w^2}(x;\lambda,\beta,k) = \log\beta + (k\beta+2)\log\lambda - \log\Gamma\left(k+\frac{2}{\beta}\right) + (k\beta+1)\log x - \lambda^\beta x^\beta \qquad (3.26)$$

The mode of Even-Power Weighted Generalized Gamma distribution is given as:

$$x = \frac{\left(k+\frac{1}{\beta}\right)^{\frac{1}{\beta}}}{\lambda} \qquad (3.27)$$

4 Simulation Study of Even-Power Weighted Generalized Gamma Distribution:

For description of this manner, we generate different random samples of size 25,50 and 100 from the Even-Power Weighted Generalized Gamma distribution, a simulation study is carried out 5,000 times for each pairs of (λ,β,k) where $(\lambda=1.0,1.5,2.0),(\beta=0.5,1.0,1.5)$ and $(k=0.5,1.0,1.0)$.

Table 4.1: AIC and BIC criteria of Even-Power Weighted Generalized Gamma Distribution

n	λ	β	**k**	Entropy	**AIC**	**BIC**
25	1.0	0.5	0.5	73.49978	14705.96	14709.61
	1.5	1.0	1.0	74.21933	14849.87	14853.52
	2.0	1.5	1.0	73.48967	14703.93	14707.59
50	1.0	0.5	0.5	39.49036	7904.071	7909.808
	1.5	1.0	1.0	37.05609	7417.219	7422.955
	2.0	1.5	1.0	36.36152	7278.304	7284.04
100	1.0	0.5	0.5	21.92278	4390.555	4398.371
	1.5	1.0	1.0	18.41898	3689.795	3697.611
	2.0	1.5	1.0	18.15234	3636.467	3644.283

From the above table 4.1, we can conclude that the Even-Power Weighted Generalized Gamma Distribution have the smallest AIC and BIC values when sample size is 100 and scale parameter is 2.0 and shape parameters are $\beta=1.5$, k=1.0.

From table 4.2, it has been observed that the Size-biased Weighted exponential distribution have the smallest AIC and BIC values as compared to other family of Even-Power Weighted Generalized Gamma Distribution, when sample sizes of distributions are 25, 50 and 100.Hence we can concluded that the Size-biased Weighted exponential distribution is the best model as compared to Even-Power Weighted exponential and Even-Power Weighted Gamma distributions. It was also been observed that if we increases the sample sizes, the corresponding Shannon's entropy, AIC and BIC values decreases.

Table 4.2: AIC and BIC criteria of different subfamilies of Even-Power Weighted Generalized Gamma Distribution

n	Distribution	Shannon's entropy	AIC	BIC
25	Even-Power Weighted exponential	5.455043	1093.009	1094.228
	Even-Power Weighted Gamma	17.64211	3532.422	3534.859
	Size-biased Weighted exponential	0.5945349	31.72674	32.94562
50	Even-Power Weighted exponential	4.418664	885.7328	887.6449
	Even-Power Weighted Gamma	9.01122	1806.244	1810.068
	Size-biased Weighted exponential	0.6134556	63.34556	65.25758
100	Even-Power Weighted exponential	3.699976	741.9952	744.6003
	Even-Power Weighted Gamma	3.996696	803.3391	808.5495
	Size-biased Weighted exponential	0.221281	46.2562	48.86137

5 Conclusion

In this research article, we have introduced a new class of Weighted Generalized Gamma distribution named as Even-power Weighted Generalized Gamma distribution (EPWGGMD). Various structural and characterizing properties of this new model has been derived and studied. From the simulation study, it has been observed that when the sample size increases from 25 to 100, the Shannon's entropy, AIC and BIC values decreases quite significantly.

A comparative study has been done between different special cases of Even-power Weighted Generalized Gamma distribution, it has been observed that the Size-biased Weighted exponential distribution have the smallest AIC and BIC values as compared to other family of Even-Power Weighted Generalized Gamma Distribution, when sample sizes of distributions increases. Hence we can conclude that the Size-biased weighted exponential distribution is the best model as compared to Even-Power Weighted exponential and Even-Power Weighted Gamma distributions.

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