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Allee Effect on a Ratio-Dependent Prey-Predator System with Disease in Prey

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Abstract: In the present paper we have constructed a ratio-dependent prey-predator system introducing additive Allee effect with disease in prey. We have analyzed the conditions of nonpersistence, permanency of the disease and also local and global stability of the system.

Keywords: Prey-Predator system. Allee Effect. Global Stability. Persistence. Permanence.

1 Introduction

In 1931 Ecologist W.C. Allee[1] introduced a phenomena in ecological systems characterized by a positive correlation between population size or density and the individuals fitness(known as 'Allee Effect'). This effect can be caused by difficulties in finding mates (population density low), social dysfunction, and increased predation risk due to failing, flocking or schooling behavior [4-6]. There are two types of Allee effects of which one is strong Allee that produces a threshold value of the Allee constant above which population grow or becomes extinct. The other one is the case of weak Allee where no threshold value of the Allee constant exists . Allee effect's has been studied by severals authors in various ecological and eco-epidemiological problems[15-21].

After the initial work by Kermack-Mackendrick in 1927 on SIRS model epidemiology become a growing field of research mathematical biology. Initial studies in this field were based on human population[7]. Later epidemiological modeling in ecological systems involving prey-predator models received lot of attention[3,4,9,14]. Chattopadhyay and Arino studied a prey-predator model with disease in prey population[9]. An additive Allee effect was introduced to study the effect of Allee on stability of reaction-diffusion prey-predator model by Wang et al (2013) without considering the infection in prey[8]. In this paper we have considered Allee effect in an eco-epidemiological model with disease in prey population. When an group of prey population is infected then the predators eat the infected prey[9] more because infected prey are less active and can be easily caught(fish and aquatic snails)[12]. Again the predator population who eat infected prey must also be infected. Peterson and Page[13] pointed that the wolf attack on moose are more often successful as the moose is heavily infected by 'Echinococcus gramulosus'.

In this paper we shall investigate the stability of an ecoepidemiological model described below with additive Allee effect on its susceptible prey population. In the eco-epidemiological model under our consideration prey population N(T) and predator population P(T) satisfying the following assumptions:

(i) The prey population N(T) has been divided into two sub-classes Suspected prey S(T) and Infected prey I(T) at time T [3].

(ii) The susceptible prey population grows according to logistic model with carrying capacity K where both non-linear interactions of susceptible and infected are considered along intrinsic growth rate $r > 0$ in the absence of predator.
(iii) An additive Allee effect has been introduced in the growth rate of susceptible prey population [8].

(iii) Only Susceptible population have reproducing

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capacity to increase population size.

(iv) The susceptible population is being infected by simple mass action law through the interaction term γSI, where $\gamma > 0$ is called transmission coefficient.Here c is the death rate of infected population. Then the SI model with additive Allee effect becomes

$$\frac{dS}{dT} = rS(1 - \frac{S+I}{K} - \frac{K}{A+S+I}) - \gamma SI \tag{1.1a}$$
$$\frac{dI}{dT} = \gamma SI - cI \tag{1.1b}$$

where the term $\frac{K}{A+S+I}$ is the Allee effect on prey, which was first introduced in[6] and applied in[10,11].

(v) Interaction between infected prey and predator follows Michaelis-Menton functional response $\xi(I,P) = \frac{IP}{mP+I}$, $m > 0$.

2 Mathematical Formulation

The scucptible prey (S),infected prey(I) and predator(P) population satisfies the following model

$$\frac{dS}{dT} = rS(1 - \frac{S+I}{K} - \frac{K}{A+S+I}) - \gamma SI \tag{2.1a}$$
$$\frac{dI}{dT} = \gamma SI - cI - \frac{\delta IP}{mP+I} \tag{2.1b}$$
$$\frac{dP}{dT} = -dP + \frac{\theta\delta IP}{mP+I} - pP \tag{2.1c}$$

where $d > 0$ is the predator natural death rate, $\delta > 0$ is the predator coefficient and θ is the conversion coefficient prey into predator and $p > 0$ is the death rate of predator population due to consumption the infected prey population.

We now express the model (2.1) in dimensionless form as

$$\frac{ds}{dt} = \alpha s(1 - s - i - \frac{1}{B+s+i}) - si \tag{2.2a}$$
$$\frac{di}{dt} = si - d_1 i - \frac{\beta iy}{my+i} \tag{2.2b}$$
$$\frac{dy}{dt} = -d_2 y + \frac{\theta\beta iy}{my+i} \tag{2.2c}$$

where

$$t = \gamma KT, s = \frac{S}{K}, i = \frac{I}{K}, y = \frac{P}{K}, \alpha = \frac{r}{\gamma K}, \tag{2.3}$$
$$B = \frac{A}{K}, d_1 = \frac{c}{\gamma K}, d_2 = \frac{d+p}{\gamma K}$$

are the dimensionless parameters.
The initial condition for(2.2)is

$$\Omega = \{(s,i,y); s \geqslant 0, i \geqslant 0, y \geqslant 0\} \tag{2.4}$$

Standard and simple arguments show that the solution of the system (2.2) always exist and stay positive[3]. The stationary solution(or steady-states) of the system (2.2) are as follows:

$E_0(0,0,0)$ [Trivial Stationary point]
$E_1(s^1,0,0)$, $E_2(s^2,i^2,0)$, $E^*(s^*,i^*,y^*)$ are also equilibrium points where
$s^1 = \frac{\sqrt{(B+3)(B-1)}-(B-1)}{2}$;
$s^2 = d_1, i^2 = \frac{\sqrt{(B+d_1)(1+\alpha)-\alpha(1-d_1)^2+4(1+\alpha)\alpha(B+d_1)(d_1-1)+1}-(B+d_1)(1+\alpha)-\alpha(1-d_1)}{2(1+\alpha)}$;
$s^* = d_1 + \frac{\theta\beta - d_2}{m\theta}$,
$i^* = \frac{\sqrt{(s^*+B)(1+\alpha)-\alpha(1-s^*)^2+4\alpha(1+\alpha)(1-s^*)(B+s^*)}-(1+\alpha)(B+s^*)-\alpha(1-s^*)}{2(1+\alpha)}$,
$y^* = \frac{\theta\beta - d_2}{m\theta} i^*$;
The existence criterion of the steady-state population (i.e.$s^* > 0$, $i^* > 0$, $y^* > 0$)demands the criteria $B \geqslant 1$ *and* $0 < (\theta\beta - d_2) < \theta m(0.62 - d_1)$.

2.1 Boundedness of solutions:

Theorem 1: If the initial conditions(2.4)of the system of equations (2.2)satisfies $s_0 + i_0 > 0.62$ where $s(0) = s_0$, $i(0) = i_0$ then either (a) $s(t)+i(t) > \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}$ and $B > 1$ $\forall t \geqslant 0$, such that $(s(t),i(t),y(t)) \to E_1 = (\frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}, 0, 0)$ as $t \to +\infty$, ,or (b)$\exists$ a $t^* > 0$ such that $s(t)+i(t) < \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}, \forall t > t^*$. On the other hand for $s_0 + i_0 < 0.62$, $s(t)+i(t) < \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}, \forall t \geqslant 0$ will be satisfied.

Proof. First we consider $s(t) + i(t) \geqslant \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}, B \geqslant 1$. From the equations (2.2a)and(2.2b) we get

$$\frac{d}{dt}(s(t)+i(t)) = \alpha s(1 - s - i - \frac{1}{B+s+i}) - d_1 i - \frac{\beta iy}{my+i} \tag{2.5}$$

Hence $\forall t \geqslant 0 \frac{d}{dt}(s(t)+i(t)) \leqslant 0$ as $s(t)+i(t) \geqslant \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}$.
Let us assume$\lim_{t\to\infty}(s(t)+i(t)) = \mu$
If $\mu > \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}, B \geqslant 1$,then by Barbalat Lemma, we get
$0 = \lim t \to \infty \frac{d}{dt}(s(t)+i(t))$
$= \lim t \to \infty[\alpha s(1-s-i-\frac{1}{B+s+i}) - d_1 i - \frac{\beta iy}{my+i}]$
$\leqslant -min\alpha(\mu + \frac{1}{B+\mu} - 1), d_1 \lim t \to \infty \frac{d}{dt}(s(t)+i(t))$
$= -\mu d_1, \alpha(\mu + \frac{1}{B+\mu} - 1), \mu > 0.62$
< 0
Obviously this is a contradiction so we say that

$$\lim_{t\to\infty}(s(t)+i(t)) = \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}, B \geqslant 1 \tag{2.6}$$

Let $\xi(t) = s(t)+i(t) \forall\ t \geqslant 0$ and $\xi(t)$ is differential and uniformly continuous function on $t \geqslant 0$. Then by Barbalat Lemma we say that

$$\lim_{t\to\infty}\frac{d}{dt}(s(t)+i(t)) = 0 \tag{2.7}$$

Again from the equations(2.2a) and(2.2b)we get $\frac{d}{dt}(s(t)+i(t)) = \alpha s(1-s-i-\frac{1}{B+s+i}) - d_1 i - \frac{\beta iy}{my+i}$ also from the equation (2.6) we conclude that

$$\lim_{t\to\infty}\frac{d}{dt}(s(t)+i(t)) = -\lim_{t\to\infty}[d_1 i + \frac{\beta iy}{my+i}] \tag{2.8}$$

From (2.7) and (2.9) we show that $\lim_{t\to\infty} i(t) = 0$. Again from (2.6) we say that $\lim_{t\to\infty} s(t) = \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}$.
From (2.2c) we get $y(t) \to 0\ as t \to \infty$. So (a) is proved.
If (a) is violated the $\exists$ a $t^* > 0$ at which for the first time $s_0 + i_0 = \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}, B \geqslant 1$.Again from (2.5) we get

$$\frac{d}{dt}(s(t)+i(t)) \mid_{t=t^*} = -d_1 i(t^*) - \frac{\beta i(t^*)y(t^*)}{my(t^*)+i(t^*)} < 0 \tag{2.9}$$

This implies that $s+i$ has entered into the interval $(0, \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2})$ then its remain bounded $\forall t > t^*$ i.e,

$$s+i < \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2} \quad t > t^* \tag{2.10}$$

Finally if $s_0 + i_0 < 0.62$, applying the same argument as above we show that $s+i < \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2},\ \ t > 0$.
This complete the proof.

2.2 Persistence Of the System

Definition The system (2.2) is said to be persistent if $min\{\lim_{t\to\infty} inf s(t),\ \lim_{t\to\infty} inf i(t),\ \lim_{t\to\infty} inf y(t)\} \geqslant 0$ for some of its positive solutions[3]. Otherwise the system is not persistence.

Theorem 2 If $1 - d_1 - \frac{\beta}{m} + d_2 < 0$, the system(2.2) is not persistent.

Proof. We know that the system(2.2) is not persistent if $min[\lim_{t\to\infty} inf s(t),\ \lim_{t\to\infty} inf i(t),\ \lim_{t\to\infty} inf y(t)] < 0$ for some positive solutions[3]. As we have $1 - d_1 - \frac{\beta}{m} + d_2 < 0$, then there is a ν such that $\frac{\beta}{m+\nu} = 1 - d_1 + d_2$. Let $\rho = \frac{i(0)}{y(0)} < \nu$ and $s(0) < 0.62$. Then $s(t) < 0.62 \forall t \geqslant 0$ by using the theorem 1. So we conclude that $\frac{i(t)}{y(t)} < \nu, \forall t \geqslant 0$ and $\lim_{t\to\infty} i(t) = 0$. Otherwise there is a first time t_1, $\frac{i(t_1)}{y(t_1)} = \nu$ and for $[0,t_1)$ $\frac{i(t)}{y(t)} < \nu$. Then $\forall\ t \in [0,t_1]$, we have from the equation (2.2b)
$\frac{di(t)}{dt} \leqslant i(t)[1 - d_1 - \frac{\beta}{m+\frac{i}{y}}]$
$\leqslant i(t)[1 - d_1 - \frac{\beta}{m+\nu}] = -d_2 i(t)$,
Which implies that $i(t) \leqslant \sigma(0)e^{(-d_2 t)}$. However, for all $t \geqslant 0$, $\frac{dy(t)}{dt} \geqslant -d_2 y(t)$,
Which implies that $y(t) \geqslant y(0)e^{(-d_2 t)}$. This shows that for$t \in [0,t_1]$,
$\frac{i(t)}{y(t)} \leqslant \frac{i(0)}{y(0)} = \rho < \nu$,
Which is a contradiction to existence of t_1,. This implies $i(t) \leqslant i(0)e^{-d_2 t} \forall t \geqslant 0$. which completes the proof.

Theorem 3 If $1 - d_1 - \frac{\beta}{m} + d_2 < 0$ then $\exists$ positive solutions $(s(t), i(t), y(t))$ of the system(2.2) such that $\lim_{t\to\infty}(s(t), i(t), y(t)) = (\frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}, 0, 0)$

Proof. If $\theta m \leqslant d_2$ then it is surely true from (2.2c) that Theorem 2 holds . Assume that $\theta m \geqslant d_2$. Again Theorem 2 implies that $\lim_{t\to\infty} i(t) = 0, \forall\ t \geqslant 0$, $\frac{i(t)}{y(t)} \leqslant \rho$, provided $\rho = \frac{i(0)}{y(0)} < \nu$, $s(0) < 0.62$, where $\nu = [\frac{l}{1-d_1+d_2}] - m$. Let $(s(t), i(t), y(t))$ be the solution of the equation(2.2) with $\frac{i(0)}{y(0)} < \nu\ and\ s(0) < 0.62$. Since the solution of the equation is bounded $0 \leqslant \beta_1 = \lim\sup y(t) < \infty$, $0 \leqslant \beta_2 = \lim\inf y(t) < \infty$.
If $\beta_2 > 0$, then we show that for large value of t, $\frac{dy(t)}{dt} < -\frac{1}{2}d_2 y(t)$, which leads to $\lim_{t\to\infty y(t)=0}$ which is a contradiction. So we must have $\beta_2 = 0$.
Now we assume $\beta_1 > 0$. Since $\lim_{t\to\infty} i(t) = 0$, then $\exists\ a\ t_1$ such that $\forall t > t_1$, $i(t) \leqslant \frac{md_2\beta_1}{3(\theta\beta - d_2)}$. From our assumption $\beta_1 > 0$, we find a $t_2 > t_1$ such that $\frac{dy(t)}{dt} > \frac{1}{3}\beta_1$ and $\frac{dy(t)}{dt} > 0$.So$\frac{dy(t)}{dt} > 0$ implies that $i(t_2) > \frac{md_2 y(t_2)}{\theta\beta - d_2} > \frac{md_2\beta_1}{3(\theta\beta - d_2)}$. This is a contradiction. Hence $\beta_1 = 0$. i.e,$\lim t \to \infty y(t) = 0$. Since $\lim_{t\to\infty} i(t) = 0$ $\exists\ t_3 > t_2$ such that $i(t) < \varepsilon \forall\ t > t_3\ and\ \varepsilon > 0$. From the

equation (2.2a) we get $\alpha s(1-s-\varepsilon-\frac{1}{B+s+\varepsilon}) < \frac{ds}{dt} < \alpha s(1-s-\frac{1}{B+s})$. This implies $\lim_{t\to\infty} s(t) = \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}$. Hence the theorem.

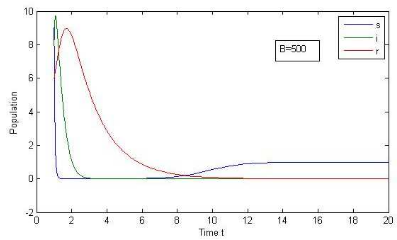

Fig. 1: MatLab generated graph of population of s, i *and* r verses time t with B=500, $\alpha = 1$, $d_1 = 0.5$, $\beta = 3$, m=1, $d_2 = 0.6$ and $\theta = 0.9$ shows nonpersistence of the disease.

In this paper we show that the system (2.2) is not persistent if $1-d_1-\frac{\beta}{m}+d_2 < 0$ condition holds i.e.,under this certain condition obviously the system is not permanent. Now in the next section we try to investigate for what condition the is permanent i.e., all classes population population exist after a long time.

2.3 Permanence Of the System

Theorem 4 The system of equations (2.2) is permanent if $B \geqslant 1$, $\theta\beta > d_2$ and$1-d_1-\frac{\beta}{m} > 0$.

Proof. We take the first two equations of the system (2.2) as

$$\frac{ds}{dt} = \alpha s(1-s-i-\frac{1}{B+s+i})-si \quad (2.11a)$$

$$\frac{di}{dt} \geqslant si-d_1 i-\frac{\beta}{m}i \quad (2.11b)$$

We consider the equations in the form:

$$\frac{du_1}{dt} = \alpha u_1(1-u_1-u_2-\frac{1}{B+u_1+u_2})-u_1u_2 \quad (2.12a)$$

$$\frac{du_2}{dt} \geqslant u_1u_2-d_1u_2-\frac{\beta}{m}u_2 \quad (2.12b)$$

With simple calculation we say that if $B \geqslant 1, \theta\beta > d_2$ and$1-d_1-\frac{\beta}{m} > 0$,

$(s', i') = (d_1+\frac{\beta}{m}, \frac{\sqrt{[(1+\alpha)(B+d_1+\frac{\beta}{m})-\alpha(1-d_1-\frac{\beta}{m})]^2+4\alpha(1+\alpha)[(1-d_1-\frac{\beta}{m})(B+d_1+\frac{\beta}{m})-1]}-[(1+\alpha)(B+d_1+\frac{\beta}{m})-\alpha(1-d_1-\frac{\beta}{m})]}{2(1+\alpha)})$

is a positive equilibrium point of (2.12) which is globally asymptotically stable. Let $u_1(0) \leqslant s(0)$, $u_2(0) \leqslant i(0)$. As $(u_1(t), u_2(t))$ is a solution of (2.12) then using comparison theorem we get $s(t) \geqslant u_1(t)$ and $i(t) \geqslant u_2(t)$ and hence $\lim_{t\to\infty} inf s(t) \geqslant d_1+\frac{\beta}{m}$,

$\lim_{t\to\infty} inf i(t) \geqslant \frac{\sqrt{[(1+\alpha)(B+d_1+\frac{\beta}{m})-\alpha(1-d_1-\frac{\beta}{m})]^2+4\alpha(1+\alpha)[(1-d_1-\frac{\beta}{m})(B+d_1+\frac{\beta}{m})-1]}-[(1+\alpha)(B+d_1+\frac{\beta}{m})-\alpha(1-d_1-\frac{\beta}{m})]}{2(1+\alpha)}$

if $B \geqslant 1$, $\theta\beta > d_2$ and $1-d_1-\frac{\beta}{m} > 0$ holds. Then $\exists$ a T such that for every small $\varepsilon > 0$ $i(t) > i'-\varepsilon \forall t > T$. Again from the equation(2.2c) we get $\frac{dy}{dt} \geqslant -d_2 y+\frac{\theta\beta(i'-\varepsilon)y}{my+(i'-\varepsilon)}$

$= \frac{y(t)}{my(t)+(i'-\varepsilon)}[-md_2y(t)+(\theta\beta-d_2)(i'-\varepsilon)]$

this shows that

$\lim_{t\to\infty} inf y(t) \geqslant \frac{(\theta\beta-d_2)(i'-\varepsilon)}{md_2}$. This completes the proof.

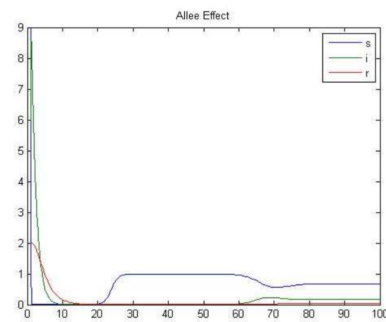

Fig. 2: MatLab generated graph of population of s, i *and* r verses time t with B=500, $\alpha = 1$, $d_1 = 0.5$, $\beta = 1.5$, m=5, $d_2 = 0.4$ and $\theta = 0.6$ shows permanence of the disease.

3 Stability Analysis

To study the stability of the system(2.2) we have calculate the Jacobian matrix in the form

$$J(s,i,y) = \begin{pmatrix} \alpha(1-s-i-\frac{1}{B+s+i})+\alpha s(-1+\frac{1}{(B+s+i)^2})-i & \alpha s(-1+\frac{1}{(B+s+i)^2})-s & 0 \\ i & \frac{\beta iy}{(my+i)^2} & \frac{-\beta i^2}{(my+i)^2} \\ 0 & \frac{\theta m\beta}{(my+i)^2} & \frac{-m\theta\beta iy}{(my+i)^2} \end{pmatrix} \quad (3.1)$$

By simple calculation at the equilibrium point E_1 reduces the jacobian matrix becomes to the following form

$$J(s^1,0,0) = \begin{pmatrix} \alpha s^1(-1+\frac{1}{(B+s^1)^2}) & -(1+\alpha)s^1+\frac{\alpha s^1}{(B+s^1)^2} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (3.2)$$

The characteristic equation of the jacobian at the equilibrium point E_1 has eigenvalues $0, 0$,and a finite

negative real number. Therefore the equilibrium point E_1 is stable.
The to find the nature of the trajectories around the equilibrium point E_2 we reduce Jacobian matrix, after some calculation as,

$$J(s^2,i^2,0)=\begin{pmatrix}\alpha s^2(-1+\frac{1}{(B+s^2+i^2)^2}) & -(1+\alpha)s^2+\frac{\alpha s^2}{(B+s^2+i^2)^2} & 0\\ i^2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{3.3}$$

After some calculation we get its characteristic equation as

$$\lambda[\lambda^2+\frac{d_2+i^2+\frac{i^2}{\alpha}}{B+d_2+i^2}\lambda+(\alpha d_2(d_2+i^2+\frac{i^2}{\alpha})(1+\frac{1}{B+d_2+i^2})+d_2)i^2]=0 \tag{3.4}$$

One of the eigenvalue is zero. As the trace of the quadratic equation $[\lambda^2+\frac{d_2+i^2+\frac{i^2}{\alpha}}{B+d_2+i^2}\lambda+(\alpha d_2(d_2+i^2+\frac{i^2}{\alpha})(1+\frac{1}{B+d_2+i^2})+d_2)i^2]=0$ is positive , so the stability condition is violated. Thus E_2 is unstable.

Theorem 5 If $1 \leqslant m\theta < \frac{md_2}{\theta\beta-d_2}$, $d_2^2(m\theta-1)(\theta\beta-d_2)^2 > m\theta^3\beta^2$ and $B \geqslant 1$ holds for the system (2.2) then the interior equilibrium point is stable.

Proof. At the interior equilibrium point E^* , the Jacobian matrix reduced into

$$J(s^*,i^*,y^*)=\begin{pmatrix}\alpha s^*(-1+\frac{1}{(B+s^*+i^*)^2}) & \alpha s^*(-1+\frac{1}{(B+s^*+i^*)^2})-s^* & 0\\ i^* & \frac{\beta i^* y^*}{(my^*+i^*)^2} & \frac{-\beta}{(my^*+i^*)^2}\\ 0 & \frac{m\theta\beta y^2}{(my^*+i^*)^2} & -\frac{m\theta\beta i^* y^*}{(my^*+i^*)^2}\end{pmatrix} \tag{3.5}$$

$$=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix} \tag{3.6}$$

where $a_{13}=a_{31}=0$,
$a_{21},a_{22},a_{32}>0$
$a_{23},a_{33}<0$.
Then the jacobian matrix is as

$$\lambda^3+c_1\lambda^2+c_2\lambda+c_3=0 \tag{3.7}$$

where

$$c_1=-(a_{11}+a_{22}+a_{33}) \tag{3.8}$$

$$c_2=a_{22}a_{33}+a_{11}a_{22}+a_{11}a_{33}-a_{12}a_{21}-a_{23}a_{32} \tag{3.9}$$

$$c_3=a_{12}a_{21}a_{33}+a_{11}a_{23}a_{32}-a_{11}a_{22}a_{33} \tag{3.10}$$

Let we assume that $a_{11}=-P, a_{12}=-Q, a_{21}=R, a_{22}=L, a_{23}=-M, a_{32}=N, a_{33}=-V$
As $i^*>y^*$ then clearly $M>L$, $V>N$ and $V>L$.It is also shown that $LV-MN=0$ and $Q>P$.
$c_1=P+V-L>0$
$c_2=P(V-L)+QR>0$
$c_3=RQV+PV(M-L)>0$
Then

$$c_1c_2-c_3=(P+V-L)[RQV+PV(M-L)]-[P(V-L)+QR] \tag{3.11}$$

$$c_1c_2-c_3=PRQV+P^2V(M-L)+QR[V(V-L)-1]+P(V-L)[V(M-L)-1] \tag{3.12}$$

.After some calculation we say that $c_1c_2-c_3>0$ provided $1 \leqslant m\theta < \frac{md_2}{\theta\beta-d_2}$, $d_2^2(m\theta-1)(\theta\beta-d_2)^2 > m\theta^3\beta^2$ and $B \geqslant 1$ holds.
Therefore the interior equilibrium point E^* stable.

Theorem 6 If Allee constant $B \geqslant 1$ and $d_1 \geqslant 1$ then the equilibrium points E_1 is globally asymptotically stable.

Proof. To prove the global stability of the disease free equilibrium point we show that if we take solution from the feasible region then after some time it remain in a interior Ⓢ . We have to first construct it. From the Theorem 1 we say that if we choose $s_0+i_0 \geqslant 0.62$ then $\exists$ a time t^* such that $s(t)+i(t) < \frac{\sqrt{(B+3)(B-1)}-(B-1)}{2}$ $\forall t>t^*$ i.e, $\lim t \to \infty s(t) = \frac{\sqrt{(B+3)(B-1)}-(B-1)}{2}$ and $\lim t \to \infty i(t)=0$. Again if $s_0+i_0<0.62$ then $s(t)+i(t) \geqslant \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}, B \geqslant 1$. Now from the equation (2.2c) we get
$\frac{dy(t)}{dt} \leqslant -d_2y(t)+\frac{\theta\beta i(t)}{m} \leqslant -d_2y(t)+\frac{\theta\beta}{m}$. If we choose a quantity as $G=\frac{\theta\beta}{md_2}$ then $y(t)<G+\varepsilon$. So the interior as
Ⓢ $=\{(s,i.y), s+i \leqslant \frac{\sqrt{(B+3)(B+1)}-(B-1)}{2}, y \leqslant G\}$.
Now we choose a scalar function as $\Gamma:\Omega \to$ Ⓢ such that $\Gamma(t)=s-i-\ln s+i$
Then

$$\begin{aligned}\frac{d\Gamma(t)}{dt} &= \alpha s(1-s-i-\frac{1}{B+s+i})-si-\alpha(1-s-i-\frac{1}{B+s+i})+i\\ &=-\alpha(1-s)(1-s-i-\frac{1}{B+s+i})+(1-d_1)i-\frac{\beta iy}{my+i}\\ &\leqslant -\alpha(1-s)(1-s-i-\frac{1}{B+s+i})+(1-d_1)i\end{aligned} \tag{3.13}$$

It is negative if $d_1>1$ in interior of Ⓢ which vanishes iff $(s,i)=(\frac{\sqrt{(B+3)(B+1)}-(B-1)}{2},0)$. Then for any arbitrary $\varepsilon>0$ there is a T such that $i(t)<\varepsilon$ forall $t>T$. From the equation(2.2c) we get
$\frac{dy(t)}{dt} \leqslant -d_2y(t)+\frac{\theta\beta i(t)}{m} \leqslant -d_2y(t)+\frac{\theta\beta}{m}\varepsilon$. which implies thatlim $\sup_{t\to\infty} \leqslant \frac{\theta\beta\varepsilon}{md_1}$. As ε is arbitrary thenlim$_{t\to\infty}=0$.
This implies that the disease free equilibrium points is globally asymptotically stable[2].

Theorem 7 If $B \geqslant 1$, $\theta\beta < d_2$ and $1-d_2-\frac{\beta}{m}>0$ then all the solution of the system(2.2) with initial value in Ω approaches to the equilibrium point $E_2=(s^2,i^2,0)$ where $s^2 = d_1, i^2 = \frac{\sqrt{(B+d_1)(1+\alpha)-\alpha(1-d_1)^2+4(1+\alpha)\alpha(B+d_1)(d_1-1)+1}-(B+d_1)(1+\alpha)-\alpha(1-d_1)}{2(1+\alpha)}$.

Proof We obtain from the theorem 4
$\liminf_{t\to\infty} i(t) \geq \frac{\sqrt{[(1+\alpha)(B+d_1+\frac{\beta}{m})-\alpha(1-d_1-\frac{\beta}{m})]^2+4\alpha(1+\alpha)[(1-d_1-\frac{\beta}{m})(B+d_1+\frac{\beta}{m})-1]}-[(1+\alpha)(B+d_1+\frac{\beta}{m})-\alpha(1-d_1-\frac{\beta}{m})]}{2(1+\alpha)}$
$=\hat{i}$ provided $B \geq 1$, $\theta\beta > d_2$ and $1-d_2-\frac{\beta}{m}>0$. Then obviously $\limsup_{t\to\infty} y(t) \leq 0$ if $\theta\beta<d_2$. As $y(t)$ is positive so $\limsup_{t\to\infty} y(t)=0$.
Since $1>d_1$ and for any small $\varepsilon>0$ such that $1-d_1-\frac{\beta\varepsilon}{\hat{i}-\varepsilon}>0$

From(3.8) we know that there is a T_1 such that $t > T_1$, $i(t) > \hat{i} - \varepsilon$. There is a $T_2 > T_1$ such that for $t > T_2$, $y(t) < \varepsilon$. Therefore(2.2b) give

$$\frac{di}{dt} \geq s(t)i(t) - d_1 i(t) - \frac{\beta i(t) y(t)}{\hat{i} - \varepsilon}$$
$$\geq i(t)[s(t) - d_1 - \frac{\beta \varepsilon}{\hat{i} - \varepsilon}]$$

Consider the comparison equation as

$$\frac{ds_1(t)}{dt} = \alpha s_1(1 - s_1 - i_1 - \frac{1}{B + s_1 + i_1}) - s_1 i_1$$
$$\frac{di_1(t)}{dt} = i_1(t)(s_1(t) - d_1 - \frac{\beta \varepsilon}{\hat{i} - \varepsilon})$$
$$\frac{ds_2(t)}{dt} = \alpha s_2(1 - s_2 - i_2 - \frac{1}{B + s_2 + i_2}) - s_2 i_2$$
$$\frac{di_2(t)}{dt} = i_2(t)(s_2(t) - d_1)$$

Let $s_1(0) \leq s(0)$, $i_1(0) \leq i(0)$. If $(s_1(t), i_1(t))$ is a solution of equation(1) with initial conditions $(s_1(0), i_1(0))$, then by comparison theorem we have $(s_1(0), s(t))$, $(i_1(t), i(t))$ $\forall t \geq 0$.
It is clear that if $1 - d_1 - \frac{\beta\varepsilon}{\hat{i}-\varepsilon} > 0$
$(s_{1*}, i_{1*}) \triangleq (d_1 + \frac{\beta\varepsilon}{\hat{i}-\varepsilon}, \frac{1}{2(1+\alpha)}[\sqrt{[(1+\alpha)(B+d_1+\frac{\beta\varepsilon}{\hat{i}-\varepsilon}) - \alpha(1-d_1-\frac{\beta\varepsilon}{\hat{i}-\varepsilon})]^2 + 4\alpha(1+\alpha)[(1-d_1-\frac{\beta\varepsilon}{\hat{i}-\varepsilon})(B+d_1+\frac{\beta\varepsilon}{\hat{i}-\varepsilon}) - 1]} - [(1+\alpha)(B+d_1+\frac{\beta\varepsilon}{\hat{i}-\varepsilon}) - \alpha(1-d_1-\frac{\beta\varepsilon}{\hat{i}-\varepsilon}]])$ is unique equilibrium of (1). Then it is globally asymptotically stable.
By comparison theorem we get,
$\liminf_{t\to\infty} s(t) \geq \liminf_{t\to\infty} s_1(t) = s_{1*}$, $\liminf_{t\to\infty} i(t) \geq \liminf_{t\to\infty} i_1(t) = i_{1*}$
similarly,
$\limsup_{t\to\infty} s(t) \geq \limsup_{t\to\infty} s_2(t) = s_{2*}$, $\limsup_{t\to\infty} i(t) \geq \limsup_{t\to\infty} i_2(t) = i_{2*}$
where $(s_{2*}, i_{2*}) =$
$(d_1, \frac{\sqrt{[(1+\alpha)(B+d_1)-\alpha(1-d_1)]^2+4\alpha(1+\alpha)[(1-d_1)(B+d_1)-1]}-[(1+\alpha)(B+d_1)-\alpha(1-d_1)]}{2(1+\alpha)})$
Because of any ε , we get
$\lim_{t\to\infty} s(t) = d_1 = s^2$
$\lim_{t\to\infty} i(t) =$
$\frac{\sqrt{(B+d_1)(1+\alpha)-\alpha(1-d_1)^2+4(1+\alpha)\alpha(B+d_1)(d_1-1)+1}-(B+d_1)(1+\alpha)-\alpha(1-d_1)}{2(1+\alpha)} = i^2$
So $(s^2, i^2, 0)$ is also globally asymptotically stable for the system(2.2).

4 Conclusion

From our detailed study of the ratio-dependent prey-predator system with disease in prey we conclude that the system is globally asymptotically stable under certain conditions[3]. When we consider the additive Allee effect on the susceptible prey population (the susceptible growth rate interrupts), the behavior of the system changes slightly i.e, the system is globally

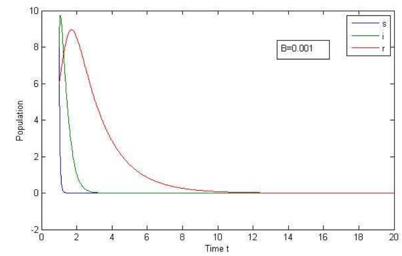

Fig. 3: MatLab generated graph of population of s, i *and* r verses time t with B=0.001, $\alpha = 1$, $d_1 = 0.5$, $\beta = 3$, m=1, $d_2 = 0.6$ and $\theta = 0.9$

asymptotically stable provided the Allee constant is greater than 1. Then after some time only susceptible population survive and other the population extinct. Again when the Allee constant less or equal to 1 then all the population vanish after a long time. This is very interesting that after imposing the Allee effect on the model the persistency condition is remaining unchanged but the system is permanent when the Allee constant is greater than 1 otherwise the permanency of the system will be lost.In our numerical simulation we have seen that from the Fig-1 susceptible population immediately falls due to the rapid infection. Infected population initially increases then all falls to zero value due to conversion of the infected individuals to recover ones. Recovered population initially increases and then decreases to zero value due to conversion of the recovered individual to susceptible ones. The susceptible population again grows after $t = 6$. Fig-1 also shows that after time $t = 12$ only susceptible population exists as some of the recovered individual has been converted to susceptible.This figure also shows that the disease is not persistence for certain values of parameters. Fig-2 shows that after long time all the population exists i.e., the disease is permanence. Fig-3 shows that for $B < 1$ i.e., for the Allee constant is less than carrying capacity all susceptible, infected and recovered population decrease to zero value which is an unstable situation. Thus for survival of population Allee constant must be greater than carrying capacity. In this paper we also shown that the endemic equilibrium exists provided the system is permanent.

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References

[1] W.C.Allee: Animal Aggregations, A study in General Sociology, AMS Press(1978).

[2] Y.Kaung, E.Beretta: Global qualitative analysis of a ratio-dependent predator-prey system, J.Math.Biol.,36,389-406(1998).

[3] Y.Xiao, L.Chen: A ratio-dependent predator-prey model with disease in the prey,App. Mathematics and Computation, 131,397-414(2002).

[4] G.Wang, X.G.Liang and F.Z.Wang: The competitive dynamics of population subject to an Allee effect, Ecological Modelling, vol.124 no.2, 183-192(1999).

[5] M.A.Mc Carthy: The Allee effect finding mates and theoretical model, Ecological Modelling, vol.103,no.1 99-102(1997).

[6] B.Dennis: Allee effects:population growth, critical density and the chance of extinction, Natural Resources Modeling, vol.3, no.3, 481-538(1989).

[7] R.M.Anderson, R.M.May: Infectious Disease of Human Dynamics and Control, Oxford University Press, Oxford(1991).

[8] W.Wang, Y.Cai, Y.Zhu and Z.Guo: Allee-Effect-Induced Instability in a reaction-diffusion Predator-prey model, Abstract and Applied Analysis,vol.2013, Article ID 487810,10 pages.

[9] J.Chattopadhaya, O.Arino: A Predator-prey model with disease in the prey,Nonlinear Analysis, 36, 749-766(1999).

[10] Y.Cai, W.Wang and J.Wang: Dynamics of a diffusive predator-prey model with additive Allee effect, Int.J.of Bio-mathematics, vol.5, no.2, Article ID 1250023(2012).

[11] P.Aguirre, F.Gonazalez-olivares and F.Saez: Two limit cycles in a Leslie-Gower Predator-prey model with additive Alle effect, Nonlinear Analysis: Real world App., vol.10,no.3, 1401-1416(2009)

[12] J.C.Holmes, W.M.Bethel: Modification of intermediate host behavior by parasites.in:E.V.Canning, C.A.Wright(Eds.), Behavioral Aspects of Parasite Transmission, Suppl. I to Zool.f.Linnean Soc.51123-149 (1972).

[13] R.O.Peterson, R.E.Page: Wolf density as a predation rate, Swedish.Wildlife Research Suppl.1,771-773(1987).

[14] S. Chatterjee, M. Bandyopadhyay and J. Chattopadhyay : Proper Predation May be Used as a Controlling Agent for Preventing Chaos in an Eco-epidemiological System - A Mathematical Study, Journal of Biological Systems, Vol. 14(4), pp. 599 - 616(2006).

[15] S.R.Zhou, Y.F.Liu and G.Wang: The stability of predator-prey systems subject to the Allee effects, Theoretical population Biology, vol.67 , no.1, 23-31(2005).

[16] M.Lewis and P.Kareiva: Allee dynamics and the spread of invading organism, Theoretical population Biology,vol.13,no.2, 141-159(1998).

[17] P.A.Stephens, W.J.sutherland: consequences of the Allee effect for behaviour, ecology and conservation, Trends in Ecology and Evolution , vol.14,no.10,pp.401-405(1999).

[18] T.H.Keeitt, M.A.Lewis: Allee effects invasion pinning and specie's borders,American Naturalists,vol.157, no.2, pp.203-216(2002).

[19] A,Morozov, S.Petrovoskii and B.L.Li:Spatiotemporal complexity of patchy invasion in a predator-prey system with the Allee effect, J.Theor. Biol.,vol.238, no.1, pp.18-35(2006).

[20] S.Petrovoskii, A.Morozov, and B.L.Li: Regimes of biological invasion in a predator-prey system with the Allee effect, Bulletin of Mathematical Biology,vol. 67,no. 3,pp. 637-661(2005).

[21] J.Zu and M.Mimura:The impact of Allee effect on predator-prey system Holling type-II functional response, Applied Mathematics and Computation.vol.217,no.7,pp.3542-3556(2010).

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