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The Generalized Inverted Generalized Exponential Distribution with an Application to a Censored Data

P. E Oguntunde[1,*] *and A. O. Adejumo*[2]

[1] Department of Mathematics, Covenant University, Ogun State, Nigeria
[2] Department of Statistics, University of Ilorin, Kwara State, Nigeria

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Abstract: We propose a two parameter Inverted Generalized Exponential (IGE) and a three parameter Generalized Inverted Generalized Exponential (GIGE) probability models as generalizations of the one-parameter Exponential distribution and some other distributions in the literature. We explore the statistical properties of the GIGE distribution and its parameters were estimated for both censored and uncensored cases using the method of maximum likelihood estimation (MLE). An application to a real data set is also provided to assess the flexibility of the GIGE distribution over some of its sub-models.

Keywords: Censored, Exponential distribution, Generalizations, Inverted Generalized Exponential, Probability models, Uncensored.

1 Introduction

([1]) and ([2]) proposed and studied a generalization of the Exponential distribution called the Generalized Exponential (GE) distribution by introducing a shape parameter to the Exponential distribution. The cumulative density function (cdf) of the Generalized Exponential (GE) distribution is given by;

$$F_{GE}(x) = (1-e^{-\lambda x})^{\alpha} \qquad ; x>0,\ \alpha>0,\ \lambda>0 \tag{1}$$

The corresponding probability density function (pdf) is given by;

$$f_{GE}(x) = \alpha\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\alpha-1} \qquad ; x>0,\ \alpha>0,\ \lambda>0 \tag{2}$$

where;
α is a shape parameter
λ is the scale parameter

Beside, inverted distributions have been studied by a number of researchers. For instance, the Inverted Exponential distribution was introduced by ([4]) and its applicability as a lifetime model has been identified by ([6]). Practically, if a random variable X has an exponential distribution, the variable $Y = \frac{1}{X}$ will have an Inverted Exponential (IE) distribution.

Motivated by the work of ([3]), we shall first introduce a two parameter probability model known as the Inverted Generalized Exponential (IGE) distribution which is practically the inverse or reciprocal of the Generalized Exponential (GE) distribution. With this understanding, we shall further propose a three parameter model named the Generalized Inverted Generalized Exponential (GIGE) distribution.

However, the rest of this article is organized as follows; In Section 2, we present both the IGE and GIGE distributions, Section 3 deals with some basic statistical properties of the proposed models coupled with the estimation of model

* Corresponding author e-mail: peluemman@yahoo.com

parameters; Section 4 provides the estimation of the model parameters using the method of maximum likelihood estimation (MLE) for both the Censored and Uncensored cases, Section 5 discusses the application of the GIGE distribution using a real data set followed by a concluding remark.

2 The Generalized Inverted Generalized Exponential Distribution

We shall start by proposing the Inverted Generalized Exponential (IGE) distribution. Let X denote a non-negative continuous random variable and given that the Generalized Exponential distribution is as defined in Equation (1) and Equation (2), therefore, the cdf and the pdf of the Inverted Generalized Exponential distribution are respectively given by;

$$F_{IGE}(x) = 1-(1-e^{-(\frac{\lambda}{x})})^{\alpha} \qquad ;x,\alpha,\lambda > 0 \tag{3}$$

$$f_{IGE}(x) = \alpha\lambda e^{-(\frac{\lambda}{x})}x^{-2}(1-e^{-(\frac{\lambda}{x})})^{\alpha-1} \qquad ;x,\alpha,\lambda > 0 \tag{4}$$

where;
α is a shape parameter
λ is the scale parameter

We can otherwise name the Inverted Generalized Exponential distribution as the Complementary or Reciprocal Generalized Exponential distribution.
We present the Reliability function and the Failure rate respectively as;

$$S_{IGE}(x) = (1-e^{-(\frac{\lambda}{x})})^{\alpha} \tag{5}$$

For $x > 0$, $\alpha > 0$ and $\lambda > 0$

$$h_{IGE}(x) = \alpha\lambda e^{-(\frac{\lambda}{x})}x^{-2}(1-e^{-(\frac{\lambda}{x})})^{-1} \tag{6}$$

For $x > 0$, $\alpha > 0$ and $\lambda > 0$

With this understanding, we can confidently propose the Generalized Inverted Generalized Exponential (GIGE) distribution by generalizing the IGE distribution as follows;

$$F_{GIGE}(x) = 1-(1-e^{-\gamma(\frac{\lambda}{x})})^{\alpha} \qquad ;x,\alpha,\lambda,\gamma > 0 \tag{7}$$

Differentiating Equation (7) with respect to x gives the pdf of the propose GIGE distribution as;

$$f_{GIGE}(x) = \alpha\lambda\gamma x^{-2}e^{-\gamma(\frac{\lambda}{x})}(1-e^{-\gamma(\frac{\lambda}{x})})^{\alpha-1} \qquad ;x,\alpha,\lambda,\gamma > 0 \tag{8}$$

where;
α and γ are shape parameters
λ is the scale parameter

For notation purposes, we write; $X \sim GIGE(\lambda,\alpha,\gamma)$

The plot for the pdf and cdf of the GIGE distribution at various parameter values are given in Figure 1 and 2 respectively.

As shown in Figure 1, the proposed GIGE distribution is positively skewed and the shape of the model is unimodal.

Special Cases:
Some known distributions in the literature are found to be sub-models of the proposed GIGE distribution. For instance,

- For $\gamma = 1$, we get the Inverted Generalized Exponential (IGE) distribution.
- For $\alpha = \gamma = 1$, we get the Inverse Exponential distribution.

Observations from the GIGE distribution with parameters α,λ,γ can be simulated using the following transformation;

$$X = \lambda\gamma[log(1-(1-U)^{\frac{1}{\alpha}})]^{-1} \tag{9}$$

where U is a random variable uniformly distributed on $(0,1)$.

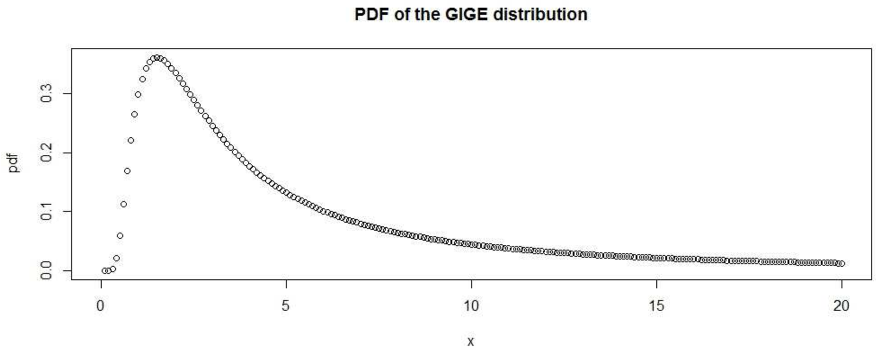

Fig. 1: Plot for the pdf of the GIGE distribution at $\alpha = 1,\ \lambda = 3,\ \gamma = 2$

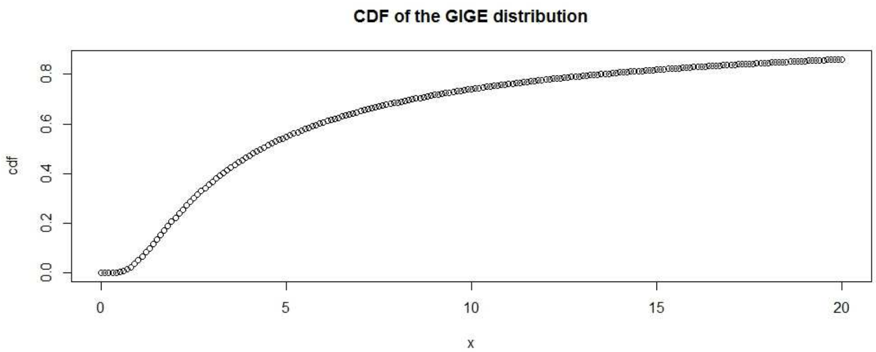

Fig. 2: Plot for the cdf of the GIGE distribution at $\alpha = 1,\ \lambda = 3,\ \gamma = 2$

3 Statistical Properties of the GIGE Distribution

This section provides some basic statistical properties of the proposed Generalized Inverted Generalized Exponential Distribution.

3.1 Reliability Analysis

The reliability (survival) function is given by;

$$S(x) = 1 - F(x) \tag{10}$$

Therefore, the reliability function for the GIGE distribution is given by;

$$S_{GIGE}(x) = (1 - e^{-\gamma(\frac{\lambda}{x})})^{\alpha} \tag{11}$$

For $x > 0, \alpha > 0, \lambda > 0, \gamma > 0$

The corresponding plot for the reliability function of the GIGE distribution is as shown in Figure 3;

The probability that a system having age $'x'$ units of time will survive up to $'x+t'$ units of time for $x > 0, \alpha > 0, \lambda > 0, \gamma > 0$ and $t > 0$ is given by;

$$S_{GIGE}(t|x) = \frac{S_{GIGE}(x+t)}{S_{GIGE}(x)} \tag{12}$$

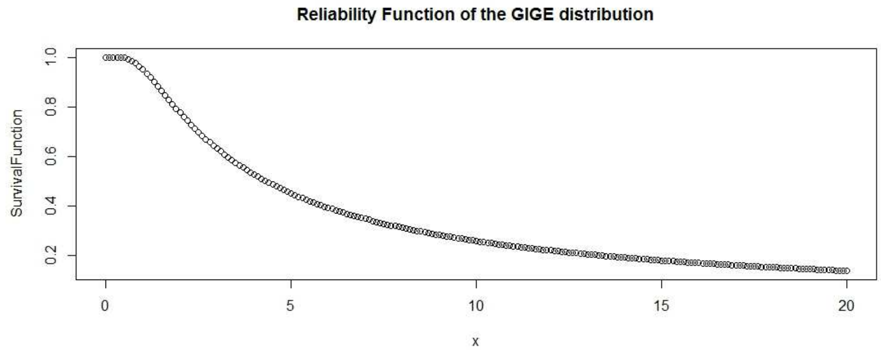

Fig. 3: Plot for the Survival function of the GIGE distribution at $\alpha = 1,\ \lambda = 3,\ \gamma = 2$

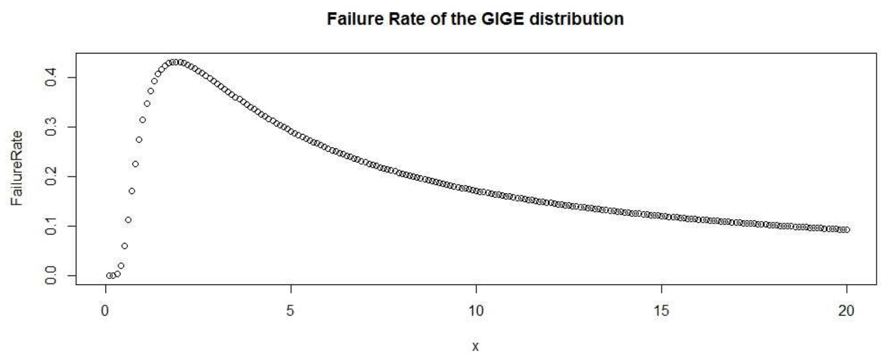

Fig. 4: Plot for the Failure rate of the GIGE distribution at $\alpha = 1,\ \lambda = 3,\ \gamma = 2$

$$S_{GIGE}(t|x) = \frac{(1 - e^{-\gamma(\frac{\lambda}{x+t})})^{\alpha}}{(1 - e^{-\gamma(\frac{\lambda}{x})})^{\alpha}} \tag{13}$$

Hazard function is given by;

$$h(x) = \frac{f(x)}{1 - F(x)} \tag{14}$$

Therefore, the corresponding failure rate (hazard function) is given by;

$$h_{GIGE}(x) = \alpha\lambda\gamma x^{-2} e^{-\gamma(\frac{\lambda}{x})}(1 - e^{-\gamma(\frac{\lambda}{x})})^{-1} \tag{15}$$

$x > 0, \alpha > 0, \lambda > 0, \gamma > 0$

The plot for the failure rate at different parameter values is provided in Figure 4.

3.2 Moments

The r-th moment of a continuous random variable X is given by;

$$\mu_r = E[X^r] = \int_0^{\infty} x^r f(x)dx \tag{16}$$

If a continuous random variable X is such that; $X \sim GIGE(\alpha,\lambda,\gamma)$,the rth moment is given by;

$$E[X^r] = \int_0^\infty \alpha\lambda\gamma x^{-2} e^{-\gamma(\frac{\lambda}{x})}(1-e^{-\gamma(\frac{\lambda}{x})})^{\alpha-1}dx \tag{17}$$

Let $\theta = \gamma(\frac{\lambda}{x})$ in Equation (17) and by following ([3]), then;

$$E[X^r] = \alpha\lambda^r\gamma^r \int_0^\infty \theta^{-r}e^{-\theta}(1-e^{-\theta})^{\alpha-1}d\theta \tag{18}$$

$$= \alpha\lambda^r\gamma^r \int_0^\infty \theta^{-r}e^{-\theta}(1+\sum_{i=1}^{\infty} a_i e^{-\theta i})d\theta$$

where;

$$a_i = \frac{\left((-1)^i(\alpha-1)(\alpha-2)...(\alpha-i)\right)}{i!} \tag{19}$$

Therefore;

$$E[X^r] = \alpha\lambda^r\gamma^r\Gamma(1-r)\left(1+\sum_{i=1}^{\infty}\frac{a_i}{(i+1)^{1-r}}\right) \tag{20}$$

The general expression for the moment generating function (mgf) of the GIGE distribution is given by;

$$M_X(t) = \alpha\sum_{r=0}^{n}\left(\frac{t^r}{r!}\lambda^r\gamma^r\Gamma(1-r)\left(1+\sum_{i=1}^{\infty}\frac{a_i}{(i+1)^{1-r}}\right)\right) \tag{21}$$

4 Parameter Estimation

This section provides the estimation of the proposed GIGE distribution both for the Censored and Uncensored cases using the method of maximum likelihood estimation (MLE) as follows;

4.1 For Censored Case:

Censoring is a situation in which the value of an observation is only partially known. It is a form of missing data problem and it is common in survival analysis.

Following ([3]), let X_i and C_i be independent random variables, where X_i denotes the lifetime of ith individual and C_i denotes the censoring time and $t_i = min(X_i, C_i)$ for $i = 1,2,3,....,n$. Each X_i is distributed according to Equation (8) with parameters α, λ and γ.

Let m denote the number of failures and let C and L denote the sets of censored and uncensored observations respectively.

The likelihood function for the censored case is given by;

$$L = \prod_{i\subset F} f(t_i) * \prod_{i\subset C} S(t_i)$$

where; $f(t_i)$ and $S(t_i)$ are the pdf and reliability function of the GIGE distribution as given in Equations (8) and (11) respectively.

Let $l = logL$;

$$l = m(\log\alpha + log\gamma + \log\lambda) - 2\sum_{i\subset F}\log t_i - \gamma\lambda\sum_{i\subset F} t_i^{-1} + (\alpha-1)\sum_{i\subset F}\log\left(1-e^{-\gamma(\frac{\lambda}{t_i})}\right) + \alpha\sum_{i\subset C}\log\left(1-e^{-\gamma(\frac{\lambda}{t_i})}\right) \tag{22}$$

Differentiating l with respect to α, λ and γ respectively gives;

$$\frac{\partial l}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i \subset F} \log\left(1 - e^{-\gamma(\frac{\lambda}{t_i})}\right)$$

$$\frac{\partial l}{\partial \lambda} = \frac{m}{\lambda} - \gamma \sum_{i \subset F} t_i + (\alpha - 1)\gamma \sum_{i \subset F} t_i^{-1}\left(\frac{e^{-\gamma(\frac{\lambda}{t_i})}}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})}\right) + \alpha\gamma \sum_{i \subset C} t_i^{-1}\left(\frac{e^{-\gamma(\frac{\lambda}{t_i})}}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})}\right)$$

$$\frac{\partial l}{\partial \gamma} = \frac{m}{\gamma} - \lambda \sum_{i \subset F} t_i + (\alpha - 1)\lambda \sum_{i \subset F} t_i^{-1}\left(\frac{e^{-\gamma(\frac{\lambda}{t_i})}}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})}\right)$$

The 3×3 observed information matrix for hypothesis testing and interval estimation for parameters α, λ and γ is given by;

$$\mathbf{J}(\theta) = -\begin{pmatrix} J_{\lambda,\lambda} & J_{\lambda,\alpha} & J_{\lambda,\gamma} \\ & J_{\alpha,\alpha} & J_{\alpha,\gamma} \\ & & J_{\gamma,\gamma} \end{pmatrix}$$

where;

$$J_{\lambda,\lambda} = \frac{\partial^2 l}{\partial \lambda^2} = -\frac{m}{\lambda^2} + (\alpha - 1)\gamma\lambda^{-1} \times \sum_{i \subset F} t_i^{-1} + (\alpha - 1)\left(\frac{e^{-\gamma(\frac{\lambda}{t_i})}}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})}\right) \times \left(-\frac{\gamma}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})}(\frac{\lambda}{t_i})\right) + \alpha\gamma\lambda^{-1} \times \sum_{i \subset C} t_i^{-1}\left(\frac{e^{-\gamma(\frac{\lambda}{t_i})}}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})}\right)\left(-\frac{\gamma}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})}(\frac{\lambda}{t_i})\right) \tag{23}$$

$$J_{\lambda,\alpha} = \frac{\partial^2 l}{\partial \lambda \partial \alpha} = \gamma \sum_{i \subset F} t_i^{-1}\left(\frac{e^{-\gamma(\frac{\lambda}{t_i})}}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})}\right) + \gamma \sum_{i \subset C} t_i^{-1}\left(\frac{e^{-\gamma(\frac{\lambda}{t_i})}}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})}\right) \tag{24}$$

$$J_{\lambda,\gamma} = \frac{\partial^2 l}{\partial \lambda \partial \gamma} = -\sum_{i \subset F} t_i^{-1} + (\alpha - 1) \times \sum_{i \subset F} t_i^{-1}\left(\frac{e^{-\gamma(\frac{\lambda}{t_i})}}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})}\right) \times \left(1 - \frac{\gamma}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})}(\frac{\lambda}{t_i})\right) + \alpha \sum_{i \subset C} t_i^{-1}\left(\frac{e^{-\gamma(\frac{\lambda}{t_i})}}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})}\right)\left(1 - \frac{\gamma}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})}(\frac{\lambda}{t_i})\right) \tag{25}$$

$$J_{\alpha,\alpha} = \frac{\partial^2 l}{\partial \alpha^2} = -\frac{m}{\alpha^2} \tag{26}$$

$$J_{\gamma,\gamma} = \frac{\partial^2 l}{\partial \gamma^2} = -\frac{m}{\gamma^2} - (\alpha - 1)\lambda^2 \sum_{i \subset F} t_i^{-2}\left(\frac{e^{-\gamma(\frac{\lambda}{t_i})}}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})^2}\right) - \alpha\lambda^2 \sum_{i \subset C} t_i^{-2}\left(\frac{e^{-\gamma(\frac{\lambda}{t_i})}}{(1 - e^{-\gamma(\frac{\lambda}{t_i})})^2}\right) \tag{27}$$

4.2 For Uncensored Case:

Let $X_1, X_2, ...X_n$ be a random sample of 'n' independently and identically distributed random variables each having a Generalized Inverted Generalized Exponential distribution defined in Equation (8), the likelihood function L is given by;

$$L(\widetilde{X}|\alpha, \lambda, \gamma) = \prod_{i=1}^{n}\left(\alpha\lambda\gamma x^{-2} e^{-\gamma(\frac{\lambda}{x})}(1 - e^{-\gamma(\frac{\lambda}{x})})^{\alpha-1}\right) \tag{28}$$

Following Oguntunde et al (2014),

Let $l = \log L(\widetilde{X}|\alpha,\lambda,\gamma)$,

$$l = n\log\alpha + n\log\gamma + n\log\lambda - 2\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}\gamma(\frac{\lambda}{x_i}) + (\alpha-1)\sum_{i=1}^{n}\log\left(1-e^{-\gamma(\frac{\lambda}{x_i})}\right) \quad (29)$$

Differentiating l with respect to α, λ and γ respectively gives;

$$\frac{\partial l}{\partial\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n}\log\left(1-e^{-\gamma(\frac{\lambda}{x_i})}\right) \quad (30)$$

$$\frac{\partial l}{\partial\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}(\frac{\lambda}{x_i}) + (\alpha-1)\sum_{i=1}^{n}\frac{(\frac{\lambda}{x_i})e^{-\gamma(\frac{\lambda}{x_i})}}{\left(1-e^{-\gamma(\frac{\lambda}{x_i})}\right)} \quad (31)$$

$$\frac{\partial l}{\partial\gamma} = \frac{n}{\gamma} - \sum_{i=1}^{n}(\frac{\lambda}{x_i}) + (\alpha-1)\sum_{i=1}^{n}\frac{(\frac{\lambda}{x_i})e^{-\gamma(\frac{\lambda}{x_i})}}{\left(1-e^{-\gamma(\frac{\lambda}{x_i})}\right)} \quad (32)$$

Setting $\frac{\partial l}{\alpha} = 0$, $\frac{\partial l}{\lambda} = 0$, $\frac{\partial l}{\gamma} = 0$ and solving the resulting nonlinear equations gives the maximum likelihood estimates of parameters α, λ and γ.

5 Application

To assess the flexibility of the proposed GIGE distribution, we make use of a censored data given by ([5]), ([8]) and ([3]). The data consist of death times (in weeks) of patients with cancer of tongue with aneuploid DNA profile. The observations are; 1, 3, 3, 4, 10, 13, 13, 16, 16, 24, 26, 27, 28, 30, 30, 32, 41, 51, **61**, 65, 67, 70, 72, 73, **74**, 77, **79**, **80**, **81**, **87**, **87**, **88**, **89**, 91, 93, **93**, 96, **97**, 100, **101**, 104, **104**, **108**, **109**, **120**, **131**, **150**, 157, 167, **231**, **240**, and **400**, where the twenty-one observations written in bold denote censored observations. The analysis in Table 1 is performed with the aid of R software. The Log-likelihood and Akaike Information Criteria (AIC) for the GIGE, IGE and IE distributions are provided.

Table 1: Descriptive Statistics on Death Times

Min	Q1	Q2	Mean	Q3	Max	Var	Skewness	Kurtosis
1.00	30.00	78.00	81.76	100.20	400.00	4774.898	2.193221	10.35995

Table 2: Performance of the GIGE, IGE and IE distributions

Distribution	Parameters	LOG-Likelihood	AIC	Rank by AIC
GIGE (Proposed)	$\widehat{\alpha} = 5.7711e-01$ $\widehat{\gamma} = 4.0605e-03$ $\widehat{\lambda} = 2.7875e+03$	-300.856	607.712	2
IGE (Proposed)	$\widehat{\alpha} = 0.5771$ $\widehat{\lambda} = 11.3186$	-300.856	605.712	1
IE ([4])	$\widehat{\lambda} = 17.3795$	-306.107	614.213	3

It is good to note that the model with the lowest AIC is ranked the best.

6 Conclusion

A two-parameter Inverted Generalized Exponential (IGE) distribution and a three-parameter Generalized Inverted Generalized Exponential (GIGE) distribution are defined in this article. The basic statistical properties of the GIGE distribution are rigorously studied. The model is positively skewed and its shape is unimodal. A number of distributions are found to be sub-models of the GIGE distribution. We estimated the parameters of the GIGE distribution for both the Censored and Uncensored cases using the method of maximum likelihood estimation (MLE). The result based on the data used, shows that the two-parameter Inverted Generalized Exponential (IGE) distribution is better than the three-parameter Generalized Inverted Generalized Exponential (GIGE) distribution. This in turn means that, generalizing the IGE distribution by introducing another shape parameter may not be needful except if the data set is more skewed.

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