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The Evaluation of the Sums of More General Series by Bernstein Polynomials

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Abstract: Let n,k be the positive integer, and let $S_k(n)$ be the sums of the k-th power of positive integers up to n: $S_k(n) = \sum_{l=1}^{n} l^k$. By means of which we consider the evaluation of the sum of more general series by Bernstein polynomials. In addition, we show reality of our idea with some examples.

Keywords: Bernoulli numbers and polynomials, Bernstein polynomials, Sums of powers of integers.

1 Introduction

The history of Bernstein polynomials depends on Bernstein in 1904. It is well known that Bernstein polynomials play a crucial important role in the area of approximation theory and the other areas of mathematics, on which they have been studied by many researchers for a long time [1, 3, 5-7, 10, 11, 16, 17]. These polynomials also take an important role in physics.

Recently the works including applications of umbral calculus to Genocchi numbers and polynomials [2], the Legendre polynomials associated with Bernoulli, Euler, Hermite and Bernstein polynomials [3], the applications of umbral calculus to extended Kim's p-adic q-deformed fermionic integrals in the p-adic integer ring [4], the integral of the product of several Bernstein polynomials [5], the generating function of Bernstein polynomials [6], a theorem concerning Bernstein polynomials [10], new generating function of the (q-) Bernstein type polynomials and their interpolation function [11], q-analogues of the sums of powers of consecutive integers, squares, cubes, quarts and quints [12-15, 18-20] have been investigated extensively.

In the complex plane, the Bernoulli polynomials $B_n(x)$ are known by the following generating series:

$$\sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!} = \frac{t}{e^t-1}e^{xt}, \ |t| < 2\pi. \tag{1.1}$$

In the case $x=0$ in (1.1), we have $B_n(0) := B_n$ that stands for Bernoulli numbers. By (1.1), we have

$$B_n(x) = \sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}. \tag{1.2}$$

The Bernoulli numbers satisfy the following identity

$$B_0 = 1 \text{ and } (B+1)^n - B_n = \delta_{1,n}$$

where $\delta_{1,n}$ stands for Kronecker's delta and we have used $B^n := B_n$ (for details, see [3], [7], [9], [17]).

Recently, Acikgoz and Araci has constructed the generating function for the Bernstein polynomials $B_{k,n}(x)$ by the rule:

$$\sum_{n=k}^{\infty} B_{k,n}(x)\frac{t^n}{n!} = \frac{(tx)^k}{k!}e^{t(1-x)} \ (t\in\mathbb{C} \text{ and } k=0,1,2,\cdots,n). \tag{1.3}$$

By (1.3), we see that

$$\sum_{n=k}^{\infty} B_{k,n}(x)\frac{t^n}{n!} = \sum_{n=k}^{\infty}\left(\binom{n}{k}x^k(1-x)^{n-k}\right)\frac{t^n}{n!}$$

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by comparing the coefficients of $\frac{t^n}{n!}$ in the above, we derive well known expression of Bernstein polynomials: For $k,n \in Z_+$

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \tag{1.4}$$

where, throughout this paper, we will assume that $x \in \mathbb{Q}$ and

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & \text{, if } n \geq k \\ 0 & \text{, if } n < k. \end{cases}.$$

It follows from (1.4) that a few Bernstein polynomials are as follows:

$$B_{0,0}(x) = 1, B_{0,1}(x) = 1-x, B_{1,1}(x) = x, B_{0,2}(x) = (1-x)^2,$$
$$B_{1,2}(x) = 2x(1-x)$$
$$B_{2,2}(x) = x^2, B_{0,3}(x) = (1-x)^3, B_{1,3}(x) = 3x(1-x)^2,$$
$$B_{2,3}(x) = 3x^2(1-x), B_{3,3}(x) = x^3.$$

In the same time, the Bernstein polynomials $B_{k,n}(x)$ have several properties of interest:

– $B_{k,n}(x) \geq 0$, for $0 \leq x \leq 1$ and $k = 0,1,\ldots,n$
– Bernstein polynomials have the symmetry property $B_{k,n}(x) = B_{n-k,n}(1-x)$
– $\sum_{k=0}^{n} B_{k,n}(x) = 1$, which is know a part of unity.
– $B_{k,n}(x) = (1-x)B_{k,n-1}(x) + xB_{k-1,n-1}(x)$ with $B_{k,n}(x) = 0$ for $k < 0$, $k > n$ and $B_{0,0}(x) = 1$ *cf.* [1], [3], [5], [6], [7], [10], [16], [17].

From (1.1), a few Bernoulli polynomials can be generated as

$$B_0(x) = 1, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6},$$
$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x.$$

For any positive integer n, followings are the most known first three sums of powers of integers:

$$1+2+3+\ldots+n = \frac{n(n+1)}{2}$$

$$1^2+2^2+3^2+\ldots+n^2 = \frac{n(n+1)(2n+1)}{6}$$

and

$$1^3+2^3+3^3+\ldots+n^3 = (1+2+3+\ldots+n)^2 = \left[\frac{n(n+1)}{2}\right]^2.$$

Formulas for sums of integer powers were first given in generalizable form by mathematician Thomas Harriot (c. 1560-1621) of England. At about the same time, Johann Faulhaber (1580-1635) of Germany gave formulas for these sums, but he did not make clear how to generalize them. Also Pierre de Fermat (1601-1665) and Blaise Pascal (1623-1662) gave the formulas for sums of powers of integers.

The Swiss mathematician Jacob Bernoulli (1654-1705) is perhaps best and most deservedly known for presenting formulas for sums of integer powers. Because he gave the most explicit sufficient instructions for finding the coefficients of the formulas [12-15, 18-20].

So, we interested in finding a method to derive a formula for the sums of powers of integers. Following an idea due to J. Bernoulli, we aim to obtain a Theorem which gives the method for the evaluation of the sums of more general series by Bernstein polynomials.

2 The Evaluation of the Sum of More General Series by Bernstein Polynomials

In the 17*th* century a topic of mathematical interest was finite sums of power of integers such as the series $1+2+3+\cdots+(n-1)$ or the series $1^2+2^2+3^2+\cdots+(n-1)^2$. The closed form for these finite sums were known, but the sums of the more general series $1^k+2^k+3^k+\ldots+(n-1)^k$ was not. It was the mathematician Jacob Bernoulli who would solve this problem with the following equality [12-15, 18-20]. The sum of the k-th powers of the first $(n-1)$ integers is given by the formula

$$1^k+2^k+3^k+\ldots+(n-1)^k = \int_1^n B_k(x)\,dx \tag{2.1}$$

using the integral of the Bernoulli polynomials $B_n(x)$ under integral from 1 to n.

Theorem 1. *Let n, k and m be positive integer and let $S_m(n)$ be $\sum_{l=1}^{n} l^m$, then we have*

$$S_m(n) = \frac{(-n^{-1})^k}{(m+k+1)!} \sum_{l=k}^{m+k+1} \binom{m+k+1}{l} B_{m+k-l+1} B_{k,l}(-n)$$
$$-\frac{1}{(m+1)!} \sum_{l=0}^{m+1} \binom{m+1}{l} 2^{m+1-l} B_l + 1.$$

Proof. To prove this Theorem, we take $\sum_{k=0}^{\infty} \frac{t^k}{k!}$ in the both sides of the Eq. (2.1), so it yields to

$$e^t + e^{2t} + \cdots + e^{(n-1)t} = \int_1^n \left(\sum_{k=0}^{\infty} B_k(x) \tfrac{t^k}{k!}\right) dx$$
$$= \int_1^n \left[\tfrac{t}{e^t-1} e^{xt}\right] dx$$
$$= \left[\sum_{m=0}^{\infty} B_m \tfrac{t^{m-1}}{m!}\right] \left[e^{nt} - e^t\right]$$
$$= \left[\sum_{m=0}^{\infty} B_m \tfrac{t^{m-1}}{m!}\right] \left[\tfrac{n^{-k}(-1)^k k!}{e^t} \sum_{m=k}^{\infty} B_{k,m}(-n) \tfrac{t^{m-k}}{m!} - e^t\right]$$

from the last identity, we see that

$$e^{2t}+e^{3t}+\cdots+e^{nt} = \frac{1}{t}\left[\sum_{m=0}^{\infty}B_m\frac{t^m}{m!}\right]\left[\frac{n^{-k}(-1)^k k!}{t^k}\sum_{m=0}^{\infty}B_{k,m}(-n)\frac{t^m}{m!}-\sum_{m=0}^{\infty}2^m\frac{t^m}{m!}\right] \quad (2.2)$$

by using Cauchy product rule in the right hand side of Eq. (2.2), we have

$$I_1=\sum_{m=0}^{\infty}\left(n^{-k}(-1)^k k!\sum_{l=k}^{m}\binom{m}{l}B_{m-l}B_{k,l}(-n)\right)\frac{t^{m-k-1}}{m!} - \sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}\binom{m}{l}2^{m-l}B_l\right)\frac{t^{m-1}}{m!}.$$

By (2.2), we derive the following

$$I_2=\sum_{m=0}^{\infty}\left(2^m+3^m+\cdots+n^m\right)\frac{t^m}{m!}.$$

When we equate I_1 and I_2, we have

$$1^m+2^m+3^m+\cdots+n^m = \frac{(-n^{-1})^k}{(m+k+1)!}\sum_{l=k}^{m+k+1}\binom{m+k+1}{l}B_{m+k-l+1}B_{k,l}(-n) - \frac{1}{(m+1)!}\sum_{l=0}^{m+1}\binom{m+1}{l}2^{m+1-l}B_l+1.$$

Thus, we complete the proof of the Teorem.

Let $m=k$ in Theorem 1, we arrive at the following Corollary 1.

Corollary 1.*Let n and k be positive integer and let $S_k(n)$ be $\sum_{l=1}^{n} l^k$, then we have*

$$S_k(n)=\frac{(-n^{-1})^k}{(2k+1)!}\sum_{l=k}^{2k+1}\binom{2k+1}{l}B_{2k-l+1}B_{k,l}(-n) - \frac{1}{(k+1)!}\sum_{l=0}^{k+1}\binom{k+1}{l}2^{k+1-l}B_l+1$$

*Example 1.*Taking $k=1$ in Corollary 1, we see that

$$1+2+3+\ldots+n=\frac{-n^{-1}}{6}\sum_{l=1}^{3}\binom{3}{l}B_{3-l}B_{1,l}(-n) - \frac{1}{2}\sum_{l=0}^{2}\binom{2}{l}2^{2-l}B_l+1 = \frac{n(n+1)}{2}.$$

For $k=2$ in Corollary 1, we have

$$1^2+2^2+3^2+\ldots+n^2=\frac{n^{-2}}{120}\sum_{l=2}^{5}\binom{5}{l}B_{5-l}B_{2,l}(-n) - \frac{1}{6}\sum_{l=0}^{3}\binom{3}{l}2^{3-l}B_l+1 = \frac{n(n+1)(2n+1)}{6}.$$

By similar way, it can be easily shown for $k=3,4,\cdots$.

3 Conclusion

We have derived the sums of the k-th power of positive integers by Bernstein polynomials and gave some examples to support Corollary 1.

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