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Numerical Solutions of the Fractional Advection-Dispersion Equation

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Abstract: In this paper we have used the homotopy analysis method (HAM) to obtain solution of space-time fractional advection-dispersion equation. The fractional derivative is described in the Caputo sense. Some illustrative examples have been presented. The obtained results using homotopy analysis method demonstrate the reliability and efficiency of the proposed algorithm.

Keywords: Advection-dispersion equation, Homotopy analysis method, Caputo fractional derivative, Riemann-Liouville fractional integral.

1 Introduction

Fractional differential equations (FDEs) have been applied in modeling many physical, engineering problems and fractional differential equations in nonlinear dynamics. Finding accurate and efficient methods for solving FDEs has been an active research undertaking. Exact solutions of most of the FDEs cannot be found easily, thus analytical and numerical methods must be used. Several methods have been used to solve Fractional differential equations, such as Laplace transform method [15], Fourier transform method [10], Adomian's decomposition method (ADM) [1,2,4], Homotopy analysis method [6,12,13] and so on [7,16,17]. The homotopy analysis method (HAM) was first proposed by Liao in his Ph.D. thesis [11]. This method has been successfully applied to solve many types of nonlinear problems [5,6, 12,13].

In this paper, we present an alternative approach based on HAM to approximate the solutions of the advection-dispersion equation with time-and space-fractional derivatives of the form:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=-v\frac{\partial^\beta u(x,t)}{\partial x^\beta}+k\frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}},\quad t>0,\ x>0,\quad 0<\alpha,\beta\leq 1 \tag{1}$$

subject to the initial conditions

$$u(0,t)=f_1(t),\quad u_x(0,t)=f_2(t),\quad u(x,0)=g(x), \tag{2}$$

where u is the concentration of contaminant, x is the spatial domain, t is time and α, β are parameters describing the order of the time- and space-fractional derivatives, respectively. Here v and k represent the average fluid velocity and the dispersion coefficient. In this paper, we consider that the fractional derivatives are taken in Caputo sense for solving the space-time fractional advection-dispersion equation.
The time fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is the Caputo fractional derivative of order α defined as

$$D_t^\alpha u(x,t)=\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\begin{cases} I^{m-\alpha}\frac{\partial^m u(x,t)}{\partial t^m}, & m-1<\alpha<m,\\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha=m\in N.\end{cases}$$

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where I^α the Riemann-Liouville fractional integral operator of order α, is defined as

$$I_t^\alpha u(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} u(x,\tau)\, d\tau, \quad \alpha > 0,\ t > 0. \tag{3}$$

Properties of the operator I^α can be found in Refs. [15], we mention only the following. For $\alpha \geq 0$ and $\mu > -1$:

1. $I_t^\alpha D_t^\alpha u(x,t) = u(x,t) - \sum_{k=0}^{m-1} \frac{\partial^k u(x,0^+)}{\partial t^k} \frac{t^k}{k!}, \quad m-1 < \alpha \leq m,\ t > 0.$ (4)

2. $I^\alpha x^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)} x^{\alpha+\mu}.$ (5)

In the case of $\alpha = \beta = 1$, Eq (1) reduces to the classical advection-dispersion equation (ADE). We assume that $\nu, k \geq 0$ so that the flow is from left to right and space-time fractional advection-dispersion equation has a unique and sufficiently smooth solution under the above initial and boundary conditions (some results on existence and uniqueness are developed in [3]). The space-time fractional advection-dispersion equation has been recently treated by a number of authors. Momani and Odibat [14] used variational iteration method and Adomian decomposition method for solving the space-time fractional advection-dispersion equation. Huang and Liu [9] considered the space-time fractional advection-dispersion equation and the solution was obtained in terms of Green functions and the representations of the Green function by applying the Fourier-Laplace transforms. Yilidrm [18] used Homotopy perturbation method for solving this equation. Also Huang et al. [8] used a finite element solution for the fractional advection-dispersion equation

The paper has been organized as follows. In Section 2 the homotopy analysis method is described. In Section 3 applying HAM for linear and nonlinear fractional diffusion-wave equation. Discussion and conclusions are presented in Section 4.

2 Homotopy analysis method

In this section the basic ideas of the homotopy analysis method are introduced. We consider the following fractional equation we extend the applications of method [13] to the following fractional equation:

$$N[u(r,t)] = 0, \tag{6}$$

where N is a nonlinear operator, $u(r,t)$ is unknown function of the independent variable r and t. By means of generalizing the traditional homotopy method, Liao [13] constructs the so-called zero-order deformation equation

$$(1-q)L[\phi(r,t;q) - u_0(r,t)] = q\hbar H(r,t) N[\phi(r,t;q)], \tag{7}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is the auxiliary parameter which increases the results convergence, $H(r,t) \neq 0$ is the auxiliary function and $L = D_t^\alpha (n-1 < \alpha \leq m)$ is an auxiliary linear operator with the following property

$$L[\phi(r,t)] = 0 \ \ when \ \ \phi(r,t) = 0, \tag{8}$$

$u_0(r,t)$ is an initial guess of $u(r,t)$, $\phi(r,t;q)$ is a unknown function, respectively. Here, we emphasize that we have freedom to choose the auxiliary linear operator L and the initial guess $u_0(r,t)$. Obviously, when $q = 0$ and $q = 1$, it holds

$$\phi(r,t;0) = u_0(r,t), \qquad \phi(r,t;1) = u(r,t) \tag{9}$$

respectively. Thus, as q increases from 0 to 1, the solution $\phi(r,t;q)$ varies from the initial guesses $u_0(r,t)$ to the solution $u(r,t)$. Expanding $\phi(r,t;q)$ in Taylor series with respect to q, we have

$$\phi(r,t;q) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t) q^m, \tag{10}$$

$$u_m(r,t) = \frac{1}{m!} \frac{\partial^m \phi(r,t;q)}{\partial q^m}\Big|_{q=0}. \tag{11}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series Eq.(10) converges at q=1, then we have

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t). \tag{12}$$

Now we define the vector of $\overrightarrow{u}_m$ as follows:

$$\overrightarrow{u}_m = \{u_0(r,t), u_1(r,t), \ldots, u_n(r,t)\}.$$

Differentiating Eq(7) for m times with respect to the embedding parameter q and setting q=0 and finally dividing by m!, we will have the so-called mth order deformation equation in the following form:

$$L[u_m(r,t) - \chi_m u_{m-1}(r,t)] = \hbar H(r,t) R_m(\overrightarrow{u}_{m-1}(r,t)), \tag{13}$$

where

$$R_m(\overrightarrow{u}_{m-1})(r,t) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(r,t;q)]}{\partial q^{m-1}}|_{q=0} \tag{14}$$

and

$$\chi_m = \begin{cases} 0, \ m \leqslant 1, \\ 1, \ m > 1. \end{cases}$$

Operating the Riemann-Liouville integral operator I^α on both side of Eq. (13), we have

$$u_m(r,t) = \chi_m u_{m-1}(r,t) - \chi_m \sum_{i=0}^{n-1} u_{m-1}^{(i)}(r,0)\frac{t^i}{i!} + \hbar H(r,t) I^\alpha R_m(\overrightarrow{u}_{m-1}(r,t)). \tag{15}$$

In this way, it is easily to obtain $u_m(r,t)$ for m $m \geq 1$, at Mth order, we have

$$u(r,t) = \sum_{m=0}^{M} u_m(r,t). \tag{16}$$

When $M \to \infty$, we get an accurate approximation of the original equation (6).

3 Application

To demonstrate the effectiveness of this method for solving space-time fractional advection dispersion equations.
Example 1: We consider the following time-fractional advection-dispersion equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = -v\frac{\partial u(x,t)}{\partial x} + k\frac{\partial^2 u(x,t)}{\partial x^2}, \ \ t > 0, \ x > 0, \quad 0 < \alpha \leq 1 \tag{17}$$

with initial conditions as

$$u(x,0) = sin(x). \tag{18}$$

To solve the Eq (17) by means of homotopy analysis method, according to the initial condition denoted in (18), it is natural to choose

$$u_0(x,t) = sin(x). \tag{19}$$

Thus, we choose the linear operator

$$L[\phi(x,t;q)] = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$$

with the property $L[c] = 0$,where c is constant.
We now define a nonlinear operator as

$$N[\phi(x,t;q)] = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + v\frac{\partial u(x,t)}{\partial x} - k\frac{\partial^2 u(x,t)}{\partial x^2}. \tag{20}$$

Using above definition, with assumption $H(x,t) = 1$ we construct the zeroth-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(t;q)]. \tag{21}$$

Obviously, when $q = 0$ and $q = 1$,

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t). \tag{22}$$

Thus, we obtain the mth-order deformation equations

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\overrightarrow{u}_{m-1}), \tag{23}$$

where

$$R_m(\overrightarrow{u}_{m-1}(x,t)) = \frac{\partial u^{\alpha}_{m-1}(x,t)}{\partial t^{\alpha}} + v\frac{\partial u_{m-1}(x,t)}{\partial x} - k\frac{\partial^2 \partial u_{m-1}(x,t)}{\partial x^2}.$$

Now the solution of the mth-order deformation equations (23)

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1} R_m(\overrightarrow{u}_{m-1}(x,t)). \tag{24}$$

Finally, we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t).$$

From (18) and (24), we obtain

$$\begin{aligned}
u_0(x,t) &= sin(x),\\
u_1(x,t) &= \hbar(k\sin(x) + v\cos(x))\frac{t^{\alpha}}{\Gamma(\alpha+1)},\\
u_2(x,t) &= \hbar(\hbar+1)(k\sin(x) + v\cos(x))\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\
&\quad + \hbar^2\left(\left(k^2 - v^2\right)\sin(x) + 2kv\cos(x)\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},\\
u_3(x,t) &= \hbar(\hbar+1)^2(k\sin(x) + v\cos(x))\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\
&\quad + 2\hbar^2(1+\hbar)\left(\left(k^2 - v^2\right)\sin(x) + 2kv\cos(x)\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&\quad + \hbar^3\left(\left(3k^2v - v^3\right)\cos(x) + k\left(k^2 - 3v^2\right)\sin(x)\right)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)},\\
&\quad \vdots
\end{aligned}$$

In the same manner the rest of components can be obtained. Consequently, we obtained the following expansion:

$$\begin{aligned}
u(x,t) = sin(x) &+ \hbar(k\sin(x) + v\cos(x))\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \hbar(1+\hbar)(k\sin(x) + v\cos(x))\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\
&+ \hbar^2\left(\left(k^2 - v^2\right)\sin(x) + 2kv\cos(x)\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\\
&+ \hbar(\hbar+1)^2(k\sin(x) + v\cos(x))\frac{t^{\alpha}}{\Gamma(\alpha+1)}\\
&+ 2\hbar^2(1+\hbar)\left(\left(k^2 - v^2\right)\sin(x) + 2kv\cos(x)\right)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots
\end{aligned} \tag{25}$$

Figs. 1-3 show the evolution results for the approximate solutions of Eq. (17) obtained for different values of α using the homotopy analysis method. It is to be noted that only five terms of the homotopy analysis series were used in evaluating the approximate solutions in Figs. 1-3.

Example 2: In this example we consider the following nonhomogeneous space-fractional equation

$$\frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} - \frac{\partial u(x,t)}{\partial x} = \frac{\partial u(x,t)}{\partial t} + (2 - 2t - 2x), \;\; t > 0, \; x > 0, \quad 0 < \beta \leq 1 \tag{26}$$

with initial conditions as

$$u(0,t) = t^2 \quad u_x(0,t) = 0 \quad u(x,0) = x^2. \tag{27}$$

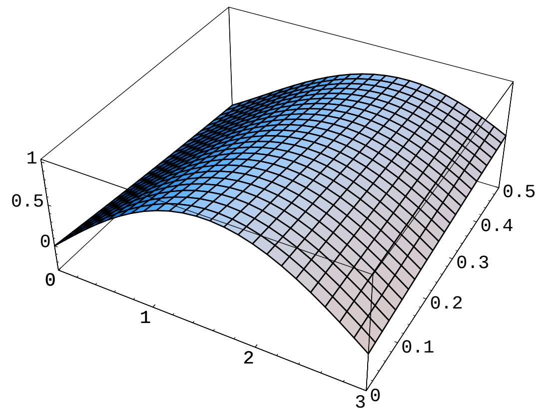

Fig. 1: Approximate solutions for $u(x,t)$ with $\alpha = 1$ and $\nu = k = 1$.

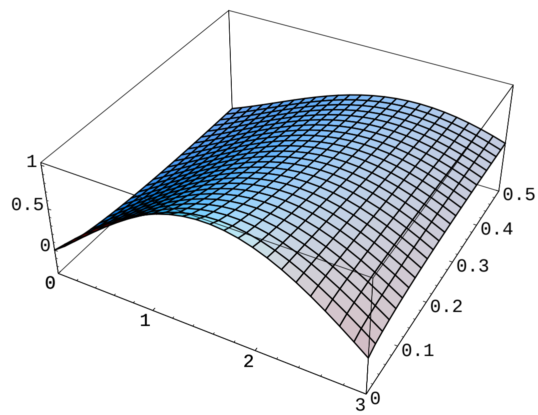

Fig. 2: Approximate solutions for $u(x,t)$ with $\alpha = 0.75$ and $\nu = k = 1$.

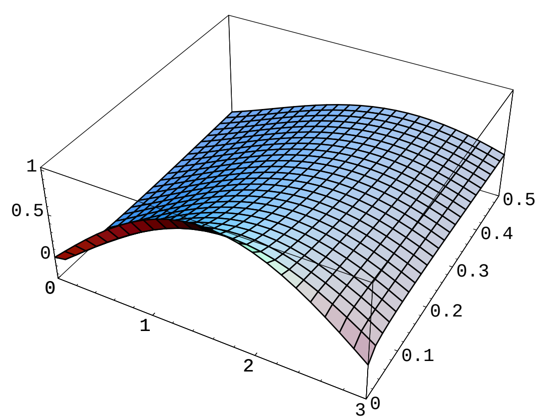

Fig. 3: Approximate solutions for $u(x,t)$ with $\alpha = 0.5$ and $\nu = k = 1$.

To solve the Eq (26), by means of homotopy analysis method, according to the initial condition denoted in (27), it is natural to choose

$$u_0(x,t) = t^2 + \frac{2-2t}{\Gamma(2\beta+1)}x^{2\beta} - \frac{2}{\Gamma(2\beta+2)}x^{2\beta+1}. \tag{28}$$

We choose the linear operator

$$L[\phi(x,t;q)] = \frac{\partial^{2\beta} u(x,t)}{\partial t^{2\beta}}$$

with the property $L[c] = 0$ with c being a constant.
We now define a nonlinear operator as

$$N[\phi(x,t;q)] = \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} - \frac{\partial u(x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial t} - (2-2t-2x). \tag{29}$$

Using above definition, with assumption $H(x,t) = 1$ we construct the zeroth-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(t;q)]. \tag{30}$$

Obviously, when $q = 0$ and $q = 1$,

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t). \tag{31}$$

Thus, we obtain the mth-order deformation equations

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\overrightarrow{u}_{m-1}), \tag{32}$$

where

$$R_m(\overrightarrow{u}_{m-1}(x,t)) = \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} - \frac{\partial u(x,t)}{\partial x} - \frac{\partial u(x,t)}{\partial t} + (1 - \chi_m(2-2t-2x)).$$

Now the solution of the mth-order deformation equations (32)

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1} R_m(\overrightarrow{u}_{m-1}(x,t)). \tag{33}$$

Finally, we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t).$$

From (27) and (43), we obtain

$$\begin{aligned}
u_0(x,t) &= t^2 + \frac{2-2t}{\Gamma(2\beta+1)} x^{2\beta} - \frac{2}{\Gamma(2\beta+2)} x^{2\beta+1},\\
u_1(x,t) &= \frac{-2\hbar t}{\Gamma(2\beta+1)} x^{2\beta} + \frac{2\hbar}{\Gamma(4\beta+1)} x^{4\beta} - \frac{\hbar(2-2t)2\beta\Gamma(2\beta)}{\Gamma(2\beta+1)\Gamma(4\beta)} x^{4\beta-1}\\
&\quad + \frac{2\hbar(2\beta+1)\Gamma(2\beta+1)}{\Gamma(2\beta+2)\Gamma(4\beta+1)} x^{4\beta},\\
&\ \ \vdots
\end{aligned}$$

and so on, in this manner the rest of components can be obtained. The solution in series form is given by

$$\begin{aligned}
u(x,t) = t^2 &+ \frac{2-2t}{\Gamma(2\beta+1)} x^{2\beta} - \frac{2}{\Gamma(2\beta+2)} x^{2\beta+1} - \frac{2\hbar t}{\Gamma(2\beta+1)} x^{2\beta} + \frac{2\hbar}{\Gamma(4\beta+1)} x^{4\beta}\\
&- \frac{\hbar(2-2t)2\beta\Gamma(2\beta)}{\Gamma(2\beta+1)\Gamma(4\beta)} x^{4\beta-1} + \frac{2\hbar(2\beta+1)\Gamma(2\beta+1)}{\Gamma(2\beta+2)\Gamma(4\beta+1)} x^{4\beta} + \cdots
\end{aligned} \tag{34}$$

It is obvious that the self-canceling noise terms appear between various components of the approximate solution. Setting $\beta = 1, \hbar = -1$ and canceling the noise terms in the HAM solution (44) yields the exact solution, for this special case, given by

$$u(x,t) = t^2 + x^2. \tag{35}$$

Figs. 4,5 show the evolution results for exact and approximate solutions of Eq. (26) when $\beta = 1$. Figs. 6 and 7 show the evolution results for the approximate solutions of Eq. (26) obtained for different values of β using the homotopy analysis method.

Example 3: We next consider the following space-time fractional advection-dispersion equation with initial conditions:

$$\frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} - \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}, \;\; t > 0,\; x > 0, \;\;\; 0 < \alpha \leq 1, \;\; 0.5 < \beta \leq 1 \tag{36}$$

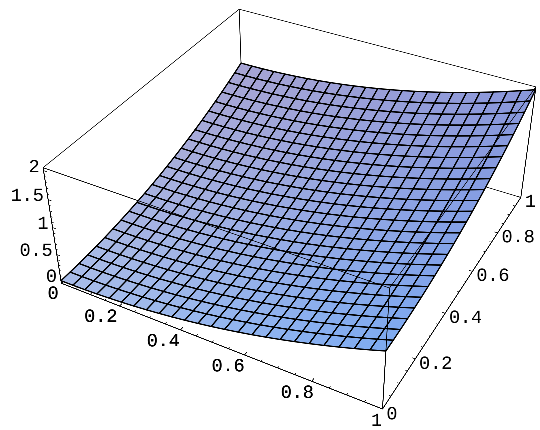

Fig. 4: the solution $u(x,t)$ when $\beta = 1$: exact solution.

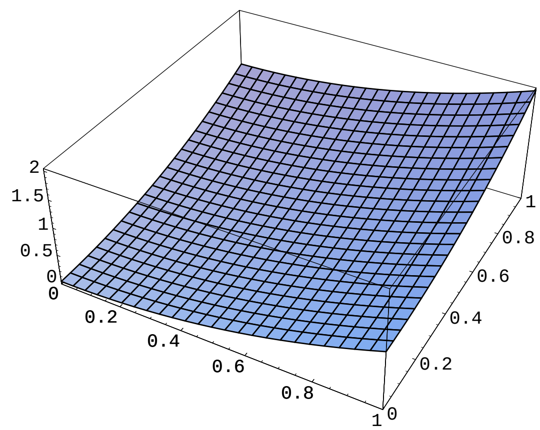

Fig. 5: Approximate solutions for $u(x,t)$ with $\alpha = 1$ and $\nu = k = 1$.

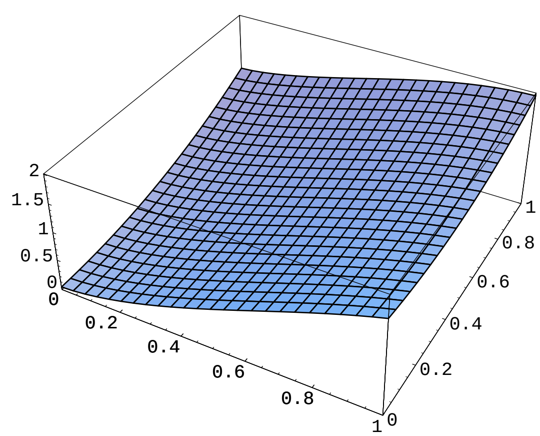

Fig. 6: Approximate solutions for $u(x,t)$ with $\alpha = 0.75$ and $\nu = k = 1$.

subject to the boundary and initial conditions

$$u(0,t) = f_1(t) \quad u_x(0,t) = f_2(t) \quad u(x,0) = g(x). \tag{37}$$

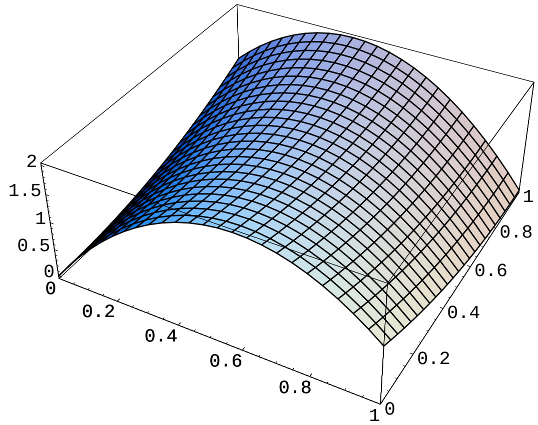

Fig. 7: Approximate solutions for $u(x,t)$ with $\alpha = 0.5$ and $v = k = 1$.

To solve the Eq (36)by using the homotopy analysis method, according to the initial condition denoted in (37), it is natural to choose

$$u_0(x,t) = f_1(t) + x f_2(t). \tag{38}$$

We choose the linear operator

$$L[\phi(x,t;q)] = \frac{\partial^{2\beta} u(x,t)}{\partial t^{2\beta}}$$

with the property $L[c] = 0$.where c is constant.
We define a nonlinear operator as

$$N[\phi(x,t;q)] = \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} - \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} - \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}. \tag{39}$$

Using above definition, with assumption $H(x,t) = 1$ we construct the zeroth-order deformation equation

$$(1-q)L[\phi(x,t;q) - u_0(x,t)] = q\hbar N[\phi(t;q)]. \tag{40}$$

It is clear that when $q = 0$ and $q = 1$,

$$\phi(x,t;0) = u_0(x,t), \qquad \phi(x,t;1) = u(x,t). \tag{41}$$

Thus, we obtain the mth-order deformation equations

$$L[u_m(x,t) - \chi_m u_{m-1}(x,t)] = \hbar R_m(\overrightarrow{u}_{m-1}), \tag{42}$$

where

$$R_m(\overrightarrow{u}_{m-1}(x,t)) = \frac{\partial^{2\beta} u(x,t)}{\partial x^{2\beta}} - \frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} - \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}.$$

Now the solution of the mth-order deformation equations (32)

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar L^{-1} R_m(\overrightarrow{u}_{m-1}(x,t)). \tag{43}$$

Finally, we have

$$u(x,t) = u_0(x,t) + \sum_{m=1}^{\infty} u_m(x,t).$$

From (27) and (43), we obtain

$$\begin{aligned}
u_0(x,t) &= f_1(t) + x f_2(t),\\
u_1(x,t) &= -\frac{\hbar f_1(t)}{\Gamma(\beta+1)} x^{\beta} - \frac{\hbar f_2(t)}{\Gamma(\beta+2)} x^{\beta+1} - \frac{\hbar D_t^{\alpha} f_1(t)}{\Gamma(2\beta+1)} x^{2\beta} - \frac{\hbar D_t^{\alpha} f_2(t)}{\Gamma(2\beta+2)} x^{2\beta+1},\\
&\ \vdots
\end{aligned}$$

and so on, in this manner the rest of components can be obtained. The solution in series form is given by

$$u(x,t) = f_1(t) + x f_2(t) - \frac{\hbar f_1(t)}{\Gamma(\beta+1)}x^{\beta} - \frac{\hbar f_2(t)}{\Gamma(\beta+2)}x^{\beta+1} - \frac{\hbar D_t^{\alpha} f_1(t)}{\Gamma(2\beta+1)}x^{2\beta} - \frac{\hbar D_t^{\alpha} f_2(t)}{\Gamma(2\beta+2)}x^{2\beta+1} + \cdots \quad (44)$$

Clear conclusion can be drawn from the analytical results in Examples 1-3 that the homotopy analysis method provides highly accurate numerical solutions without spatial discretization for the problem. It is evident that the efficiency of these approaches can be dramatically enhanced by computing further terms or further components of $u(x,t)$ when the homotopy analysis method is used.

4 Conclusions

In this paper, the homotopy analysis method has been applied to study the fractional partial differential equations. The explicit series solutions the space-time fractional advection dispersion equation are obtained, which are the same as those results given by VIM and ADM [14] and HPM [18]. It is worth pointing out that this method presents a rapid convergence for the solutions. HAM also do not require large computer memory and discretization of the variables t and x. The results show that HAM is powerful mathematical tool for solving fractional partial differential equations having wide applications in engineering. Mathematica has been used for computations in this paper.

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