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Upper and Lower Solutions Method for Partial Fractional Differential Inclusions with Not Instantaneous Impulses

Saïd Abbas[1], *Mouffak Benchohra*[2,3,*] and *Juan J. Trujillo*[4]

[1] Laboratory of Mathematics, University of Saïda, PO Box 138, 20000 Saïda, Algeria
[2] Laboratory of Mathematics, University of Sidi Bel-Abbes, P.O. Box 89, Sidi Bel-Abbès 22000, Algeria
[3] Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia
[4] Departamento de Análisis Matemático, Universidad de La Laguna, 38271 La Laguna, Tenerife, Spain

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Abstract: In this paper we use the upper and lower solutions method combined with a fixed point theorem for condensing multivalued maps due to Martelli to investigate the existence of solutions of a class of partial hyperbolic differential inclusions with not instantaneous impulses.

Keywords: Fractional differential inclusion, left-sided mixed Riemann-Liouville integral, Caputo fractional order derivative, Darboux problem, upper solution, lower solution, fixed point, not instantaneous impulses.

1 Introduction

The fractional calculus represents a powerful tool in applied mathematics to study a myriad of problems from different fields of science and engineering, with many break-through results found in mathematical physics, finance, hydrology, biophysics, thermodynamics, control theory, statistical mechanics, astrophysics, cosmology and bioengineering. There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Abbas *et al.* [9,10], Kilbas *et al.* [23], Miller and Ross [27], Zhou [30], the papers of Abbas *et al.* [1,2,3,7,8,11], Diethelm [15], Kilbas and Marzan [21], Vityuk and Golushkov [29] and the references therein.

The method of upper and lower solutions has been successfully applied to study the existence of solutions for fractional order ordinary and partial differential equations and inclusions. See the monographs by Benchohra *et al.* [13], Heikkila and Lakshmikantham [18], Ladde *et al.* [24], the papers of Abbas and Benchohra [1,2,4], and the references therein.

In [3,4,5,8], Abbas *et al.* used the upper and lower solutions method combined with some fixed point theorems for investigate the existence of solutions of some classes of impulsive partial hyperbolic differential equations and inclusions at fixed moments of impulse. In pharmacotherapy, the above instantaneous impulses can not describe the certain dynamics of evolution processes. For example, one considers the hemodynamic equilibrium of a person, the introduction of the drugs in the bloodstream and the consequent absorption for the body are gradual and continuous process. From the viewpoint of general theories, Hernández and O'Regan [19] initially offered to study a new class of abstract semilinear impulsive differential equations with not instantaneous impulses in a *PC*-normed Banach space. Meanwhile, in [6,28] the authors continue to study other new classes of differential equations with not instantaneous impulses.

* Corresponding author e-mail: benchohra@univ-sba.dz

In this paper, we use the method of upper and lower solutions for the existence of solutions of the following partial fractional differential inclusions with not instantaneous impulses

$$\begin{cases} {}^cD^r_{\theta_k}u(t,x) \in F(t,x,u(t,x)); \text{ if } (t,x) \in I_k, \ k = 0,\dots,m, \\ u(t,x) = g_k(t,x,u(t,x)); \text{ if } (t,x) \in J_k, \ k = 1,\dots,m, \\ u(t,0) = \varphi(t); \ t \in [0,a], \\ u(0,x) = \psi(x); \ x \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{1}$$

where $I_k := (s_k,t_{k+1}] \times [0,b]$, $J_k := (t_k,s_k] \times [0,b]$, $a,b > 0$, $\theta_k = (s_k,0)$; $k = 0,\dots,m$, ${}^cD^r_{\theta_k}$ is the fractional Caputo derivative of order $r = (r_1,r_2) \in (0,1] \times (0,1]$, $0 = s_0 < t_1 \le s_1 \le t_2 < \cdots < s_{m-1} \le t_m \le s_m \le t_{m+1} = a$, $F : I_k \times \mathbb{R}^n \to \mathscr{P}(\mathbb{R}^n)$; $k = 0,\dots,m$ is a compact valued multi-valued map, $\mathscr{P}(\mathbb{R}^n)$ is the family of all nonempty subsets of $\mathbb{R}^n$, $g_k : J_k \times \mathbb{R}^n \to \mathbb{R}^n$; $k = 1,\dots,m$ are given continuous functions, $\varphi : [0,a] \to \mathbb{R}^n$ and $\psi : [0,b] \to \mathbb{R}^n$ are given absolutely continuous functions. Our approach in this paper is based on a combination of a fixed-point theorem for condensing multivalued maps due to Martelli (see[26]) with the concept of upper and lower solutions. This paper initiates the application of upper and lower solutions method to this new class of problems.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. Let $J = [0,a] \times [0,b]$; $a,b > 0$, denote $L^1(J)$ the space of Lebesgue-integrable functions $u : J \to \mathbb{R}^n$ with the norm

$$\|u\|_{L^1} = \int_0^a \int_0^b \|u(t,x)\| dxdt,$$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$.
As usual, by $AC(J)$ we denote the space of absolutely continuous functions from J into $\mathbb{R}^n$, and $\mathscr{C} := C(J)$ is the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm $\|.\|_\infty$ defined by

$$\|u\|_\infty = \sup_{(t,x)\in J} \|u(t,x)\|.$$

Consider the Banach space

$$\begin{aligned} PC = \big\{ u : J \to \mathbb{R}^n : \ & u \in C((t_k,t_{k+1}] \times [0,b]); \ k = 0,1,\dots,m, \text{ and there} \\ & \text{exist } u(t_k^-,x) \text{ and } u(t_k^+,x); \ k = 1,\dots,m, \\ & \text{with } u(t_k^-,x) = u(t_k,x) \text{ for each } x \in [0,b] \big\}, \end{aligned}$$

with the norm

$$\|u\|_{PC} = \sup_{(t,x)\in J} \|u(t,x)\|.$$

Definition 1.*[9,29]. Let $\theta = (0,0)$, $r = (r_1,r_2)$; $r_1,r_2 > 0$ and $u \in L^1(J)$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by the expression*

$$(I^r_\theta u)(t,x) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^t \int_0^x (t-\tau)^{r_1-1}(x-\xi)^{r_2-1} u(\tau,\xi) d\xi d\tau,$$

where $\Gamma(.)$ is the (Euler's) Gamma function defined by $\Gamma(\varsigma) = \int_0^\infty t^{\varsigma-1}e^{-t}dt$; $\varsigma > 0$.

In particular,

$$(I^\theta_\theta u)(t,x) = u(t,x), \ (I^\sigma_\theta u)(t,x) = \int_0^t \int_0^x f(\tau,\xi) d\xi d\tau; \text{ for almost all } (t,x) \in J,$$

where $\sigma = (1,1)$.

For instance, $I^r_\theta u$ exists for all $r_1,r_2 \in (0,\infty)$, when $u \in L^1(J)$. Note also that when $u \in C(J)$, then $(I^r_\theta u) \in C(J)$, moreover

$$(I^r_\theta u)(t,0) = (I^r_\theta u)(0,x) = 0; \ t \in [0,a], \ x \in [0,b].$$

*Example 1.*Let $\lambda,\omega\in(-1,0)\cup(0,\infty)$, $r=(r_1,r_2)$, $r_1,r_2\in(0,\infty)$ and $h(t,x)=t^{\lambda}x^{\omega};\ \ (t,x)\in J$. We have $h\in L^1(J)$, and we get

$$(I_\theta^r h)(t,x)=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)}t^{\lambda+r_1}x^{\omega+r_2},\ \text{for almost all } (t,x)\in J.$$

By $1-r$ we mean $(1-r_1,1-r_2)\in[0,1)\times[0,1)$. Denote by $D_{tx}^2:=\frac{\partial^2}{\partial t\partial x}$, the mixed second order partial derivative.

Definition 2.*[9,29]. Let $r\in(0,1]\times(0,1]$ and $u\in L^1(J)$. The Caputo fractional-order derivative of order r of u is defined by the expression*

$${}^cD_\theta^r u(t,x)=(I_\theta^{1-r}D_{tx}^2u)(t,x)=\frac{1}{\Gamma(1-r_1)\Gamma(1-r_2)}\int_0^t\int_0^x\frac{D_{\tau\xi}^2u(\tau,\xi)}{(t-\tau)^{r_1}(x-\xi)^{r_2}}d\xi d\tau.$$

The case $\sigma=(1,1)$ is included and we have

$$({}^cD_\theta^\sigma u)(t,x)=(D_{tx}^2u)(t,x);\ \text{for almost all } (t,x)\in J.$$

*Example 2.*Let $\lambda,\omega\in(-1,0)\cup(0,\infty)$ and $r=(r_1,r_2)\in(0,1]\times(0,1]$, then

$${}^cD_\theta^r t^\lambda x^\omega=\frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda-r_1)\Gamma(1+\omega-r_2)}t^{\lambda-r_1}x^{\omega-r_2};\ \text{for almost all } (t,x)\in J.$$

Let $a_1\in[0,a]$, $z^+=(a_1,0)\in J$, $J_z=(a_1,a]\times[0,b]$, $r_1,r_2>0$ and $r=(r_1,r_2)$. For $u\in L^1(J_z)$, the expression

$$(I_{z^+}^r u)(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{a_1^+}^t\int_0^x(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}u(\tau,\xi)d\xi d\tau,$$

is called the left-sided mixed Riemann-Liouville integral of order r of u.

Definition 3.*[9,29]. For $u\in L^1(J_z)$ where D_{tx}^2u is Bochner integrable on J_z, the Caputo fractional order derivative of order r of u is defined by the expression*

$$({}^cD_{z^+}^r u)(t,x)=(I_{z^+}^{1-r}D_{tx}^2u)(t,x).$$

Let (X,d) be a metric space. We use the following notations:

$$\mathscr{P}_{bd}(X)=\{Y\in\mathscr{P}(X):Y\ bounded\},\ \mathscr{P}_{cl}(X)=\{Y\in\mathscr{P}(X):Y\ closed\},$$

$$\mathscr{P}_{cp}(X)=\{Y\in\mathscr{P}(X):Y\ compact\},\ \ and\ \ \mathscr{P}_{cv}(X)=\{Y\in\mathscr{P}(X):Y\ convex\}.$$

A multivalued map $G:X\to\mathscr{P}(X)$ has *convex (closed) values* if $G(x)$ is convex (closed) for all $x\in X$. We say that G is *bounded* on bounded sets if $G(B)$ is bounded in X for each bounded set B of X, i.e.,

$$\sup_{x\in B}\{\sup\{\ \|u\|:\ u\in G(x)\}\}<\infty.$$

G is called upper semi-continuous (u.s.c.) on X if for each $x_0\in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0)\subseteq N$. Finally, we say that G has a *fixed point* if there exists $x\in X$ such that $x\in G(x)$.

Definition 4.*[12] An upper semicontinuous map $G:X\to\mathscr{P}(X)$ is said to be condensing, if for any bounded subset $B\subseteq X$ with $\alpha(B)\neq0$, we have $\alpha(G(B))<\alpha(B)$, where α denotes the Kuratowski measure of noncompactness.*

*Remark.*A completely continuous multivalued map is the easiest example of a condensing map.

For each $u \in PC$ let the set $S_{F,u}$ known as *the set of selectors* from F defined by

$$S_{F,u} = \{v \in L^1(I_k) : v(t,x) \in F(t,x,u(t,x)))\ ,\ a.e.\ (t,x) \in I_k;\ k = 0,\dots,m\}.$$

For more details on multivalued maps we refer to the books of Deimling [14], Djebali *et al.* [16], Hu and Papageorgiou [20], Kisielewicz [22], Górniewicz [17].

Definition 5. *A multivalued map $F : J \times \mathbb{R}^n \to \mathscr{P}(\mathbb{R}^n)$ is said to be Carathéodory if*

(i) $(t,x) \longmapsto F(t,x,u)$ *is measurable for each* $u \in \mathbb{R}^n$;
(ii) $u \longmapsto F(t,x,u)$ *is upper semicontinuous for almost all* $(t,x) \in J$.

F is said to be L^1-Carathéodory if $(i),(ii)$ and the following condition holds;

(iii) for each $c > 0$, there exists $\sigma_c \in L^1(J,\mathbb{R}_+)$ such that

$$\begin{aligned}\|F(t,x,u)\|_{\mathscr{P}} &= \sup\{\|f\| : f \in F(t,x,u)\}\\ &\le \sigma_c(t,x)\ \text{ for all } \|u\| \le c \text{ and for } a.e.\ (t,x) \in J.\end{aligned}$$

Lemma 1. *[20]. Let G be a completely continuous multivalued map with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $u_n \to u,\ w_n \to w,\ w_n \in G(u_n)$ imply $w \in G(u)$).*

Lemma 2. *[25]. Let X be a Banach space. Let $F : J \times X \longrightarrow \mathscr{P}_{cp,cv}(X)$ be an L^1-Carathéodory multivalued map and let Λ be a linear continuous mapping from $L^1(J,X)$ to $C(J,X)$, then the operator*

$$\begin{aligned}\Lambda \circ S_F : C(J,X) &\longrightarrow \mathscr{P}_{cp,cv}(C(J,X)),\\ u &\longmapsto (\Lambda \circ S_F)(u) := \Lambda(S_{F,u})\end{aligned}$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

Lemma 3. *(Martelli)[26]. Let X be a Banach space and $N : X \to \mathscr{P}_{cl,cv}(X)$ be an u. s. c. and condensing map. If the set $\Omega := \{u \in X : \lambda u \in N(u)\ for\ some\ \lambda > 1\}$ is bounded, then N has a fixed point.*

3 Main Result

Definition 6. *A function $u \in PC$ whose r-derivative exists is said to be a solution of (1) if there exists a function $f \in S_{F,u}$ such that u satisfies $({}^cD^r_{\theta_k}u)(t,x) = f(t,x)$ on I_k; $k = 0,\dots,m$, and $u(t,x) = g_k(t,x,u(t,x))$ on J_k, $k = 1,\dots,m$, and conditions $u(t,0) = \varphi(t)$; $t \in [0,a]$, $u(0,x) = \psi(x)$; $x \in [0,b]$, $\varphi(0) = \psi(0)$ are satisfied.*

Let $z,\ \overline{z} \in C(J)$ be such that

$$z(t,x) = (z_1(t,x), z_2(t,x),\dots,z_n(t,x)),\ (t,x) \in J,$$

and

$$\overline{z}(t,x) = (\overline{z}_1(t,x), \overline{z}_2(t,x),\dots,\overline{z}_n(t,x));\ (t,x) \in J.$$

The notation $z \le \overline{z}$ means that

$$z_i(t,x) \le \overline{z}_i(t,x);\ i = 1,\dots,n.$$

Definition 7. *A function $z \in PC$ is said to be a lower solution of (1) if there exists a function $f \in S_{F,z}$ such that z satisfies*

$$\begin{cases}({}^cD^r_{\theta_k}z)(t,x) \le f(t,x);\ \text{ on } I_k;\ k = 0,\dots,m,\\ z(x_k^+,y) \le z(x_k^-,y) + I_k(z(x_k^-,y));\ \text{ if } y \in [0,b],\ k = 1,\dots,m,\\ z(t,x) \le g_k(t,x,z(t,x));\ \text{ on } J_k,\ k = 1,\dots,m,\\ z(t,0) \le \varphi(t);\ t \in [0,a],\ z(0,x) \le \psi(x);\ x \in [0,b],\ \text{ and }\ z(0,0) \le \varphi(0).\end{cases}$$

The function z is said to be an upper solution of (1) if the reversed inequalities hold.

As a consequence of Lemma 2.14 in [9], we have the following lemma

Lemma 4. *Let $r_1, r_2 \in (0,1]$, $\mu(t,x) = \varphi(t) + \psi(x) - \varphi(0)$. A function $u \in PC(J)$ is solution of the problem (1), if and only if there exists $f \in S_{F,u}$, such that u satisfies*

$$\begin{cases} u(t,x) = \mu(t,x) \\ + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi) d\xi d\tau; \;\; if \;\; (t,x) \in [0,t_1] \times [0,b], \\ u(t,x) = \varphi(t) + g_k(s_k, x, u(s_k,x)) - g_k(s_k, 0, u(s_k,0)) \\ + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi) d\xi d\tau; \;\; if \;\; (t,x) \in I_k, \; k = 1, \dots, m, \\ u(t,x) = g_k(t,x,u(t,x)); \; if \; (t,x) \in J_k, \; k = 1, \dots, m, \\ u(t,0) = \varphi(t); \; t \in [0,a], \; u(0,x) = \psi(x); \; x \in [0,b] \; and \;\; \varphi(0) = \psi(0). \end{cases} \tag{2}$$

For the study of the problem (1), we first list the following hypotheses:

(H_1) $F : I_k \times \mathbb{R}^n \longrightarrow \mathscr{P}_{cp,cv}(\mathbb{R}^n)$; $k = 0, \dots, m$ is L^1-Carathéodory,
(H_2) There exist v and $w \in PC$ lower and upper solutions for the problem (1) such that $v(t,x) \leq w(t,x)$ for each $(t,x) \in J$,
(H_3)For each $(t,x) \in J_k$; $k = 1, \dots, m$ we have

$$\begin{aligned} v(t,x) &\leq \min_{u \in [v(t,x), w(t,x)]} g_k(t,x,(gu)) \\ &\leq \max_{u \in [v(t,x), w(t,x)]} g_k(t,x,(gu)) \\ &\leq w(t,x). \end{aligned}$$

Theorem 1. *Assume that hypotheses $(H_1) - (H_3)$ hold. Then the problem (1) has at least one solution u such that*

$$v(t,x) \leq u(t,x) \leq w(t,x); \; for\ all \; (t,x) \in J.$$

Proof. Transform the problem (1) into a fixed point problem. Consider the following modified problem

$$\begin{cases} {}^cD^r_{\theta_k} u(t,x) \in F(t,x,(gu)(t,x)); \text{ if } (t,x) \in I_k, \; k = 0, \dots, m, \\ u(t,x) = g_k(t,x,(gu)(t,x)); \text{ if } (t,x) \in J_k, \; k = 1, \dots, m, \\ u(t,0) = \varphi(t); \; t \in [0,a], \\ u(0,x) = \psi(x); \; x \in [0,b], \\ \varphi(0) = \psi(0), \end{cases} \tag{3}$$

where $g : PC \longrightarrow PC$ be the truncation operator defined by

$$(gu)(t,x) = \begin{cases} v(t,x); & u(t,x) < v(t,x), \\ u(t,x); & v(t,x) \leq u(t,x) \leq w(t,x), \\ w(t,x); & w(t,x) < u(t,x). \end{cases}$$

A solution to (3) is a fixed point of the operator $N : PC \to \mathscr{P}(PC)$ defined by

$$(Nu)(t,x) = \left\{ h \in PC : \begin{cases} h(t,x) = \mu(t,x) \\ + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi) d\xi d\tau; \\ \text{ if } \;\; (t,x) \in [0,t_1] \times [0,b], \\ \\ h(t,x) = \varphi(t) \\ + g_k(s_k, x, (gu)(s_k,x)) - g_k(s_k, 0, (gu)(s_k,0)) \\ + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi) d\xi d\tau; \\ \text{ if } \;\; (t,x) \in I_k, \; k = 1, \dots, m, \\ \\ h(t,x) = g_k(t,x,(gu)(t,x)); \\ \text{if } (t,x) \in J_k, \; k = 1, \dots, m, \end{cases} \right\}$$

where

$$f \in \tilde{S}^1_{F,g(u)} = \{f \in S^1_{F,g(u)} : f(t,x) \geq f_1(t,x) \text{ on } A_1 \text{ and } f(t,x) \leq f_2(t,x) \text{ on } A_2\},$$
$$A_1 = \{(t,x) \in J : u(t,x) < v(t,x) \leq w(t,x)\},$$
$$A_2 = \{(t,x) \in J : u(t,x) \leq w(t,x) < u(t,x)\},$$

and

$$S^1_{F,g(u)} = \{f \in L^1(I_k) : f(t,x) \in F(t,x,(gu)(t,x)), \text{ for } (t,x) \in I_k;\ k = 0,\ldots,m\}.$$

Remark. (A)For each $u \in PC$, the set $\tilde{S}_{F,g(u)}$ is nonempty. In fact, (H_1) implies there exists $f_3 \in S_{F,g(u)}$, so we set

$$f = f_1\chi_{A_1} + f_2\chi_{A_2} + f_3\chi_{A_3},$$

where χ_{A_i} is the characteristic function of $A_i;\ i = 1,2,3$ and

$$A_3 = \{(t,x) \in J : v(t,x) \leq u(t,x) \leq w(t,x)\}.$$

Then, by decomposability, $f \in \tilde{S}_{F,g(u)}$.

(B)By the definition of g it is clear that $F(.,.,(gu)(.,.))$ is an L^1-Carathéodory multi-valued map with compact convex values and there exists $\phi \in C(I_k,\mathbb{R}_+);\ k = 0,\ldots,m$ such that

$$\|F(t,x,(gu)(t,x))\|_{\mathscr{P}} \leq \phi(t,x); \text{ for each } u \in \mathbb{R}^n \text{and } (t,x) \in I_k;\ k = 0,\ldots,m.$$

Set

$$\phi^* := \max_{k=1,\ldots,m} \sup_{(t,x)\in I_k} \phi(t,x).$$

(C)By the definition of g and from (H_3) we have

$$v(t,x) \leq g_k(t,x,(gu)(t,x)) \leq w(t,x);\ (t,x) \in J_k;\ k = 1,\ldots,m.$$

Set

$$L := \max_{k=1,\ldots,m} \max_{(t,x)\in J_k} (\|v(t,x)\|, \|w(t,x)\|).$$

From Lemma 4 and the fact that $g(u) = u$ for all $v \leq u \leq w$, the problem of finding the solutions of problem (1) is reduced to finding the solutions of the operator inclusion $u \in N(u)$. We shall show that N is a completely continuous multivalued map, u.s.c. with convex closed values. The proof will be given in several steps.

Step 1: *$N(u)$ is convex for each $u \in PC$.*
Indeed, if $h_1,\ h_2$ belong to $N(u)$, then there exist $f_1,\ f_2 \in \tilde{S}^1_{F,g(u)}$ such that for each $(t,x) \in J$ we have

$$\begin{cases} h_i(t,x) = \mu(t,x) \\ + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f_i(\tau,\xi)d\xi d\tau; \text{ if } (t,x) \in [0,t_1]\times[0,b],\ i = 1,2, \\ h_i(t,x) = \varphi(t) + g_k(s_k,x,(gu)(s_k,x)) - g_k(s_k,0,(gu)(s_k,0)) \\ + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f_i(\tau,\xi)d\xi d\tau; \text{ if } (t,x) \in I_k,\ k = 1,\ldots,m,\ i = 1,2, \\ h_i(t,x) = g_k(t,x,(gu)(t,x)); \text{ if } (t,x) \in J_k,\ k = 1,\ldots,m,\ i = 1,2. \end{cases}$$

Let $0 \leq \xi \leq 1$. Then, for each $(t,x) \in J$, we have

$$\begin{cases} (\xi h_1 + (1-\xi)h_2)(t,x) = \mu(t,x) \\ + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} ((\xi f_1 + (1-\xi)f_2))(\tau,\xi)d\xi d\tau; \text{ if } (t,x) \in [0,t_1]\times[0,b], \\ (\xi h_1 + (1-\xi)h_2)(t,x) = \varphi(t) + g_k(s_k,x,(gu)(s_k,x)) - g_k(s_k,0,(gu)(s_k,0)) \\ + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} (\xi f_1 + (1-\xi)f_2)(\tau,\xi)d\xi d\tau; \text{ if } (t,x) \in I_k,\ k = 1,\ldots,m, \\ (\xi h_1 + (1-\xi)h_2)(t,x) = g_k(t,x,(gu)(t,x)); \text{ if } (t,x) \in J_k,\ k = 1,\ldots,m. \end{cases}$$

Since $\tilde{S}^1_{F,g(u)}$ is convex (because F has convex values), we have

$$\xi h_1 + (1-\xi)h_2 \in N(u).$$

Step 2: *N sends bounded sets of PC into bounded sets.*
Indeed, we can prove that $N(PC)$ is bounded. It is enough to show that there exists a positive constant ℓ such that for each $h \in N(u),\ u \in PC$ one has $\|h\|_\infty \leq \ell$.
If $h \in N(u)$, then there exists $f \in \tilde{S}^1_{F,g(u)}$ such that for each $(t,x) \in J$ we have

$$\begin{cases} h(t,x) = \mu(t,x) \\ + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi)d\xi d\tau; \text{ if } (t,x) \in [0,t_1]\times[0,b], \\ h(t,x) = \varphi(t) + g_k(s_k,x,(gu)(s_k,x)) - g_k(s_k,0,(gu)(s_k,0)) \\ + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi)d\xi d\tau; \text{ if } (t,x) \in I_k,\ k=1,\dots,m, \\ h(t,x) = g_k(t,x,(gu)(t,x)); \text{ if } (t,x) \in J_k,\ k=1,\dots,m. \end{cases}$$

Then, we get

$$\begin{cases} \|h(t,x)\| \leq \|\mu(t,x)\| \\ + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \phi(\tau,\xi)d\xi d\tau; \text{ if } (t,x) \in [0,t_1]\times[0,b], \\ \|h(t,x)\| \leq \|\varphi(t)\| + \|g_k(s_k,x,(gu)(s_k,x))\| + \|g_k(s_k,0,(gu)(s_k,0))\| \\ + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \phi(\tau,\xi)d\xi d\tau; \text{ if } (t,x) \in I_k,\ k=1,\dots,m, \\ \|h(t,x)\| \leq \|g_k(t,x,(gu)(t,x))\|; \text{ if } (t,x) \in J_k,\ k=1,\dots,m. \end{cases}$$

Then, we obtain

$$\begin{cases} \|h(t,x)\| \leq \|\mu\|_\infty \\ + \int_0^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \phi^* d\xi d\tau; \text{ if } (t,x) \in [0,t_1]\times[0,b], \\ \|h(t,x)\| \leq \|\varphi\|_\infty + 2L \\ + \int_{s_k}^t \int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \phi^* d\xi d\tau; \text{ if } (t,x) \in I_k,\ k=1,\dots,m, \\ \|h(t,x)\| \leq L; \text{ if } (t,x) \in J_k,\ k=1,\dots,m. \end{cases}$$

Thus

$$\begin{cases} \|h\|_\infty \leq \|\mu\|_\infty + \frac{a^{r_1} b^{r_2} \phi^*}{\Gamma(1+r_1)\Gamma(1+r_2)} := \ell_1; \text{ if } (t,x) \in [0,t_1]\times[0,b], \\ \|h\|_\infty \leq \|\varphi\|_\infty + 2L + \frac{a^{r_1} b^{r_2} \phi^*}{\Gamma(1+r_1)\Gamma(1+r_2)} := \ell_2; \text{ if } (t,x) \in I_k,\ k=1,\dots,m, \\ \|h\|_\infty \leq L; \text{ if } (t,x) \in J_k,\ k=1,\dots,m. \end{cases}$$

Hence

$$\|h\|_\infty \leq \max\{L,\ell_1,\ell_2\} := \ell.$$

Step 3: *N sends bounded sets of PC into equi-continuous sets.*
Let $(\tau_1,\xi_1),\ (\tau_2,\xi_2) \in J,\ \tau_1 < \tau_2,\ \xi_1 < \xi_2$ and $B_\rho = \{u \in PC : \|u\|_\infty \leq \rho\}$ be a bounded set of PC. For each $u \in B_\rho$ and

$h \in N(u)$, there exists $f \in \tilde{S}^1_{F,g(u)}$ such that for each $(t,x) \in [0,t_1] \times [0,b]$ we have

$$\|h(\tau_2,\xi_2)-h(\tau_1,\xi_1)\| \leq \|\mu(\tau_2,\xi_2)-\mu(\tau_1,\xi_1)\|$$
$$+\int_0^{\tau_1}\int_0^{\xi_1}\frac{[(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}-(\tau_1-\tau)^{r_1-1}(x_1-\xi)^{r_2-1}]}{\Gamma(r_1)\Gamma(r_2)}\|f(\tau,\xi)\|d\xi d\tau$$
$$+\int_{\tau_1}^{\tau_2}\int_{\xi_1}^{\xi_2}\frac{(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|f(\tau,\xi)\|d\xi d\tau$$
$$+\int_0^{\tau_1}\int_{\xi_1}^{\xi_2}\frac{(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|f(\tau,\xi)\|d\xi d\tau$$
$$+\int_{\tau_1}^{\tau_2}\int_0^{\xi_1}\frac{(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|f(\tau,\xi)\|d\xi d\tau.$$

Then, we get

$$\|h(\tau_2,\xi_2)-h(\tau_1,\xi_1)\| \leq \|\mu(\tau_2,\xi_2)-\mu(\tau_1,\xi_1)\|$$
$$+\phi^*\int_0^{\tau_1}\int_0^{\xi_1}\frac{[(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}-(\tau_1-\tau)^{r_1-1}(x_1-\xi)^{r_2-1}]}{\Gamma(r_1)\Gamma(r_2)}d\xi d\tau$$
$$+\phi^*\int_{\tau_1}^{\tau_2}\int_{\xi_1}^{\xi_2}\frac{(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}d\xi d\tau$$
$$+\phi^*\int_0^{\tau_1}\int_{\xi_1}^{\xi_2}\frac{(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}d\xi d\tau$$
$$+\phi^*\int_{\tau_1}^{\tau_2}\int_0^{\xi_1}\frac{(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}d\xi d\tau.$$

As $\tau_1 \longrightarrow \tau_2$ and $\xi_1 \longrightarrow \xi_2$, the right-hand side of the above inequality tends to zero.

Also, for each $(t,x) \in I_k,\ k=1,\dots,m$, we have

$$\|h(\tau_2,\xi_2)-h(\tau_1,\xi_1)\| \leq \|\varphi(\tau_1)-\varphi(\tau_2)\|$$
$$+\|g_k(s_k,\xi_1,(gu)(s_k,\xi_1))-g_k(s_k,\xi_2,(gu)(s_k,\xi_2))\|$$
$$+\int_{s_k}^{\tau_1}\int_0^{\xi_1}\frac{[(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}-(\tau_1-\tau)^{r_1-1}(x_1-\xi)^{r_2-1}]}{\Gamma(r_1)\Gamma(r_2)}\|f(\tau,\xi)\|d\xi d\tau$$
$$+\int_{\tau_1}^{\tau_2}\int_{\xi_1}^{\xi_2}\frac{(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|f(\tau,\xi)\|d\xi d\tau$$
$$+\int_{s_k}^{\tau_1}\int_{\xi_1}^{\xi_2}\frac{(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|f(\tau,\xi)\|d\xi d\tau$$
$$+\int_{\tau_1}^{\tau_2}\int_0^{\xi_1}\frac{(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|f(\tau,\xi)\|d\xi d\tau.$$

Thus, we get

$$\|h(\tau_2,\xi_2)-h(\tau_1,\xi_1)\| \leq \|\varphi(\tau_1)-\varphi(\tau_2)\|$$
$$+\|g_k(s_k,\xi_1,(gu)(s_k,\xi_1))-g_k(s_k,\xi_2,(gu)(s_k,\xi_2))\|$$
$$+\phi^*\int_{s_k}^{\tau_1}\int_0^{\xi_1}\frac{[(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}-(\tau_1-\tau)^{r_1-1}(x_1-\xi)^{r_2-1}]}{\Gamma(r_1)\Gamma(r_2)}d\xi d\tau$$
$$+\phi^*\int_{\tau_1}^{\tau_2}\int_{\xi_1}^{\xi_2}\frac{(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}d\xi d\tau$$
$$+\phi^*\int_{s_k}^{\tau_1}\int_{\xi_1}^{\xi_2}\frac{(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}d\xi d\tau$$
$$+\phi^*\int_{\tau_1}^{\tau_2}\int_0^{\xi_1}\frac{(\tau_2-\tau)^{r_1-1}(\xi_2-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}d\xi d\tau.$$

As $\tau_1 \longrightarrow \tau_2$ and $\xi_1 \longrightarrow \xi_2$, the right-hand side of the above inequality tends to zero. Again, for each $(t,x) \in J_k,\ k = 1,\dots,m,$ we have

$$\|h(\tau_2,\xi_2)-h(\tau_1,\xi_1)\| \le \|g_k(\tau_2,\xi_2,(gu)(\tau_2,\xi_2))-g_k(\tau_1,\xi_1,(gu)(\tau_1,\xi_1))\|.$$

and as $\tau_1 \longrightarrow \tau_2$ and $\xi_1 \longrightarrow \xi_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that N is completely continuous and therefore a condensing multi-valued map.

Step 4: *N has a closed graph.*
Let $u_n \to u_*,\ h_n \in N(u_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(u_*)$.
$h_n \in N(u_n)$ means that there exists $f_n \in \tilde{S}^1_{F,g(u_n)}$ such that, for each $(t,x) \in J$, we have

$$\begin{cases} h_n(t,x)=\mu(t,x)\\ +\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f_n(\tau,\xi)d\xi d\tau;\ \text{ if }\ (t,x)\in[0,t_1]\times[0,b],\\ h_n(t,x)=\varphi(t)+g_k(s_k,x,(gu_n)(s_k,x))-g_k(s_k,0,(gu_n)(s_k,0))\\ +\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f_n(\tau,\xi)d\xi d\tau;\ \text{ if }\ (t,x)\in I_k,\ k=1,\dots,m,\\ h_n(t,x)=g_k(t,x,(gu_n)(t,x));\ \text{if } (t,x)\in J_k,\ k=1,\dots,m. \end{cases}$$

We must show that there exists $f_* \in \tilde{S}^1_{F,g(u_*)}$ such that, for each $(t,x) \in J$,

$$\begin{cases} h_*(t,x)=\mu(t,x)\\ +\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f_*(\tau,\xi)d\xi d\tau;\ \text{ if }\ (t,x)\in[0,t_1]\times[0,b],\\ h_*(t,x)=\varphi(t)+g_k(s_k,x,(gu_*)(s_k,x))-g_k(s_k,0,(gu_*)(s_k,0))\\ +\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f_*(\tau,\xi)d\xi d\tau;\ \text{ if }\ (t,x)\in I_k,\ k=1,\dots,m,\\ h_*(t,x)=g_k(t,x,(gu_*)(t,x));\ \text{if } (t,x)\in J_k,\ k=1,\dots,m. \end{cases}$$

Now, we consider the linear continuous operator

$$\begin{aligned} \Lambda: L^1(J)\ &\longrightarrow C(J),\\ f\ &\longmapsto \Lambda f \end{aligned}$$

defined by

$$\begin{cases} (\Lambda f)(t,x)=\mu(t,x)\\ +\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f(\tau,\xi)d\xi d\tau;\ \text{ if }\ (t,x)\in[0,t_1]\times[0,b],\\ (\Lambda f)(t,x)=\varphi(t)+g_k(s_k,x,(gu)(s_k,x))-g_k(s_k,0,(gu)(s_k,0))\\ +\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} f_*(\tau,\xi)d\xi d\tau;\ \text{ if }\ (t,x)\in I_k,\ k=1,\dots,m,\\ (\Lambda f)(t,x)=g_k(t,x,(gu)(t,x));\ \text{if } (t,x)\in J_k,\ k=1,\dots,m. \end{cases}$$

From Lemma 2, it follows that $\Lambda \circ \tilde{S}^1_F$ is a closed graph operator. Clearly we have

$$\begin{cases} \|h_n(t,x)-h_*(t,x)\| \\ \leq \int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|f_n(\tau,\xi)-f_*(\tau,\xi)\|d\xi d\tau \\ \to 0 \ as\ n\to\infty;\ \text{ if }\ (t,x)\in[0,t_1]\times[0,b], \\ \\ \|[h_n(t,x)-g_k(s_k,x,(gu_n)(s_k,x))+g_k(s_k,0,(gu_n)(s_k,0))] \\ -[h_*(t,x)-g_k(s_k,x,(gu_*)(s_k,x))+g_k(s_k,0,(gu_*)(s_k,0))]\| \\ \leq \int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}\|f_n(\tau,\xi)-f_*(\tau,\xi)\|d\xi d\tau \\ \to 0 \ as\ n\to\infty;\ \text{ if }\ (t,x)\in I_k,\ k=1,\dots,m, \\ \\ \|[h_n(t,x)-g_k(t,x,(gu_n)(t,x))] \\ -[h_*(t,x)-g_k(t,x,(gu_*)(t,x))]\|=0;\ \text{if } (t,x)\in J_k,\ k=1,\dots,m. \end{cases}$$

Moreover, from the definition of Λ, we have

$$\begin{cases} \|h_n(t,x)-h_*(t,x)\|\in\Lambda(\tilde{S}^1_{F,g(u_n)});\ \text{ if }\ (t,x)\in[0,t_1]\times[0,b], \\ \\ \|[h_n(t,x)-g_k(s_k,x,(gu_n)(s_k,x))+g_k(s_k,0,(gu_n)(s_k,0))] \\ \in\Lambda(\tilde{S}^1_{F,g(u_n)});\ \text{ if }\ (t,x)\in I_k,\ k=1,\dots,m, \\ \\ \|[h_n(t,x)-g_k(t,x,(gu_n)(t,x))]\in\Lambda(\tilde{S}^1_{F,g(u_n)});\ \text{if } (t,x)\in J_k,\ k=1,\dots,m. \end{cases}$$

Since $u_n\to u_*$, it follows from Lemma 2 that, for some $f_*\in\Lambda(\tilde{S}^1_{F,g(u_*)})$, we have

$$\begin{cases} h_*(t,x)=\mu(t,x) \\ +\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f_*(\tau,\xi)d\xi d\tau;\ \text{ if }\ (t,x)\in[0,t_1]\times[0,b], \\ \\ h_*(t,x)=\varphi(t)+g_k(s_k,x,(gu_*)(s_k,x))-g_k(s_k,0,(gu_*)(s_k,0)) \\ +\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f_*(\tau,\xi)d\xi d\tau;\ \text{ if }\ (t,x)\in I_k,\ k=1,\dots,m, \\ \\ h_*(t,x)=g_k(t,x,(gu_*)(t,x));\ \text{if } (t,x)\in J_k,\ k=1,\dots,m. \end{cases}$$

From Lemma 1, we can conclude that N is u.s.c.

Step 5: *The set* $\Omega=\{u\in PC:\lambda u\in N(u)\ for\ some\ \lambda>1\}$ *in bounded.*
Let $u\in\Omega$. Then, there exists $f\in\Lambda(\tilde{S}^1_{F,g(u)})$, such that

$$\begin{cases} \lambda u(t,x)=\mu(t,x) \\ +\int_0^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(\tau,\xi)d\xi d\tau;\ \text{ if }\ (t,x)\in[0,t_1]\times[0,b], \\ \\ \lambda u(t,x)=\varphi(t)+g_k(s_k,x,(gu)(s_k,x))-g_k(s_k,0,(gu)(s_k,0)) \\ +\int_{s_k}^t\int_0^x \frac{(t-\tau)^{r_1-1}(x-\xi)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)}f(\tau,\xi)d\xi d\tau;\ \text{ if }\ (t,x)\in I_k,\ k=1,\dots,m, \\ \\ \lambda u(t,x)=g_k(t,x,(gu)(t,x));\ \text{if } (t,x)\in J_k,\ k=1,\dots,m. \end{cases}$$

As in Step 2, this implies that for each $(t,x)\in J$, we have

$$\|u\|_\infty\leq\frac{\ell}{\lambda}<\ell.$$

This shows that Ω is bounded. As a consequence of Lemma 3, we deduce that N has a fixed point which is a solution of (3) on J.

Step 6: *The solution u of (3) satisfies*

$$v(t,x) \le u(t,x) \le w(t,x); \text{ for all } (t,x) \in J.$$

Let u be the above solution to (3).

Case 1. If $(t,x) \in J_k$; $k = 1,\dots,m$, Then from (H_3) it is clear that

$$v(t,x) \le u(t,x) = g_k(t,x,(gu)(t,x)) \le w(t,x); \ k = 1,\dots,m.$$

Case 2. Now, we prove that the solution u of (3) satisfies

$$v(t,x) \le u(t,x) \le w(t,x); \text{ for all } (t,x) \in I_k, \ k = 0,\dots,m.$$

First, we prove that

$$u(t,x) \le w(t,x) \text{ for all } (t,x) \in I_k, \ k = 0,\dots,m.$$

Assume that $u-w$ attains a positive maximum on $(s_k^+, t_{k+1}^-] \times [0,b]$ at $(\bar{t}_k, \bar{x}) \in (s_k^+, t_{k+1}^-] \times [0,b]$, for some $k = 0,...,m$, that is,

$$(u-w)(\bar{t}_k,\bar{x}) = \max\{u(t,x) - w(t,x) : (t,x) \in (s_k^+, t_{k+1}^-] \times [0,b]\} > 0,$$

for some $k = 0,...,m$. There exists $(t_k^*, x^*) \in (s_k^+, t_{k+1}^-) \times [0,b]$ such that

$$[u(t,x^*) - w(t,x^*)] + [u(t_k^*,x) - w(t_k^*,x)]$$

$$-[u(t_k^*,x^*) - w(t_k^*,x^*)] \le 0; \text{ for all } (t,x) \in ([t_k^*,\bar{t}_k] \times \{x^*\}) \cup (\{t_k^*\} \times [x^*,b]), \tag{4}$$

and

$$u(t,x) - w(t,x) > 0; \text{ for all } (t,x) \in (t_k^*,\bar{t}_k] \times (x^*,b]. \tag{5}$$

By the definition of g one has

$${}^cD_\theta^r u(t,x) \in F(t,x,w(t,x)); \text{ for all } (t,x) \in [t_k^*,\bar{t}_k] \times [x^*,b]. \tag{6}$$

An integration of (6) on $[t_k^*,t] \times [x^*,x]$ for each $(t,x) \in [t_k^*,\bar{t}_k] \times [x^*,b]$ yields
$u(t,x) + u(t_k^*,x^*) - u(t,x^*) - u(t_k^*,x)$

$$= \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{t_k^*}^{t} \int_{x^*}^{x} (t-s)^{r_1-1}(x-y)^{r_2-1} f(s,y)\,dyds, \tag{7}$$

where $f(t,x) \in F(t,x,w(t,x))$. From (7) and using the fact that w is an upper solution to (1), we get

$$u(t,x) + u(t_k^*,x^*) - u(t,x^*) - u(t_k^*,x) \le w(t,x) + w(t_k^*,x^*) - w(t,x^*) - w(t_k^*,x),$$

which gives,

$$u(t,x) - w(t,x)$$

$$\le [u(t,x^*) - w(t,x^*)] + [u(t_k^*,x) - w(t_k^*,x)] - [u(t_k^*,x^*) - w(t_k^*,x^*)]. \tag{8}$$

Thus from (4), (5) and (8) we obtain the contradiction

$$0 < [u(t,x) - w(t,x)] \le [u(t,x^*) - w(t,x^*)]$$

$$+[u(t_k^*,x) - w(t_k^*,x)] - [u(t_k^*,x^*) - w(t_k^*,x^*)] \le 0; \text{ for all } (t,x) \in [t_k^*,\bar{t}_k] \times [x^*,b].$$

Thus

$$u(t,x) \le w(t,x) \text{ for all } (t,x) \in I_k, \ k = 0,\dots,m.$$

Analogously, we can prove that

$$u(t,x) \ge v(t,x), \text{ for all } (t,x) \in I_k, \ k = 0,\dots,m.$$

From cases 1 and 2 we get

$$v(t,x) \le u(t,x) \le w(t,x), \text{ for all } (t,x) \in J.$$

This shows that the problem (3) has a solution u satisfying $v \le u \le w$ which is solution of (1).

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