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Coincidence Point Results with Mappings Satisfying Rational Inequality in Partially Ordered Complex Valued Metric Spaces

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Abstract: In this paper we prove certain coincidence point and fixed point results in partially ordered complex valued metric spaces for a pair of compatible mappings which satisfy certain rational weak inequality involving two control functions. The results are supported with examples.

Keywords: Partial order; Complex-valued metric space; Coincidence point; Fixed point; Compatible mappings; Weak contraction; Rational inequality.

1 Introduction

In this paper we prove certain coincidence point and fixed point results in partially ordered complex valued metric spaces for a pair of compatible mappings which satisfy certain rational weak inequality involving two control functions. Coincidence points are natural extensions of fixed points when we deal with more than one mappings. Metric fixed point theory is widely recognized as have been originated in the work of S. Banach in 1922 [6] where he proved the famous contraction mapping principle. Fixed point theory in partially ordered metric spaces is of relatively recent origin. An early result in this direction is due to Turinici [37] in which fixed point problems were studied in partially ordered uniform spaces. Later, this branch of fixed point theory has developed through a number of works some of which are in [8,9,13,14,20,21,22,28,30,31].

Also there are large efforts for generalizing metric spaces by changing the form and interpretation of the metric function. Ghaler [18] introduced 2-metric spaces where a real number is assigned to any three points of the space. Probabilistic metric spaces were introduced by Schweiter et al. [33,34] in which any pair of points is assigned to a suitable distribution function making possible a probabilistic sense of distance. Fuzzy metric spaces were introduced in more than one ways by various means of fuzzification as, for example in [19] by assigning any pair of points to a suitable fuzzy set and spelling out the triangular inequality by using a t-norm. Another example is in the work of Kaleva et al. [26] where any pair of points is assigned to a fuzzy number. G-metric space [29] is another generalization in which every triplet of points is assigned to a non-negative real number but in a different way than in 2-metric spaces. Cone metric spaces [23] are introduced by allowing the metric to assume values in Banach spaces. There are also other extension of the metric which are not mentioned above. It can be seen that in recent times efforts of extending the concept of metric space has continued in a rapid manner. Simultaneously, metric fixed point and coincidence point theory have been extended rapidly in these spaces over the recent years.

Our interest is in some fixed point and coincidence point problems in partially ordered complex valued metric spaces. These spaces are generalizations of metric space where the metric function takes values from the field of complex numbers, thus opening the scope of the concepts from complex analysis for incorporation in the

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metric space structure. The space was originally introduced by Azam et al. [4]. Fixed point theory has been studied in this space in a good number of papers, some of which we mention in [1,7,15,35,36].

Weak contraction principle is a generalization of Banach's contraction principle which was first given by Alber et al. in Hilbert spaces [2]. It was subsequently extended to metric spaces by Rhoades [32]. Weak contraction and weak contractive type conditions have been used and further generalized by many researchers to establish fixed point and coincidence point results in metric and generalized metric spaces [10,11,12,13,14,15,17,28,38].

Dass and Gupta [16] generalized the Banach's contraction mapping principle by using a contractive condition of rational type. Fixed point theorems for contractive type conditions satisfying rational inequalities in metric spaces and complex valued metric spaces have been developed in a number of works [1,7,8,9,15,22,24].

The concept of compatibility was introduced by Jungck [25]. In common fixed point and coincidence point problems, this concept and its generalizations have been used extensively. References [3,5,14,27] are some examples of such works.

2 Mathematical Preliminaries

Let $\mathscr{C}$ be the set of complex numbers and $z_1,\ z_2 \in \mathscr{C}$. Define a partial order $\precsim$ on $\mathscr{C}$ as follows:

$z_1 \precsim z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.

It follows that $z_1 \precsim z_2$ if one of the following conditions is satisfied:

(i) $Re(z_1) = Re(z_2),\ \ Im(z_1) < Im(z_2)$,

(ii) $Re(z_1) < Re(z_2),\ \ Im(z_1) = Im(z_2)$,

(iii) $Re(z_1) < Re(z_2),\ \ Im(z_1) < Im(z_2)$,

(iv) $Re(z_1) = Re(z_2),\ \ Im(z_1) = Im(z_2)$.

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i), (ii) and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied.

By the notations $\mathscr{P}$ and $Int\ (\mathscr{P})$, we denote the following subsets of $\mathscr{C}$.

$$\mathscr{P} = \{z \in \mathscr{C} : 0 \precsim z\} = \{z = x + i\,y \in \mathscr{C} : x \geq 0,\ y \geq 0\},$$

and

$$Int\ (\mathscr{P}) = \{z \in \mathscr{C} : 0 \prec z\} = \{z = x + i\,y \in \mathscr{C} : x > 0,\ y > 0\}.$$

In $\mathscr{P}$ every increasing sequence which is bounded from above is convergent (or every decreasing sequence which is bounded from below is convergent).

Note that

(i) $z_1 \precsim z_2,\ \ z_2 \prec z_3 \implies z_1 \prec z_3$,

(ii) if $0 \precsim x_n \precsim y_n$, for all $n \in \mathbf{N}$, then $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y \implies 0 \precsim x \precsim y$,

(iii) if $x_n \precsim y_n \precsim z_n$, for all $n \in \mathbf{N}$, then $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} z_n = x \implies \lim\limits_{n\to\infty} y_n = x$.

Definition 2.1.[4] Let X be a nonempty set. Suppose that a mapping $d : X \times X \to \mathscr{C}$ satisfies:

(i) $0 \precsim d(x,\ y)$, for all $x,\ y \in X$ and $d(x,y) = 0$ if and only if $x = y$

(ii) $d(x,\ y) = d(y,\ x)$, for all $x,\ y \in X$

(iii) $d(x,\ y) \precsim d(x,\ z) + d(z,\ y)$, for all $x,y,z \in X$.

Then d is called a complex valued metric on X and $(X,\ d)$ is called a complex valued metric space.

Definition 2.2. Let $(X,\ d)$ be a complex valued metric space, $\{x_n\}$ be a sequence in X and $x \in X$.

(i) If for every $c \in \mathscr{C}$ with $0 \prec c$ there exists $n_0 \in \mathbf{N}$ such that for all $n > n_0$, $d(x_n,\ x) \prec c$, then $\{x_n\}$ said to be convergent, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$. We denote this by $\lim\limits_{n\to\infty} x_n = x$, or $x_n \to x$ as $n \to \infty$.

(ii) If for every $c \in \mathscr{C}$ with $0 \prec c$ there exists $n_0 \in \mathbf{N}$ such that for all $n,\ m > n_0$, $d(x_n,\ x_m) \prec c$, then $\{x_n\}$ is said to be a Cauchy sequence.

(iii) If every Cauchy sequence in X is convergent, then $(X,\ d)$ is a complete complex valued metric space.

Lemma 2.1.[4] Let $(X,\ d)$ be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|\ d(x_n,\ x)\ | \to 0$ as $n \to \infty$.

Note 2.1. We can also replace the limit in lemma 2.1 by the equivalent limiting condition $d(x_n,\ x) \longrightarrow 0$ as $n \longrightarrow \infty$.

Lemma 2.2.[4] Let $(X,\ d)$ be a complex valued metric space and $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|\ d(x_n,\ x_m)\ | \to 0$ as $n,m \to \infty$.

Note 2.2. We can also replace the limit in lemma 2.2 by the equivalent limiting condition $d(x_n,\ x_m) \longrightarrow 0$ as $n,\ m \longrightarrow \infty$.

Definition 2.3.[25] Let $(X,\ d)$ be a metric space and $f,\ g : X \longrightarrow X$. The pair $(f,\ g)$ is said to be compatible if

$$\lim_{n\to\infty} d(fgx_n,\ gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = t$, for some $t \in X$.

In the following we give the definition of compatible mappings in complex valued metric spaces as follows.

Definition 2.4. Let $(X,\ d)$ be a complex valued metric space and $f,\ g : X \longrightarrow X$. The pair $(f,\ g)$ is said to be compatible if

$$\lim_{n\to\infty} d(fgx_n,\ gfx_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = t$, for some $t \in X$.

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Definition 2.5.[14] Let f and g be self-maps of a set X (i.e., $f,\ g : X \longrightarrow X$). If $fx = gx$, for some $x \in X$, then x is called a coincidence point of f and g.

Definition 2.6.[14] Let $(X, \preceq)$ be a partially ordered set, $f : X \longrightarrow X$ and $g : X \longrightarrow X$. The mapping f is said to be g - nondecrasing if for all $x,\ y \in X$, $gx \preceq gy$ implies $fx \preceq fy$ and g - nonincreasing if for all $x,\ y \in X$, $gx \preceq gy$ implies $fx \succeq fy$.

Definition 2.7. Let $(X,\ d)$ be a complex valued metric space, $f : X \to X$ and $x_0 \in X$. Then the function f is continuous at x_0 if for any sequence $\{x_n\}$ in X, $x_n \to x_0$ implies $fx_n \to fx_0$.

Definition 2.8. A subset S of a complex valued metric space X is closed if for every sequence $\{x_n\}$ in S which converges to some $x \in X$ implies that $x \in S$.

Definition 2.9. Let $\psi : \mathscr{P} \to \mathscr{P}$ be a function.
(i)We say $\psi$is strongly monotone increasing if for $x,\ y \in \mathscr{P}, x \precsim y \Longleftrightarrow \psi(x) \precsim \psi(y)$.
$(ii)\psi$is said to be continuous at $x_0 \in \mathscr{P}$ if for any sequence $\{x_n\}$ in $\mathscr{P}$, $x_n \to x_0 \Rightarrow \psi(x_n) \to \psi(x_0)$.

In our results in the following section we will use the following class of functions.

We denote by Ψ the set of all functions $\psi : Int\ (\mathscr{P}) \cup \{0\} \longrightarrow Int\ (\mathscr{P}) \cup \{0\}$ satisfying

(i_ψ) ψ is continuous and strongly monotonic increasing,
(ii_ψ) $\psi(t) = 0$ if and only if $t = 0$;

and by Φ we denote the set of all functions $\phi : Int\ (\mathscr{P}) \cup \{0\} \longrightarrow Int\ (\mathscr{P}) \cup \{0\}$ such that

(i_ϕ) $\phi(t) = 0$ if and only if $t = 0$,
(ii_ϕ) $\phi(t) \prec t$ for $t \in Int\ (\mathscr{P})$.

Recently Choudhury and Metiya [15] proved these following lemmas which will be used in our results.
Lemma 2.3.[15] Let $(X,\ d)$ be a complex valued metric space such that $d(x,y) \in$ Int $(\mathscr{P})$, for $x,y \in X$ with $x \neq y$. Let $\phi \in \Phi$ be such that either $\phi(t) \precsim d(x,y)$ or $d(x,y) \precsim \phi(t)$, for $t \in$ Int $(\mathscr{P})$ and $x,y \in X$. Let $\{x_n\}$ be a sequence in X for which $\{d(x_n,\ x_{n+1})\}$ is monotonic decreasing. Then $\{d(x_n,\ x_{n+1})\}$ is convergent to either $r = 0$ or $r \in Int\ (\mathscr{P})$.

Lemma 2.4.[15] Let $(X,\ d)$ be a complex valued metric space, $\{x_n\}$ a sequence in X and $\phi \in \Phi$. Then the sequence $\{x_n\}$ is a Cauchy sequence if and only if for every $c \in \mathscr{C}$ with $0 \prec c$ there exists $n_0 \in \mathbf{N}$ such that $d(x_n,\ x_m) \prec \phi(c)$, for all $m,n > n_0$.

3 Main Results

Theorem 3.1. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a complex valued metric d on X such that $(X,\ d)$ is a complete complex valued metric space with $d(x,\ y) \in$ Int $(\mathscr{P})$ for $x,y \in X$ with $x \neq y$. Let f and g be two continuous self mappings on X such that $f(X) \subseteq g(X)$, f is g - nondecrasing with respect to $\preceq$ and $(f,\ g)$ is compatible pair. Suppose there exist $\psi \in \Psi$ and a continuous function $\phi \in \Phi$ such that

$$(i)\quad \phi(t) \precsim d(x,\ y) \text{ or } d(x,\ y) \precsim \phi(t)$$

$$(ii)\quad \psi(d(fx,\ fy)) \precsim \psi(u(x,\ y)) - \phi(d(gx,\ gy)), \text{for all } x,\ y \in X \text{ with } gy \preceq gx,$$

where

$$u(x,y) = \frac{d(fy,\ gy)\ d(fy,\ gx)}{1 + d(gx,\ gy)} + d(gx,\ gy).$$

If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point in X.

Proof. Let $x_0 \in X$ be such that $gx_0 \preceq fx_0$. Since $f(X) \subseteq g(X)$, we can choose $x_1 \in X$ such that $gx_1 = fx_0$. Again we can choose $x_2 \in X$ such that $gx_2 = fx_1$. Continuing this process we construct a sequence $\{x_n\}$ in X such that

$$gx_{n+1} = fx_n, \quad \text{for all } n \geq 0. \tag{1}$$

Since $gx_0 \preceq fx_0$ and $gx_1 = fx_0$, we have $gx_0 \preceq gx_1$ which implies that $fx_0 \preceq fx_1$. Now, $fx_0 \preceq fx_1$, that is, $gx_1 \preceq gx_2$ implies that $fx_1 \preceq fx_2$. Again, $fx_1 \preceq fx_2$, that is, $gx_2 \preceq gx_3$ implies that $fx_2 \preceq fx_3$. Continuing this process, we have

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq gx_3 \preceq \ldots \preceq gx_n \preceq gx_{n+1} \preceq \ldots,$$

and

$$fx_0 \preceq fx_1 \preceq fx_2 \preceq fx_3 \preceq \ldots \preceq fx_n \preceq fx_{n+1} \preceq \ldots.$$

Since for $x = x_n$ and $y = x_{n-1}$, $gx_{n-1} \preceq gx_n$, applying the condition (ii) of the theorem, we have

$$\begin{aligned}\psi(d(gx_{n+1},\ gx_n)) &= \psi(d(fx_n,\ fx_{n-1}))\\ &\precsim \psi(u(x_n,\ x_{n-1})) - \phi(d(gx_n,\ gx_{n-1})),\end{aligned}$$

where

$$\begin{aligned}u(x_n,\ x_{n-1}) &= \frac{d(fx_{n-1},\ gx_{n-1})d(fx_{n-1},gx_n)}{1 + d(gx_n,\ gx_{n-1})} + d(gx_n,\ gx_{n-1})\\ &= \frac{d(fx_{n-1},\ gx_{n-1})\ d(gx_n,\ gx_n)}{1 + d(gx_n,\ gx_{n-1})} + d(gx_n,\ gx_{n-1})\\ &= d(gx_n,\ gx_{n-1}).\end{aligned}$$

Then it follows that

$$\psi(d(gx_{n+1},\ gx_n)) \precsim \psi(d(gx_n,\ gx_{n-1})) - \phi(d(gx_n,\ gx_{n-1})). \tag{2}$$

Using a property of ϕ, we have

$$\psi(d(gx_{n+1},\ gx_n)) \precsim \psi(d(gx_n,\ gx_{n-1})), \text{ for all } n \geq 1,$$

which, by a property of ψ, implies that

$$d(gx_{n+1},\ gx_n) \precsim d(gx_n,\ gx_{n-1}), \text{ for all } n \geq 1.$$

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Therefore, $\{d(gx_{n+1},\ gx_n)\}$ is a monotone decreasing sequence. Hence by lemma 2.3, there exists an $r \in P$ with either $r = 0$ or $r \in Int\ (\mathscr{P})$ such that

$$d(gx_{n+1},\ gx_n) \longrightarrow r \text{ as } n \longrightarrow \infty. \tag{3}$$

Taking the limit as $n \to \infty$ in (2), using (3) and continuities of ϕ and ψ, we have

$$\psi(r) \precsim \psi(r) - \phi(r) \Longrightarrow \phi(r) \precsim 0,$$

which is a contradiction unless $r = 0$. Therefore,

$$d(gx_{n+1},\ gx_n) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{4}$$

Next we show that $\{gx_n\}$ is a Cauchy sequence. If $\{gx_n\}$ is not a Cauchy sequence, then there exists $c \in \mathscr{C}$ with $0 \prec c$, for all $n_0 \in \mathbf{N}$, $\exists\ n,\ m \in \mathbf{N}$ with $n > m \geq n_0$ such that

$$d(gx_m,\ gx_n) \not\prec \phi(c).$$

Hence by a property of ϕ in the theorem, $\phi(c) \precsim d(gx_m,\ gx_n)$. Therefore, there exist two sequences $\{m(k)\}$ and $\{n(k)\}$ in $\mathbf{N}$ such that for all positive integers k,

$$n(k) > m(k) > k \text{ and } d(gx_{m(k)},\ gx_{n(k)}) \succsim \phi(c).$$

Assuming that $n(k)$ is the smallest such positive integer, we get

$$d(gx_{n(k)},\ gx_{m(k)}) \succsim \phi(c) \text{ and } d(gx_{n(k)-1},\ gx_{m(k)}) \prec \phi(c).$$

Now,

$$\phi(c) \precsim d(gx_{n(k)},\ gx_{m(k)}) \precsim d(gx_{n(k)},\ gx_{n(k)-1}) + d(gx_{n(k)-1},\ gx_{m(k)}),$$

that is,

$$\phi(c) \precsim d(gx_{n(k)},\ gx_{m(k)}) \precsim d(gx_{n(k)},\ gx_{n(k)-1}) + \phi(c).$$

Letting $k \longrightarrow \infty$ in the above inequality and using (4), we have

$$\lim_{k\to\infty} d(gx_{n(k)},\ gx_{m(k)}) = \phi(c). \tag{5}$$

Again,

$$d(gx_{n(k)},\ gx_{m(k)}) \precsim d(gx_{n(k)},\ gx_{n(k)+1}) + d(gx_{n(k)+1},\ gx_{m(k)+1}) + d(gx_{m(k)+1},\ gx_{m(k)})$$

and

$$d(gx_{n(k)+1},\ gx_{m(k)+1}) \precsim d(gx_{n(k)+1},\ gx_{n(k)}) + d(gx_{n(k)},\ gx_{m(k)}) + d(gx_{m(k)},\ gx_{m(k)+1}).$$

Letting $k \longrightarrow \infty$ in above inequalities, using (4) and (5), we have

$$\lim_{k\to\infty} d(gx_{n(k)+1},\ gx_{m(k)+1}) = \phi(c). \tag{6}$$

Again,

$$d(gx_{n(k)},\ gx_{m(k)}) \precsim d(gx_{n(k)},\ gx_{m(k)+1}) + d(gx_{m(k)+1},\ gx_{m(k)})$$

and

$$d(gx_{n(k)},\ gx_{m(k)+1}) \precsim d(gx_{n(k)},\ gx_{m(k)}) + d(gx_{m(k)},\ gx_{m(k)+1}).$$

Letting $k \longrightarrow \infty$ in above inequalities, using (4) and (5), we have

$$\lim_{k\to\infty} d(gx_{n(k)},\ gx_{m(k)+1}) = \phi(c). \tag{7}$$

Since for $x = x_{n(k)}$ and $y = x_{m(k)}$, $gx_{m(k)} \preceq gx_{n(k)}$, applying the condition (ii) of the theorem, we have

$$\begin{aligned}\psi(d(gx_{n(k)+1},\ gx_{m(k)+1})) &= \psi(d(fx_{n(k)},\ fx_{m(k)})) \\ &\precsim \psi(u(x_{n(k)},\ x_{m(k)})) - \phi(d(gx_{n(k)},\ gx_{m(k)})),\end{aligned} \tag{8}$$

where

$$\begin{aligned}u(x_{n(k)},\ x_{m(k)}) &= \frac{d(fx_{m(k)},\ gx_{m(k)})d(fx_{m(k)},\ gx_{n(k)})}{1 + d(gx_{n(k)},\ gx_{m(k)})} \\ &\quad + d(gx_{n(k)},\ gx_{m(k)}) \\ &= \frac{d(gx_{m(k)+1},\ gx_{m(k)})d(gx_{m(k)+1},\ gx_{n(k)})}{1 + d(gx_{n(k)},\ gx_{m(k)})} \\ &\quad + d(gx_{n(k)},\ gx_{m(k)}).\end{aligned}$$

Now

$$\lim_{k\to\infty} u(x_{n(k)},\ x_{m(k)}) = \phi(c) \quad (\text{using } (4), (5) \text{ and } (7)). \tag{9}$$

Letting $k \longrightarrow \infty$ in (8), using (5), (6), (9) and the continuities of ϕ and ψ, we have

$$\psi(\phi(c)) \precsim \psi(\phi(c)) - \phi(\phi(c)) \Longrightarrow \phi(\phi(c)) \precsim 0,$$

which is a contradiction by virtue of a property of ϕ. Hence $\{gx_n\}$ is a Cauchy sequence. From the completeness of X there exists $z \in X$ such that

$$fx_n = gx_{n+1} \longrightarrow z \text{ as } n \longrightarrow \infty. \tag{10}$$

Since f and g are compatible, and $gx_n \longrightarrow z$, $fx_n \longrightarrow z$ as $n \longrightarrow \infty$, we have

$$\lim_{n\to\infty} d(fgx_n,\ gfx_n) = 0,$$

which, by the continuities of f and g, implies that $d(fz,\ gz) = 0$, that is, $fz = gz$, that is, $z \in X$ is a coincidence point of f and g.

In our next theorem we relax the continuity and compatibility assumption of the mappings f and g in Theorem 3.1 by considering $g(X)$ to be a closed subset of X and imposing the following order condition of the complex valued metric space X:

If $\{x_n\}$ is a non-decreasing sequence in X such that $x_n \longrightarrow x$, then $x_n \preceq x$, for all $n \in \mathbf{N}$.

Theorem 3.2. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a complex valued metric d on X such that $(X,\ d)$ is a complete complex valued metric space with $d(x,\ y) \in \text{Int}\ (\mathscr{P})$ for $x, y \in X$ with $x \neq y$.

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Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \longrightarrow x$, then $x_n \preceq x$, for all $n \in \mathbf{N}$. Let f and g be two self mappings on X such that $f(X) \subseteq g(X)$, f is g - nondecrasing with respect to $\preceq$ and $g(X)$ is closed in X. Suppose that the conditions (i) and (ii) of theorem 3.1 hold, where the conditions upon $(\phi,\ \psi)$ are the same as in theorem 3.1. If there exists $x_0 \in X$ such that $gx_0 \preceq fx_0$, then f and g have a coincidence point in X.

Proof. We take the same sequence $\{x_n\}$ as in the proof of theorem 3.1. Then we have

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq gx_3 \preceq \ldots \preceq gx_n \preceq gx_{n+1} \preceq \ldots,$$

and

$$fx_0 \preceq fx_1 \preceq fx_2 \preceq fx_3 \preceq \ldots \preceq fx_n \preceq fx_{n+1} \preceq \ldots.$$

Arguing similarly as in the proof of the theorem 3.1, we can prove that sequence $\{fx_n\}$, that is, $\{gx_{n+1}\}$ satisfies (10), that is, there exists $z \in X$ such that

$$fx_n = gx_{n+1} \longrightarrow z \ \text{ as } \ n \longrightarrow \infty.$$

Since $\{fx_n\}$, that is, $\{gx_{n+1}\}$ is a sequence in $g(X)$ and $g(X)$ is a closed subset of X, we have that $z \in g(X)$. So, there exists $w \in X$ such that $z = gw$. Now, by the condition of the theorem, $gx_n \preceq z = gw$, for all $n \in \mathbf{N}$. Applying the condition (ii) of the theorem 3.1 for $x = w, y = x_n$, we have

$$\psi(d(fw,\ fx_n)) \precsim \psi(u(w,\ x_n)) - \phi(d(gw,\ gx_n)), \qquad (11)$$

where

$$u(w,\ x_n) = \frac{d(fx_n,\ gx_n)d(fx_n,\ gw)}{1 + d(gw,\ gx_n)} + d(gw,\ gx_n).$$

Now

$$\lim_{n\to\infty} u(w,\ x_n) = d(gw,\ z) = d(z,\ z) = 0 \ \ (\text{using } (10)). \qquad (12)$$

Taking the limit as $n \longrightarrow \infty$ in (11), using (10), (12) and the properties of ϕ and ψ, we have

$$\psi(d(fw,\ z)) \precsim 0, \text{ that is, } \ \psi(d(fw,\ gw)) \precsim 0.$$

It follows by a property of ψ that $d(fw,\ gw) = 0$, that is, $fw = gw$, that is, w is a coincidence point of f and g.

Considering g to be the identity function in theorems 3.1 and 3.2, we have following corollaries.

corollary 3.3. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a complex valued metric d on X such that $(X,\ d)$ is a complete complex valued metric space with $d(x,\ y) \in \text{Int}\ (\mathscr{P})$ for $x,y \in X$ with $x \neq y$. Let $f : X \longrightarrow X$ be a continuous and nondecrasing mapping with respect to $\preceq$ such that for all $x,\ y \in X$ with $y \preceq x$,

$$\psi(d(fx,\ fy)) \precsim \psi(u(x,\ y)) - \phi(d(x,\ y)), \qquad (13)$$

where $u(x,y) = \dfrac{d(fy,\ y)d(fy,\ x)}{1 + d(x,\ y)} + d(x,\ \ y)$, and the conditions upon $(\phi,\ \psi)$ are the same as in theorem 3.1. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point in X.

corollary 3.4. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a complex valued metric d on X such that $(X,\ d)$ is a complete complex valued metric space with $d(x,\ y) \in \text{Int}\ (\mathscr{P})$ for $x,y \in X$ with $x \neq y$. Assume that if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \longrightarrow x$, then $x_n \preceq x$, for all $n \in \mathbf{N}$. Let $f : X \longrightarrow X$ be a nondecrasing mapping with respect to $\preceq$. Suppose that (13) holds, where the conditions upon $(\phi,\ \psi)$ are the same as in theorem 3.1. If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$, then f has a fixed point in X.

Theorem 3.5. In addition to the hypotheses of Corollary 3.3 and Corollary 3.4, in both of the corollaries, suppose that for every $x,\ y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$. Then f has a unique fixed point.

Proof. It follows from the corollary 3.3 or corollary 3.4, the set of fixed points of f is non-empty. If possible, let $x,\ y \in X$ $(x \neq y)$ be two fixed points of f, that is, $x = fx$ and $y = fy$. We distinguish two cases:

Case 1.

If $y \preceq x$, then by the condition (13), we have for all $n \geq 1$,

$$\psi(d(x,\ y)) = \psi(d(fx,\ fy)) \precsim \psi(u(x,\ y)) - \phi(d(x,\ y)),$$

where

$$\begin{aligned} u(x,\ y) &= \frac{d(fy,\ y)d(fy,\ x)}{1 + d(x,\ y)} + d(x,\ y) \\ &= \frac{d(y,\ y)d(y,\ x)}{1 + d(x,\ y)} + d(x,\ y) \\ &= d(x,\ y). \end{aligned}$$

Then it follows that

$$\psi(d(x,\ y)) \precsim \psi(d(x,\ y)) - \phi(d(x,\ y)) \Longrightarrow \phi(d(x,\ y)) \precsim 0,$$

which is a contradiction by a property of ϕ, unless $d(x,\ y) = 0$, that is, $x = y$.

Case 2.

If $y \npreceq x$, then there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$. Monotonicity of f implies that $f^n x = x \preceq f^n z$ and $f^n y = y \preceq f^n z$, for $n = 0,\ 1,\ 2, \ldots$

By the condition ((13), we have for all $n \geq 1$,

$$\begin{aligned} \psi(d(f^n z,\ x)) &= \psi(d(f^n z,\ f^n x)) \\ &\precsim \psi(u(f^{n-1} z,\ f^{n-1} x)) - \phi(d(f^{n-1} z,\ f^{n-1} x)) \\ &\precsim \psi(u(f^{n-1} z,\ x)) - \phi(d(f^{n-1} z,\ x)), \end{aligned}$$

where

$$u(f^{n-1}z,\ x) = \frac{d(fx,\ x)d(fx,\ f^{n-1}z)}{1 + d(f^{n-1}z,\ f^{n-1}x)} + d(f^{n-1}z,\ fx) = d(f^{n-1}z,\ x).$$

Hence it follows from the above inequality

$$\psi(d(f^n z,\ x)) \precsim \ \psi(d(f^{n-1} z,\ x)) - \phi(d(f^{n-1} z,\ x)). \qquad (14)$$

Using a property of ϕ, we have

$$\psi(d(f^n z,\ x)) \precsim \psi(d(f^{n-1} z,\ x)),$$

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which, by monotone property of ψ, implies that

$$d(f^n z,\, x) \precsim d(f^{n-1} z,\, x).$$

Therefore, $\{d(f^n z, \;\; x)\}$ is a monotone decreasing sequence. Following the lemma 2.3, it can be proved that there exists $q \in Int\,(\mathscr{P}) \cup \{0\}$ such that

$$\lim_{n\to\infty} d(f^n z,\, x) = q. \tag{15}$$

Letting $n \longrightarrow \infty$ in (14), using (15) and the continuities of ϕ and ψ, we have

$$\psi(q) \precsim \psi(q) - \phi(q) \Longrightarrow \phi(q) \precsim 0,$$

which is a contradiction unless $q = 0$.
Hence

$$\lim_{n\to\infty} d(f^n z,\, x) = 0.$$

Similarly, it can proved that

$$\lim_{n\to\infty} d(f^n z,\, y) = 0.$$

Finally, the uniqueness of the limit gives us $x = y$.

From above two cases we have that fixed point of f is unique.

Example 3.6. Let $X = [0,\, 1]$ with usual partial order $'\leq'$. Let $d : X \times X \longrightarrow C$ be given as

$$d(x,\, y) = |x-y|\,\sqrt{2}\,e^{i\frac{\pi}{4}} = |x-y|(1+i), \text{ for } x,\, y \in X.$$

Then $(X,\, d)$ is a complex valued metric space with the required properties of theorem 3.1 and theorem 3.2.
Let $\psi,\ \phi : \text{Int}\ (\mathscr{P}) \cup \{0\} \longrightarrow \text{Int}\ (\mathscr{P}) \cup \{0\}$ be defined respectively as follows:

for $z = x + i\,y \in \text{Int}\,(\mathscr{P}) \cup \{0\}$,

$$\psi(z) = \begin{cases} 0, \text{ if } \quad x = 0 \text{ and } \quad y = 0, \\ x + i\,y, \text{ if } \quad 0 < x \leq 1 \text{ and } \quad 0 < y \leq 1, \\ x^2 + i\,y, \text{ if } \quad x > 1 \text{ and } \quad 0 < y \leq 1, \\ x + i\,y^2, \text{ if } \quad 0 < x \leq 1 \text{ and } \quad y > 1, \\ x^2 + i\,y^2, \text{ if } \quad x > 1 \text{ and } \quad y > 1, \end{cases}$$

and

$$\phi(z) = \frac{v}{2} + i\,\frac{v}{2}, \text{ where } \quad v = \min\,\{x,\, y\}.$$

Then ψ and ϕ have the properties mentioned in theorem 3.1 and theorem 3.2.
Let $f,\, g : X \longrightarrow X$ be defined respectively as follows:

$$fx = \frac{x}{32} \text{ and } g(x) = \frac{x}{4}, \text{ for } x \in X.$$

Then f and g have the required properties mentioned in theorem 3.1 and theorem 3.2.
It can be verified that for all $x,\ y \in X$ with $gy \preceq gx$, condition (ii) of Theorem 3.1 and Theorem 3.2 are satisfied. Hence the conditions of theorem 3.1 and theorem 3.2 are satisfied and it is seen that 0 is a coincidence points of f and g.

*Remark.*Complex valued metric spaces have close similarities with cone metric spaces in its structure, although conceptually they are very different. In cone metric spaces the metric takes up values in linear spaces over the real field where the linear space may be infinite dimensional, whereas in the case of complex valued metric spaces the metric values are in the set of complex number which is a one dimensional vector space over the complex field. The type of rational inequality we consider here is not meaningful in a cone metric space. This is an instance which implies why fixed point theory should be pursued independently in a complex valued metric space.

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