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A Matrix Inequality Concerning Weakly Connected and Balanced Digraphs

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Abstract: Based on spectral properties of Laplacian matrix, we present a new matrix inequality concerning weakly connected and balanced digraphs.

Keywords: Laplacian matrix, balanced digraph, positive definite

1 Introduction

Let $G=(V(G),E(G),A(G))$ denote a weighted digraph (directed graph) of order n with the set of vertices $V(G)=\{1,2,\cdots,n\}$, edges $E(G)\subseteq V(G)\times V(G)$, and the $n\times n$ weighted adjacency matrix $A(G)=(a_{ij})$. A directed edge from j to i exists if and only if $a_{ij}>0$. We assume that $a_{ii}=0$ for all $i\in V(G)$. The graph Laplacian (or Laplacian matrix) $L(G)=(l_{ij})$ induced by the digraph G is defined by (see e.g. [1])

$$l_{ij}=\begin{cases}-a_{ij}, & i\neq j,\\ \sum_{k=1}^{n}a_{ik}, & i=j.\end{cases} \tag{1}$$

A digraph G is called balanced [2] if $\sum_{j=1}^{n}a_{ij}=\sum_{j=1}^{n}a_{ji}$ for all $i\in V(G)$. In other words, a digraph is balanced if and only if the total weight of edges entering a vertex and leaving the same vertex are equal for all vertices. By definition, any undirected graph is balanced. An important property of balanced digraphs is that $\mathbf{1}=(1,\cdots,1)^T\in\mathbb{R}^n$ is a left eigenvector of the Laplacian, i.e., $\mathbf{1}^TL(G)=0$.

Recall that a digraph is strongly connected if, between every pair of distinct vertices, there is a directed path. On the other hand, a digraph is called weakly connected if it is connected when viewed as a graph (replacing directed edges by undirected ones). An interesting result is that a balanced digraph is weakly connected if and only if it is strongly connected [3]. Moreover, weakly connected and balanced digraphs play an important role in the consensus coordination of multi-agent systems. It is shown that ([2] or [4, Theorem 3.17].) the agreement protocol over a digraph reaches the average consensus for every initial condition if and only if it is weakly connected and balanced.

The goal of this paper is to present a matrix inequality concerning weakly connected and balanced digraphs by using spectral properties of Laplacian matrix. It is hoped that the result may find potential applications in multi-agent coordination (see the concluding remarks in Section 2).

2 The matrix inequality

We begin this section with some notations and definitions. A nonnegative matrix $A=(a_{ij})$ with all entries on the main diagonal equal to zero can be associated naturally with a digraph $G=(V,E,A)$ in such a way that $(j,i)\in E$ if and only if $a_{ij}>0$. Consider two symmetric matrices X and Y of the same dimension, we say $X>Y$ if $X-Y$ is positive definite. For $X\in\mathbb{R}^{n\times m}$, X can be viewed as a linear map $X:\mathbb{R}^m\to\mathbb{R}^n$ with kernel defined by $\mathrm{Ker}X=\{x\in\mathbb{R}^m: Xx=0\}$.

For an undirected graph G, $L(G)$ is a symmetric matrix with real eigenvalues and, hence, the set of eigenvalues of $L(G)$ can be ordered sequentially in an ascending order as

$$0=\lambda_1(L(G))\le\lambda_2(L(G))\le\cdots\le\lambda_n(L(G)). \tag{2}$$

G is connected if and only if $\lambda_2(L(G))>0$ [1]. For a digraph G, the following lemma is shown in [2].

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Lemma 2.1. ([2]) Assume that G is a strongly connected digraph. Then all eigenvalues but one simple eigenvalue at zero of $L(G)$ have positive real-parts.

Theorem 2.1. Assume that G_1 and G_2 are two digraphs of order n. If the digraph associated with $A(G_1)-A(G_2)$ is weakly connected and balanced, for any matrix $F\in\mathbb{R}^{n\times m}$ satisfying $KerF=0$ and $\mathbf{1}^TF=0$,

$$F^T(L(G_1)+L(G_1)^T)F>F^T(L(G_2)+L(G_2)^T)F. \quad (3)$$

Proof. Let G be the digraph associated with $A(G_1)-A(G_2)$. Thus, G is weakly connected and balanced, and $L(G)=L(G_1)-L(G_2)$. It suffices to show that

$$F^T(L(G)+L(G)^T)F>0. \quad (4)$$

According to the aforementioned comment, we obtain $\mathbf{1}^TL(G)=0$. Since $L(G)\mathbf{1}=0$, it follows that $\mathbf{1}^T(L(G)+L(G)^T)=(L(G)+L(G)^T)\mathbf{1}=0$. Hence, the digraph $\hat{G}$ with the Laplacian matrix $L(G)+L(G)^T$ is also balanced. On the other hand, it is clear that $\hat{G}$ is weakly connected (and automatically strongly connected, by our above comment).

Lemma 2.1 then implies that $\lambda_2(L(G)+L(G)^T)>0$, where

$$\begin{aligned}0=&\lambda_1(L(G)+L(G)^T)<\lambda_2(L(G)+L(G)^T)\\ \le&\cdots\le\lambda_n(L(G)+L(G)^T)\end{aligned} \quad (5)$$

are the eigenvalues of $L(G)+L(G)^T$. By the Courant-Fischer theorem [1], we obtain

$$x^T(L(G)+L(G)^T)x\ge\lambda_2(L(G)+L(G)^T)x^Tx, \quad (6)$$

for $x\in\mathbb{R}^n$ satisfying $\mathbf{1}^Tx=0$. For any $y\in\mathbb{R}^m$ and $y\neq 0$, we know that $\mathbf{1}^T(Fy)=0$ by the assumption $\mathbf{1}^TF=0$. Therefore, we obtain

$$\begin{aligned}&y^TF^T(L(G)+L(G)^T)Fy\\ =&(Fy)^T(L(G)+L(G)^T)(Fy)\\ \ge&\lambda_2(L(G)+L(G)^T)(Fy)^T(Fy)\\ >&0,\end{aligned} \quad (7)$$

where the second inequality follows from (6), and the last one follows from (5) and the assumption $\mathrm{Ker}F=0$. This implies (4), and the proof of Theorem 2.1 is complete. □

We give some remarks here.

Remark 2.1. If we take G_2 as an empty graph, i.e., $A(G_2)=0$, we have the following corollary: Assume that G_1 of order n is weakly connected and balanced, then we have

$$F^T(L(G_1)+L(G_1)^T)F>0 \quad (8)$$

for any matrix $F\in\mathbb{R}^{n\times m}$ satisfying $\mathrm{Ker}F=0$ and $\mathbf{1}^TF=0$.

Remark 2.2. The digraph $\hat{G}$ with the Laplacian $L(G)+L(G)^T$ is essentially undirected with the new weights given by $\hat{a}_{ij}=\hat{a}_{ji}=a_{ij}+a_{ji}$. $\hat{G}$ is also known as disoriented digraph [4], which often appears in multi-agent coordination (see e.g. [5,6,7]).

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