Applied Mathematics & Information Sciences **4**(3) (2010), 395–412
– An International Journal

Investigations of the Essential Spectrum of a Hamiltonian in Fock Space

Tulkin H. Rasulov

Samarkand State University, Samarkand, Uzbekistan.
E-mail Address: rth@mail.ru

Received November 13, 2008; Revised April 20, 2009

In the present paper, we precisely describe the location and structure of the essential spectrum of a Hamiltonian (model operator) H associated to a system describing four particles in interaction, without conservation of the number of particles, in the quasi-momentum representation. It is also established that the essential spectrum of H consists of no more than seven bounded closed intervals.

Keywords: Model operator, Fock space, conservation of number of particles, channel operator, Birman-Schwinger principle, eigenvalue, essential spectrum.

2010 Mathematics Subject Classification: 81Q10, 35P20, 47N50.

1 Introduction

This paper is a continuation of [19], where the model operator H associated to a system describing four particles in interaction, without conservation of the number of particles, acting in the *four-particle cut subspace* of Fock space, was considered and its essential spectrum was described by the spectrum of channel operators. Here an analogue of the Hunziker-van Winter-Zhislin (HWZ) theorem for the operator H was proven and a connection between the spectrum of H and a variational approach to find boundaries of essential spectrum and some interior eigenvalues was given. In the present paper we prove that the essential spectrum of this operator consists of no more than seven bounded closed intervals and we study the location of these intervals.

The location of the essential spectrum of N-body Schrödinger operators for particles moving in $\mathbf{R}^3$ has been extensively studied in many works, see for example [8,20,23]. The Hamiltonians of systems of three quantum particles moving on the three dimensional lattice $\mathbf{Z}^3$ were considered in [1, 2, 9, 10, 14] and the essential spectrum has been investigated. In particular, in [2] it is shown that, the essential spectrum of the three-particle discrete Schrödinger operator, consists of no more than four bounded closed intervals and the main result of [1] is that the essential spectrum of the three-particle discrete Schrödinger operator consists of only finitely many bounded closed intervals, although the corresponding two-particle operators might posses infinitely many eigenvalues for some value of the two-particle quasi-momentum. The essential spectrum of discrete Schrödinger operators on lattice $\mathbf{Z}^N$ by means of the limit operators method was studied in [15].

In quantum field theory, condensed matter physics and the theory of chemical reactions, naturally occur in quantum systems, where the particle number is finite, but not conserved. The study of these systems is reduced to the study of spectral properties of self-adjoint operators, acting in the *cut subspace* $\mathcal{H}^{(n)}$ of Fock space, consisting of $r \leq n$ particles [13, 21, 24]. We note that the location and structure of the model operators acting in $\mathcal{H}^{(3)}$ are studied in detail in [4, 5, 11, 12, 17, 18, 22].

The paper is organized as follows. In Section 2 the model operator H is described as a bounded self-adjoint operator in $\mathcal{H}^{(4)}$. In Section 3 the main results are formulated (Theorems 3.1-3.3) and for completeness, we here reproduce some useful arguments, which have been proven in [19]. In Section 4 we study some spectral properties of the corresponding families of the operators. Section 5 is devoted to the proof of the main results.

We recall that for the three-particle continuous Schrödinger operators the three-particle continuum of the essential spectrum coincides with the semi-axis $[0; +\infty)$. Two-particle branches fill the interval $[\kappa; +\infty)$, where $\kappa \leq 0$ is the lowest eigenvalue of the two-particle subhamiltonians. Thus, there are no gaps in the essential spectrum. In lattice case the "two-particle" and "three-particle" branches of essential spectrum fill finite-length segments and might overlap. Theorems 3.2 and 3.3 show that under some natural conditions there exist gaps of the essential spectrum of H.

Throughout this paper we adopt the following convention: Denote by $\mathbf{T}^3$ the three-dimensional torus, the cube $(-\pi, \pi]^3$ with appropriately identified sides. The torus $\mathbf{T}^3$ will always be considered as an abelian group with respect to the addition and multiplication by real numbers regarded as operations on the three-dimensional space $\mathbf{R}^3$ modulo $(2\pi\mathbf{Z})^3$.

For each sufficiently small $\delta > 0$ the notation $U_\delta(p_0) = \{p \in \mathbf{T}^3 : |p - p_0| < \delta\}$ stands for a $\delta > 0$ neighborhood of the point $p_0 \in \mathbf{T}^3$.

2 The Model Operator

2.1 The model operator in quasi-momentum representation

Let us introduce some notations used in this work. Let $\mathbf{C}$ be the field of complex numbers and $L_2((\mathbf{T}^3)^n)$, $n = 1, 2, 3$ be the Hilbert space of square-integrable (complex) functions defined on $(\mathbf{T}^3)^n$, $n = 1, 2, 3$.

Denote

$$\mathcal{H}_0 = \mathbf{C},\ \mathcal{H}_1 = L_2(\mathbf{T}^3),\ \mathcal{H}_2 = L_2((\mathbf{T}^3)^2),\ \mathcal{H}_3 = L_2((\mathbf{T}^3)^3),$$

$$\mathcal{H}^{(n,m)} = \bigoplus_{i=n}^{m} \mathcal{H}_i,\ 0 \le n < m \le 3.$$

The space $\mathcal{H}^{(4)} \equiv \mathcal{H}^{(0,3)}$ is called the *four-particle cut subspace* of Fock space.

Let H_{ij} be annihilation (creation) operators [6] defined in the Fock space for $i < j$ $(i > j)$. In this paper we consider the case, where the number of annihilations and creations of the particles of the considering system equal to 1. It means that $H_{ij} \equiv 0$ for all $|i-j| > 1$. So, a model operator H associated to a system describing four particles in interaction, without conservation of the number of particles, acts in the Hilbert space $\mathcal{H}^{(0,3)}$ as a matrix operator

$$H = \begin{pmatrix} H_{00} & H_{01} & 0 & 0 \\ H_{10} & H_{11} & H_{12} & 0 \\ 0 & H_{21} & H_{22} & H_{23} \\ 0 & 0 & H_{32} & H_{33} \end{pmatrix},$$

where its components $H_{ij} : \mathcal{H}_j \to \mathcal{H}_i,\ \ i,j = 0,1,2,3$ are defined by the rule

$$(H_{00}f_0)_0 = w_0 f_0,\ (H_{01}f_1)_0 = \int_{\mathbf{T}^3} v_1(s) f_1(s) ds,\ (H_{10}f_0)_1(p) = v_1(p) f_0,$$

$$(H_{11}f_1)_1(p) = w_1(p) f_1(p), \quad (H_{12}f_2)_1(p) = \int_{\mathbf{T}^3} v_2(s) f_2(p,s) ds,$$

$$(H_{21}f_1)_2(p,q) = v_2(q) f_1(p),\ H_{22} = H_{22}^0 - V_{21} - V_{22},$$

$$(H_{22}^0 f_2)_2(p,q) = w_2(p,q) f_2(p,q),\ (V_{21}f_2)_2(p,q) = v_{21}(p) \int_{\mathbf{T}^3} v_{21}(s) f_2(s,q) ds,$$

$$(V_{22}f_2)_2(p,q) = v_{22}(q) \int_{\mathbf{T}^3} v_{22}(s) f_2(p,s) ds,\ (H_{23}f_3)_2(p,q) = \int_{\mathbf{T}^3} v_3(s) f_3(p,q,s) ds,$$

$$(H_{32}f_2)_3(p,q,t) = v_3(t) f_2(p,q),\ (H_{33}f_3)_3(p,q,t) = w_3(p,q,t) f_3(p,q,t).$$

Here $f_i \in \mathcal{H}_i,\ i = \overline{0,3},\ w_0$ is a real number, $v_i(\cdot),\ i = 1,2,3,\ v_{2j}(\cdot),\ j = 1,2,\ w_1(\cdot)$ are real-analytic (nonzero) functions on $\mathbf{T}^3$ and $w_2(\cdot,\cdot)$ resp. $w_3(\cdot,\cdot,\cdot)$ is a real-analytic (nonzero) function on $(\mathbf{T}^3)^2$ resp. $(\mathbf{T}^3)^3$.

Under these assumptions the operator H is bounded and self-adjoint in $\mathcal{H}^{(0,3)}$.

2.2 The channel operators and direct integral decompositions

Let us introduce the channel operators $H_n,\ n = 1,3$ resp. H_2 acting in $\mathcal{H}^{(2,3)}$ resp. $\mathcal{H}^{(1,3)}$ by the following rule

$$H_1 = \begin{pmatrix} H_{22}^0 - V_{21} & H_{23} \\ H_{32} & H_{33} \end{pmatrix}, \qquad H_2 = \begin{pmatrix} H_{11} & H_{12} & 0 \\ H_{21} & H_{22}^0 - V_{22} & H_{23} \\ 0 & H_{32} & H_{33} \end{pmatrix},$$

$$H_3 = \begin{pmatrix} H_{22}^0 & H_{23} \\ H_{32} & H_{33} \end{pmatrix}.$$

First we consider the channel operator H_3, which commutes with any multiplication operator $U_\alpha^{(3)}$ by the bounded function $\alpha(\cdot,\cdot)$ on $(\mathbf{T}^3)^2$

$$U_\alpha^{(3)} \begin{pmatrix} g_2(p,q) \\ g_3(p,q,t) \end{pmatrix} = \begin{pmatrix} \alpha(p,q)g_2(p,q) \\ \alpha(p,q)g_3(p,q,t) \end{pmatrix}, \begin{pmatrix} g_2 \\ g_3 \end{pmatrix} \in \mathcal{H}^{(2,3)}.$$

Therefore the decomposition [20] of the space $\mathcal{H}^{(2,3)}$ into the direct integral

$$\mathcal{H}^{(0,1)} = \int_{(\mathbf{T}^3)^2} \oplus \mathcal{H}^{(2,3)} dpdq$$

yields the decomposition into the direct integral

$$H_3 = \int_{(\mathbf{T}^3)^2} \oplus h_3(p,q)dpdq, \tag{2.1}$$

where a family of the generalized Friedrichs models $h_3(p,q),\ p,q \in \mathbf{T}^3$ acts in $\mathcal{H}^{(0,1)}$ as

$$h_3(p,q) = \begin{pmatrix} h_{00}^{(3)}(p,q) & h_{01}^{(3)} \\ h_{10}^{(3)} & h_{11}^{(3)}(p,q) \end{pmatrix}.$$

Here

$$(h_{00}^{(3)}(p,q)f_0)_0 = w_2(p,q)f_0, \quad (h_{01}^{(3)}f_1)_0 = \int_{\mathbf{T}^3} v_3(s)f_1(s)ds,$$
$$(h_{10}^{(3)}f_0)_1(t) = v_3(t)f_0, \quad (h_{11}^{(3)}(p,q)f_1)_1(t) = w_3(p,q,t)f_1(t).$$

In analogy with the operator H_3 one can give the decomposition

$$H_n = \int_{\mathbf{T}^3} \oplus h_n(p)dp,\ n = 1,2, \tag{2.2}$$

where a family of the operators $h_1(p),\ p \in \mathbf{T}^3$ resp. $h_2(p),\ p \in \mathbf{T}^3$ acts in $\mathcal{H}^{(1,2)}$ resp. $\mathcal{H}^{(0,2)}$ as

$$h_1(p) = \begin{pmatrix} h_{11}^{(1)}(p) & h_{12}^{(1)} \\ h_{21}^{(1)} & h_{22}^{(1)}(p) \end{pmatrix} \quad \text{resp.} \quad h_2(p) = \begin{pmatrix} h_{00}^{(2)}(p) & h_{01}^{(2)} & 0 \\ h_{10}^{(2)} & h_{11}^{(2)}(p) & h_{12}^{(1)} \\ 0 & h_{21}^{(1)} & h_{22}^{(1)}(p) \end{pmatrix}$$

with the entries

$$(h_{11}^{(1)}(p)f_1)_1(q) = w_2(p,q)f_1(q) - v_{21}(q)\int_{\mathbf{T}^3} v_{21}(s)f_1(s)ds,$$
$$(h_{12}^{(1)}f_2)_1(q) = \int_{\mathbf{T}^3} v_3(s)f_2(q,s)ds,\ (h_{22}^{(1)}(p)f_2)_2(q,t) = w_3(p,q,t)f_2(q,t),$$

$$(h_{21}^{(1)}f_1)_2(q,t)=v_3(t)f_1(q),\ (h_{00}^{(2)}(p)f_0)_0=w_1(p)f_0,\ (h_{01}^{(2)}f_1)_0=\int_{\mathbf{T}^3}v_2(s)f_1(s)ds,$$

$$(h_{10}^{(2)}f_0)_1(q)=v_2(q)f_0,\ (h_{11}^{(2)}(p)f_1)_1(q)=w_2(p,q)f_1(q)-v_{22}(q)\int_{\mathbf{T}^3}v_{22}(s)f_1(s)ds.$$

Let us introduce the notations

$$m=\min_{p,q,t\in\mathbf{T}^3}w_3(p,q,t),\quad M=\max_{p,q,t\in\mathbf{T}^3}w_3(p,q,t),$$

$$\sigma_{two}(H_n)=\bigcup_{p\in\mathbf{T}^3}\sigma_{disc}(h_n(p)),\ n=1,2,$$

$$\sigma_{three}(H_n)=\bigcup_{p,q\in\mathbf{T}^3}\sigma_{disc}(h_3(p,q)),\quad \sigma_{four}(H_n)=[m;M],\ n=1,2,3.$$

The following theorem describes the essential spectrum of H (see [19]).

Theorem 2.1. *For the essential spectrum $\sigma_{ess}(H)$ of H the following equality*

$$\sigma_{ess}(H)=\sigma_{two}(H)\cup\sigma_{three}(H)\cup\sigma_{four}(H)$$

holds, where $\sigma_{two}(H)=\sigma_{two}(H_1)\cup\sigma_{two}(H_2)$, $\sigma_{three}(H)=\sigma_{three}(H_3)$ and $\sigma_{four}(H)=\sigma_{four}(H_3)$.

The sets $\sigma_{two}(H)$, $\sigma_{three}(H)$ and $\sigma_{four}(H)$ are called two-particle, three-particle and four-particle branches of the essential spectrum of H, respectively.

2.3 Main assumptions

Throughout this paper we assume that the function $w_3(\cdot,\cdot,\cdot)$ has a unique non-degenerate minimum (resp. maximum) at the point $(p_0,q_0,t_0)\in(\mathbf{T}^3)^3$ (resp. $(p_1,q_1,t_1)\in(\mathbf{T}^3)^3$) and for simplicity we also assume that for any $p\in\mathbf{T}^3$ the operator $h_2(p)$ has no eigenvalues lying in the intervals $(-\infty;m)$, $(M;+\infty)$.

Note that if for any $p\in\mathbf{T}^3$ the operator $h_2(p)$ has no eigenvalues lying in the intervals $(-\infty;m)$ and $(M;+\infty)$, then $\sigma_{two}(H_2)\subset[m;M]$ (see Lemma 4.6).

For any fixed $p,q\in\mathbf{T}^3$ we define an analytic function $\Delta_3(p,q\,;\cdot)$ resp. $\Delta_1(p\,;\cdot)$ in $\mathbf{C}\setminus\sigma_{ess}(h_3(p,q))$ resp. $\mathbf{C}\setminus\sigma_{ess}(h_1(p))$ by

$$\Delta_3(p,q\,;z)=w_2(p,q)-z-\int_{\mathbf{T}^3}\frac{v_3^2(s)ds}{w_3(p,q,s)-z}$$

resp.

$$\Delta_1(p\,;z)=1-\int_{\mathbf{T}^3}\frac{v_{21}^2(s)ds}{\Delta_3(p,s\,;z)}$$

(the Fredholm determinant associated with the operator $h_3(p,q),p,q\in\mathbf{T}^3$ resp. $h_1(p),p\in\mathbf{T}^3$).

Since for any fixed $p, q \in \mathbf{T}^3$ the function $\Delta_3(p, q\,; \cdot)$ is decreasing in the intervals $(-\infty; m)$, $(M; +\infty)$ and the function has a unique non-degenerate minimum (resp. maximum) at the point $(p_0, q_0, t_0) \in (\mathbf{T}^3)^3$ (resp. $(p_1, q_1, t_1) \in (\mathbf{T}^3)^3$) by the dominated convergence theorem for any fixed $p, q \in \mathbf{T}^3$ there exist the following finite limits

$$\lim_{z\to m-0} \Delta_3(p, q\,; z) = \Delta_3(p, q\,; m) \quad \text{and} \quad \lim_{z\to M+0} \Delta_3(p, q\,; z) = \Delta_3(p, q\,; M).$$

Assumption 2.1. There exist positive numbers $\delta_1,\, \delta_2 > 0$ and $C_1,\, C_2 > 0$ such that for all $(p, q) \in U_{\delta_1}(p_0) \times U_{\delta_1}(q_0)$ resp. $(p, q) \in U_{\delta_2}(p_1) \times U_{\delta_2}(q_1)$ the following inequality

$$|\Delta_3(p, q\,; m)| \geq C_1(|p - p_0|^\alpha + |q - q_0|^\alpha)$$

resp.

$$|\Delta_3(p, q\,; M)| \geq C_2(|p - p_1|^\beta + |q - q_1|^\beta)$$

holds for some $0 \leq \alpha,\, \beta \leq 2$.

Remark 2.1. The class of functions $\Delta_3(\cdot, \cdot\,; m)$ and $\Delta_3(\cdot, \cdot\,; M)$ satisfying the conditions of Assumption 2.1 is nonempty (see Lemma 4.12).

Analogously if Assumption 2.1 is fulfilled, then for any fixed $p \in \mathbf{T}^3$ there exists the following finite limit $\lim_{z\to m-0} \Delta_1(p\,; z) = \Delta_1(p\,; m)$. Therefore the functions $\Delta_3(\cdot, \cdot\,; m)$, $\Delta_3(\cdot, \cdot\,; M)$ and $\Delta_1(\cdot\,; m)$ are continuous on $(\mathbf{T}^3)^2$ and $\mathbf{T}^3$, respectively.

3 Statement of the Main Results

In this section we formulate main results of the paper.

Theorem 3.1. *The essential spectrum of the operator H consists of no more than seven bounded closed intervals.*

Let us introduce the following notations:

$$\begin{aligned} &a_1 = \min \sigma_{two}(H), && b_1 = \max \sigma_{two}(H), \\ &a_2 = \min \sigma_{three}(H) \cap (-\infty; m], && b_2 = \max \sigma_{three}(H) \cap (-\infty; m], \\ &a_3 = \min \sigma_{three}(H) \cap [M; +\infty), && b_3 = \max \sigma_{three}(H) \cap [M; +\infty). \end{aligned}$$

The location and structure of the essential spectrum of H can be precisely described in the following theorems:

Theorem 3.2. *Let Assumption 2.1 be fulfilled and* $\min_{p\in\mathbf{T}^3} \Delta_1(p\,; m) \geq 0$.

I. *Assume that* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p, q\,; M) \leq 0$.

(1.1) *If* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p, q\,; m) \geq 0$, *then* $\sigma_{ess}(H) = [m; M]$.

(1.2) *If* $\min_{p,q\in\mathbf{T}^3} \Delta_2(p,q\,;m) < 0$ *and* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) \geq 0$, *then* $\sigma_{ess}(H) = [a_2; M]$ *with* $a_2 < m$.

(1.3) *If* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) < 0$, *then* $\sigma_{ess}(H) = [a_2; b_2] \cup [m; M]$ *with* $b_2 < m$.

II. *Assume that* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;M) \leq 0$ *and* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;M) > 0$.

(2.1) *If* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) \geq 0$, *then* $\sigma_{ess}(H) = [m; b_3]$ *with* $b_3 > M$.

(2.2) *If* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) < 0$ *and* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) \geq 0$, *then* $\sigma_{ess}(H) = [a_2; b_3]$ *with* $a_2 < m$ *and* $b_3 > M$.

(2.3) *If* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) < 0$, *then* $\sigma_{ess}(H) = [a_2; b_2] \cup [m; b_3]$ *with* $b_2 < m$ *and* $b_3 > M$.

III. *Assume that* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;M) > 0$.

(3.1) *If* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) \geq 0$, *then* $\sigma_{ess}(H) = [m; M] \cup [a_3; b_3]$ *with* $a_3 > M$.

(3.2) *If* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) < 0$ *and* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) \geq 0$, *then* $\sigma_{ess}(H) = [a_2; M] \cup [a_3; b_3]$ *with* $a_2 < m$ *and* $a_3 > M$.

(3.3) *If* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) < 0$, *then* $\sigma_{ess}(H) = [a_2; b_2] \cup [m; M] \cup [a_3; b_3]$ *with* $b_2 < m$ *and* $a_3 > M$.

Theorem 3.3. *Let Assumption 2.1 be fulfilled and* $\min_{p\in\mathbf{T}^3} \Delta_1(p\,;m) < 0$, $\max_{p\in\mathbf{T}^3} \Delta_1(p\,;m) \geq 0$.

I. *Assume that* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;M) \leq 0$.

(1.1) *If* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) \geq 0$, *then* $\sigma_{ess}(H) = [a_1; M]$ *with* $a_1 < m$.

(1.2) *If* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) < 0$ *and* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) \geq 0$, *then* $\sigma_{ess}(H) = [a; M]$ *with* $a = \min\{a_1, a_2\} < m$.

(1.3) *If* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) < 0$, *then* $\sigma_{ess}(H) = [a_2; b_2] \cup [a_1; M]$ *with* $b_2 < m$ *and* $a_1 < m$.

II. *Assume that* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;M) \leq 0$ *and* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;M) > 0$.

(2.1) *If* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) \geq 0$, *then* $\sigma_{ess}(H) = [a_1; b_3]$ *with* $a_1 < m$ *and* $b_3 > M$.

(2.2) *If* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) < 0$ *and* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) \geq 0$, *then* $\sigma_{ess}(H) = [a; b_3]$ *with* $a = \min\{a_1, a_2\} < m$ *and* $b_3 > M$.

(2.3) *If* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) < 0$, *then* $\sigma_{ess}(H) = [a_2; b_2] \cup [a_1; b_3]$ *with* $b_2 < m$, $a_1 < m$ *and* $b_3 > M$.

III. *Assume that* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;M) > 0$.

(3.1) *If* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) \geq 0$, *then* $\sigma_{ess}(H) = [a_1; M] \cup [a_3; b_3]$ *with* $a_1 < m$ *and* $a_3 > M$.

(3.2) *If* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) < 0$ *and* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) \geq 0$, *then* $\sigma_{ess}(H) = [a; M] \cup [a_3; b_3]$ *with* $a = \min\{a_1, a_2\} < m$ *and* $a_3 > M$.

(3.3) *If* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) < 0$, *then* $\sigma_{ess}(H) = [a_2; b_2] \cup [a_1; M] \cup [a_3; b_3]$ *with* $b_2 < m$, $a_1 < m$ *and* $a_3 > M$.

Remark 3.1. We recall that if Assumption 2.1 is fulfilled and the following inequality $\max_{p\in \mathbf{T}^3} \Delta_1(p\,;m) < 0$ holds, then we can formulate theorem analogously to Theorems 3.2 and 3.3, which obtains from Theorem 3.3 if we replace $[a_1; M]$ by $[a_1; b_1] \cup [m; M]$ with $b_1 < m$.

Remark 3.2. We also remark that Theorems 3.2 and 3.3 play a crucial role in the proof of the existence of finitely many or infinitely many eigenvalues lying in the gaps of the essential spectrum of H (see for example [16], where the finiteness of the discrete spectrum of a model operator acting in the three-particle cut subspace of Fock space was proved).

4 Some Spectral Properties of the Families of Operators $h_n(p)$, $h_3(p,q)$, $n = 1,2$, $p,q \in \mathrm{T}^3$

In this section we study some spectral properties of the families of operators $h_n(p)$, $p \in \mathbf{T}^3$, $n = 1, 2$ resp. $h_3(p,q)$, $p,q \in \mathbf{T}^3$.

The following statement was proven in [19].

Lemma 4.1. *The following equalities hold:*

$$\sigma_{disc}(h_n(p)) = \{z \in \mathbf{C} \setminus \sigma_{ess}(h_n(p)) : \Delta_n(p\,;z) = 0\},\ n = 1,2,\ p \in \mathbf{T}^3, \tag{4.1}$$

$$\sigma_{disc}(h_3(p,q)) = \{z \in \mathbf{C} \setminus \sigma_{ess}(h_3(p,q)) : \Delta_3(p,q\,;z) = 0\},\ p,q \in \mathbf{T}^3. \tag{4.2}$$

First for the study of some spectral properties of $h_2(p)$, $p \in \mathbf{T}^3$ we rewrite the operator $h_{11}^{(2)}(p)$, $p \in \mathbf{T}^3$ in the form

$$h_{11}^{(2)}(p) = h_{11}^{(2,1)}(p) + h_{11}^{(2,2)},\ p \in \mathbf{T}^3$$

with

$$(h_{11}^{(2,1)}(p)f_1)(q) = w_2(p,q)f_1(q), \quad (h_{11}^{(2,2)}f_1)(q) = -v_{22}(q)\int_{\mathbf{T}^3} v_{22}(s)f_1(s)ds.$$

Then the operator $h_2(p)$, $p \in \mathbf{T}^3$ can be written in the form

$$h_2(p) = h_2^0(p) + V(p),\ p \in \mathbf{T}^3$$

with

$$h_2^0(p) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & h_{11}^{(2,1)}(p) & h_{12}^{(1)} \\ 0 & h_{21}^{(1)} & h_{22}^{(1)}(p) \end{pmatrix} \quad \text{and} \quad V(p) = \begin{pmatrix} h_{00}^{(2)}(p) & h_{01}^{(2)} & 0 \\ h_{10}^{(2)} & h_{11}^{(2,2)} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to show that the perturbation $V(p)$, $p \in \mathbf{T}^3$ is a bounded self-adjoint operator of rank of no more than 3 and hence, it is compact. It is easy to see that for any $p \in \mathbf{T}^3$ the equality $\sigma_{ess}(V(p)) = \{0\}$ holds. Therefore, for any $p \in \mathbf{T}^3$ the operator $V(p)$ may have only positive and negative discrete eigenvalues.

Lemma 4.2. *Let $w_1(\cdot)$ be a positive function on $\mathbf{T}^3$. For any fixed $p \in \mathbf{T}^3$ the operator $V(p)$ has no more than two negative (resp. one positive) simple eigenvalues.*

Proof. Let us consider the equation $V(p)f = zf$, $z \neq 0$, $f \in \mathcal{H}^{(0,2)}$, $p \in \mathbf{T}^3$ or the system of equations

$$\begin{cases} (w_1(p) - z)f_0 + (v_2, f_1)_1 = 0 \\ v_2(q)f_0 - v_{22}(q)(v_{22}, f_1)_1 = zf_1, \end{cases} \tag{4.3}$$

where $(\cdot,\cdot)_1$ is the scalar product in $\mathcal{H}_1$.

Since $z \neq 0$ from the second equation of (4.3) we find

$$f_1(q) = \frac{v_2(q)}{z} f_0 - \frac{v_{22}(q)}{z} C_{f_1}, \tag{4.4}$$

where

$$C_{f_1} = \int_{\mathbf{T}^3} v_{22}(s) f_1(s) ds. \tag{4.5}$$

Substituting the expression (4.4) for f_1 into the first equation of the system of equations (4.3) and the equality (4.5) we have that the system of equations (4.3) has a solution if and only if $P_{(v_2,v_{22})_1}(p\,;z) = 0$, where $\|\cdot\|_1$ is the norm in $\mathcal{H}_1$ and

$$P_{(v_2,v_{22})_1}(p\,;z) = -(z^2 - w_1(p)z - \|v_2\|_1^2)(z + \|v_{22}\|_1^2) - (v_2, v_{21})_1^2,\ z \neq 0,\ p \in \mathbf{T}^3.$$

We note that, if $v_2(\cdot)$ and $v_{21}(\cdot)$ are linear dependent, then $|(v_2, v_{22})_1| = \|v_2\|_1 \|v_{22}\|_1$. Therefore, $P_{(v_2,v_{22})_1}(p\,;z) = P_0(p\,;z) - |(v_2,v_{22})_1|^2$ and

$$P_{\|v_2\|_1\|v_{22}\|_1}(p\,;z) = P_0(p\,;z) - \|v_2\|_1^2 \|v_{22}\|_1^2.$$

By the inequality $|(v_2, v_{22})_1| \leq \|v_2\|_1 \|v_{22}\|_1$ we obtain that

$$P_0(p\,;z) \geq P_{(v_2,v_{22})_1}(p\,;z) \geq P_{\|v_2\|_1\|v_{22}\|_1}(p\,;z).$$

There are three cases possible: 1) $v_2(\cdot)$ and $v_{22}(\cdot)$ are orthogonal; 2) $v_2(\cdot)$ and $v_{22}(\cdot)$ are parallel; 3) $v_2(\cdot)$ and $v_{22}(\cdot)$ are neither orthogonal and nor parallel.

Let $v_2(\cdot)$ and $v_{22}(\cdot)$ be orthogonal.

Then $P_0(p\,;z) = P_{(v_2,v_{22})_1}(p\,;z) > P_{\|v_2\|_1\|v_{22}\|_1}(p\,;z)$. In this case the numbers

$$\hat{z}_1(p) = -\|v_{22}\|_1^2 < 0, \quad \hat{z}_2(p) = \frac{w_1(p) - \sqrt{w_1^2(p) + 4\|v_2\|_1^2}}{2} < 0$$

and

$$\hat{z}_3(p) = \frac{w_1(p) + \sqrt{w_1^2(p) + 4\|v_2\|_1^2}}{2} > 0$$

are zeroes of $P_{(v_2,v_{22})_1}(p\,;z) = P_0(p;z)$, $p \in \mathbf{T}^3$, i.e., the eigenvalues of $V(p)$.

We remark that the numbers $\hat{z}_n(p)$, $n = 1,2,3$ are also zeroes of $P_0(p\,;\cdot)$, $p \in \mathbf{T}^3$ in the case where $v_2(\cdot)$ and $v_{22}(\cdot)$ are not orthogonal.

Let $v_2(\cdot)$ and $v_{22}(\cdot)$ be parallel. Then

$$P_0(p\,;z) > P_{(v_2,v_{22})_1}(p\,;z) = P_{\|v_2\|_1\|v_{22}\|_1}(p\,;z).$$

In this case the polynomial $P_{(v_2,v_{22})_1}(p\,;z)$ can be written in the form

$$P_{(v_2,v_{22})_1}(p\,;z) = -z(z^2 - z(\|v_{22}\|_1^2 - w_1(p)) - (\|v_2\|_1^2 + w_1(p)\|v_{22}\|_1^2)).$$

From here it follows that the numbers

$$\tilde{z}_1(p) = 0, \quad \tilde{z}_2(p) = \frac{\|v_{22}\|_1^2 - w_1(p) - \sqrt{(\|v_{22}\|_1^2 + w_1(p))^2 + 4\|v_2\|_1^2}}{2} < 0$$

and

$$\tilde{z}_3(p) = \frac{\|v_{22}\|_1^2 - w_1(p) + \sqrt{(\|v_{22}\|_1^2 + w_1(p))^2 + 4\|v_2\|_1^2}}{2} > 0$$

are zeroes of $P_{(v_2,v_{22})_1}(p\,;z) = P_{\|v_2\|_1\|v_{22}\|_1}(p\,;z)$, $p \in \mathbf{T}^3$, i.e., the eigenvalues of $V(p)$, where the number $\tilde{z}_2(p)$ is negative, because the function $w_1(\cdot)$ is positive function on $\mathbf{T}^3$.

We remark that the numbers $\tilde{z}_n(p)$, $n = 1, 2, 3$ are also zeroes of $P_{\|v_2\|_1\|v_{22}\|_1}(p\,;\cdot)$, $p \in \mathbf{T}^3$ in the case where $v_2(\cdot)$ and $v_{22}(\cdot)$ are not parallel.

Let $v_2(\cdot)$ and $v_{22}(\cdot)$ be neither orthogonal and nor parallel. Then we have

$$P_0(p\,;z) > P_{(v_2,v_{22})_1}(p\,;z) > P_{\|v_2\|_1\|v_{22}\|_1}(p\,;z).$$

Set $a_1(p) = \min\{\hat{z}_1(p), \hat{z}_2(p)\}$, $a_2(p) = \max\{\hat{z}_1(p), \hat{z}_2(p)\}$, $p \in \mathbf{T}^3$.

Without loss of generality (otherwise we would be prove the following facts in the same way) we assume that for any $p \in \mathbf{T}^3$ the inequalities $\tilde{z}_2(p) < \hat{z}_2(p)$, $\tilde{z}_2(p) < \hat{z}_1(p)$, $\hat{z}_3(p) < \tilde{z}_3(p)$ hold. Then it follows that

$$\tilde{z}_2(p) < a_1(p) \le a_2(p) < \tilde{z}_1(p) = 0 < \hat{z}_3(p) < \tilde{z}_3(p), \ p \in \mathbf{T}^3.$$

Since the numbers $\tilde{z}_2(p)$ and $a_1(p)$ are zeroes of $P_{\|v_2\|_1\|v_{22}\|_1}(p\,;\cdot)$ and $P_0(p\,;\cdot)$, respectively, we have $P_{(v_2,v_{22})_1}(p\,;\tilde{z}_2(p)) > P_{\|v_2\|_1\|v_{22}\|_1}(p\,;\tilde{z}_2(p)) = 0$ and $0 = P_0(p\,;a_1(p)) < P_{(v_2,v_{22})_1}(p\,;a_1(p))$, i.e., on the boundary of $[\tilde{z}_2(p), a_1(p)]$ the polynomial $P_{(v_2,v_{22})_1}(p\,;\cdot)$ has a different sign. Hence, there exists a point $z_1(p)$, such that $\tilde{z}_2(p) < z_1(p) < a_1(p)$ and $P_{(v_2,v_{22})_1}(p\,;z_1(p)) = 0$.

Analogously one can prove that there exist the numbers $z_2(p) \in (a_2(p), \tilde{z}_1(p))$ and $z_3(p) \in (\hat{z}_3(p), \tilde{z}_3(p))$, which are zeroes of the polynomial $P_{(v_2,v_{22})_1}(p\,;\cdot)$.

Since $P_{(v_2,v_{22})_1}(p\,;\cdot)$ is a polynomial of degree 3 these zeroes are simple.

One can see $z_1(p) < z_2(p) < 0$ and $z_3(p) > 0$.

Lemma 4.2 is completely proved. □

Let us introduce the notations

$$|V(p)| = \sqrt{(V(p))^2}, \ V_+(p) = \tfrac{1}{2}\{|V(p)| + V(p)\} \text{ and } V_-(p) = \tfrac{1}{2}\{|V(p)| - V(p)\},$$

where $\sqrt{(V(p))^2}$ is a nonnegative square root of $(V(p))^2$. Then $V_+(p) \geq 0$, $V_-(p) \leq 0$ and $V(p) = V_+(p) + V_-(p)$.

Since the operators $V_+(p)$ and $-V_-(p)$ are non negative, there exist non negative square roots $V_+^{1/2}(p)$ and $(-V_-(p))^{1/2}$, respectively.

Let

$$m_2(p) = \min \sigma_{ess}(h_2(p)), \quad M_2(p) = \max \sigma_{ess}(h_2(p)).$$

For any fixed $z < m_2(p)$ resp. $z > M_2(p)$ the operator $h_2^0(p) - zI + V_+(p)$ resp. $h_2^0(p) - zI + V_-(p)$ is invertible and positive resp. negative, where I is an identical operator in $\mathcal{H}^{(0,2)}$.

Set

$$r_+(p;z) = (h_2^0(p) - zI + V_+(p))^{-1}, \; r_+^{1/2}(p;z) = (h_2^0(p) - zI + V_+(p))^{-1/2}, \; z < m_2(p),$$

$$r_-(p;z) = (h_2^0(p) - zI + V_-(p))^{-1}, \; r_-^{1/2}(p;z) = (h_2^0(p) - zI + V_-(p))^{-1/2}, \; z > M_2(p).$$

Let us denote by $N_-(p;z)$ resp. $N_+(p;z)$ the number of eigenvalues of $h_2(p)$ lying below $z < m_2(p)$ resp. upper $z > M_2(p)$.

For any bounded self-adjoint operator A, acting in Hilbert space $\mathcal{H}$ not having any essential spectrum on the right of the point z we denote by $\mathcal{H}_A(z)$ the subspace such that $(Af, f) > z(f, f)$ for any $f \in \mathcal{H}_A(z)$ and set

$$n(z, A) = \sup_{\mathcal{H}_A(z)} dim \mathcal{H}_A(z).$$

By the definitions of $N_-(p\,; z)$ and $N_+(p\,; z)$ we have

$$N_-(p\,; z) = n(-z, -h_2(p)), \; -z > -m_2(p),$$
$$N_+(p\,; z) = n(z, h_2(p)), \; z > M_2(p).$$

The following lemma is a realization of the well-known Birman-Schwinger principle for the operator $h_2(p)$ (see. [4, 10]).

Lemma 4.3. *For any $z < m_2(p)$ the operator $(-V_-(p))^{1/2} r_+(p;z)(-V_-(p))^{1/2}$ is compact and*

$$N_-(p\,; z) = n(1, (-V_-(p))^{1/2} r_+(p\,; z)(-V_-(p))^{1/2}). \tag{4.6}$$

Proof. Since $(-V_-(p))^{1/2}$ is a finite rank operator and $r_+(p;z)(-V_-(p))^{1/2})$ is a bounded operator, the operator $-V_-(p))^{1/2} r_+(p;z)(-V_-(p))^{1/2}$ is compact.

The operator $h_2(p)$ can be decomposed as

$$h_2(p) = h_2^0(p) + V_+(p) + V_-(p).$$

Assume that $u \in \mathcal{H}_{-h_2(p)}(-z)$, i.e., $((h_2^0(p) - zI + V_+(p))u, u) < ((-V_-(p))u, u)$. Then

$$(r_+^{1/2}(p\,; z)(-V_-(p)) r_+^{1/2}(p\,; z)g, g) > 0, \quad g = (h_2^0(p) - zI + V_-(p))^{1/2} u.$$

Thus $N_-(p\,;z) \le n(1, r_+^{1/2}(p\,;z)(-V_-(p))r_+^{1/2}(p\,;z))$. Reserving the argument we get the opposite inequality, which proves the equality

$$N_-(p\,;z) = n(1, r_+^{1/2}(p\,;z)(-V_-(p))r_+^{1/2}(p\,;z)). \tag{4.7}$$

Now we use the following well-known fact (see [7]).

Proposition 4.1. *Let T_1, T_2 be bounded operators. If $\lambda \ne 0$ is an eigenvalue of T_1T_2, then λ is an eigenvalue for T_2T_1 as well of the same algebraic and geometric multiplicities.*

By Proposition 4.1 the discrete spectrum of $r_+^{1/2}(p;z)(-V_-(p))r_+^{1/2}(p;z)$, away from zero, coincides with the discrete spectrum of $(-V_-(p))^{1/2}r_+(p;z)(-V_-(p))^{1/2}$. Therefore,

$$n(1, r_+^{1/2}(p\,;z)(-V_-(p))r_+^{1/2}(p\,;z)) = n(1, (-V_-(p))^{1/2}r_+(p\,;z)(-V_-(p))^{1/2}). \tag{4.8}$$

Taking into account the equalities (4.7) and (4.8) we obtain (4.6). Lemma 4.3 is completely proved. □

The following lemma can be proved similarly to Lemma 4.3.

Lemma 4.4. *For any $z > M_2(p)$ the operator $V_+^{1/2}(p)r_-(p\,;z)V_+^{1/2}(p)$ is compact and*

$$N_+(p\,;z) = n(-1, -V_+^{1/2}(p)r_-(p\,;z)V_+^{1/2}(p)).$$

Now we are ready to get the proof of the following lemma.

Lemma 4.5. *Let $w_1(\cdot)$ be a positive function on $\mathbf{T}^3$. For any fixed $p \in \mathbf{T}^3$ the operator $h_2(p)$ has no more than two (resp. one) simple eigenvalues lying on the l.h.s. of $m_2(p)$ resp. on the r.h.s. of $M_2(p)$.*

Proof. By Lemma 4.2 the following inequalities hold

$$n(1, (-V_-(p))^{1/2}r_+(p\,;z)(-V_-(p))^{1/2}) \le 2,$$
$$n(-1, -V_+^{1/2}(p)r_-(p\,;z)V_+^{1/2}(p)) \le 1.$$

From Lemmas 4.3, 4.4 and the latter inequalities it follows that

$$N_-(p\,;z) \le 2,\ z < m_2(p) \quad \text{and} \quad N_+(p\,;z) \le 1,\ z > M_2(p).$$

Lemma 4.5 is completely proved. □

Remark 4.1. If in Lemmas 4.2 and 4.5 for some $p' \in \mathbf{T}^3$ the number $w_1(p')$ is negative, then the number $\tilde{z}_2(p')$ either negative or positive, i.e., for some $p' \in \mathbf{T}^3$, the operator $h_2(p')$ may have two positive eigenvalues. But in the proof of Theorem 3.1 we use only the fact that for any $p \in \mathbf{T}^3$ the operator $h_2(p)$ has no more than three eigenvalues.

Lemma 4.6. *If for any $p \in \mathbf{T}^3$ the operator $h_2(p)$ has no eigenvalues lying in the intervals $(-\infty; m)$ and $(M; +\infty)$, then $\sigma_{two}(H_2) \subset [m; M]$.*

Proof. Let the condition of the lemma be fulfilled. Then for any $p \in \mathbf{T}^3$ the inclusion $\sigma_{disc}(h_2(p)) \subset [m; M]$ holds. Then the assertion

$$\sigma_{two}(H_2) = \bigcup_{p \in \mathbf{T}^3} \sigma_{disc}(h_2)(p) \subset [m; M]$$

and Theorem 2.1 complete the proof. □

Next we will study the operator $h_3(p, q)$.

Lemma 4.7. *For any fixed $p, q \in \mathbf{T}^3$ the operator $h_3(p, q)$ has no more than one simple eigenvalue lying on the l.h.s. of $m_3(p, q)$ resp. on the r.h.s. of $M_3(p, q)$.*

Proof. Since for any fixed $p, q \in \mathbf{T}^3$ the function $\Delta_3(p, q\,; \cdot)$ is monotone decreasing on $(-\infty; m_3(p, q))$ and $(M_3(p, q); +\infty)$, Lemma 4.1 completes the proof of lemma. □

The following lemma describes the set of eigenvalues of $h_3(p, q)$.

Lemma 4.8. 1) *Assume that* $\min_{p,q \in \mathbf{T}^3} \Delta_3(p, q\,; m) \geq 0$. *Then for any $p, q \in \mathbf{T}^3$ the operator $h_3(p, q)$ has no eigenvalues lying on the l.h.s. of m.*

2) *Assume that* $\min_{p,q \in \mathbf{T}^3} \Delta_3(p, q\,; m) < 0$ *and* $\max_{p,q \in \mathbf{T}^3} \Delta_3(p, q\,; m) \geq 0$. *Then there exists a non void open set $D \subset (\mathbf{T}^3)^2$ such that $D \neq (\mathbf{T}^3)^2$ and for any $(p, q) \in D$ the operator $h_3(p, q)$ has a unique eigenvalue lying on the l.h.s. of m and for any $(p, q) \in (\mathbf{T}^3)^2 \setminus D$ the operator $h_3(p, q)$ has no eigenvalues lying on the l.h.s. of m.*

3) *Assume that* $\max_{p,q \in \mathbf{T}^3} \Delta_3(p, q\,; m) < 0$. *Then for any $p, q \in \mathbf{T}^3$ the operator $h_3(p, q)$ has a unique eigenvalue lying on the l.h.s. of m.*

Proof. First we prove part 2). Let

$$\min_{p,q \in \mathbf{T}^3} \Delta_3(p, q\,; m) < 0, \quad \max_{p,q \in \mathbf{T}^3} \Delta_3(p, q\,; m) \geq 0.$$

Introduce the notation: $D \equiv \{(p, q) \in (\mathbf{T}^3)^2 : \Delta_3(p, q\,; m) < 0\}$.

Since $(\mathbf{T}^3)^2$ is compact and the function $\Delta_3(\cdot, \cdot\,; m)$ is continuous on $(\mathbf{T}^3)^2$, there exist points $(p', q'), (p'', q'') \in (\mathbf{T}^3)^2$ such that the inequalities

$$\min_{p,q \in \mathbf{T}^3} \Delta_3(p, q\,; m) = \Delta_3(p', q'\,; m) < 0,$$

$$\max_{p,q \in \mathbf{T}^3} \Delta_3(p, q\,; m) = \Delta_3(p'', q''\,; m) \geq 0$$

hold. Hence we have that D is a non void open set and $D \neq (\mathbf{T}^3)^2$.

For any $p, q \in \mathbf{T}^3$ the function $\Delta_3(p, q\,; \cdot)$ is continuous and decreasing on $(-\infty; m]$ and $\lim_{z \to -\infty} \Delta_3(p, q\,; z) = +\infty$.

Then there exists a unique point $z(p,q) \in (-\infty; m)$ such that $\Delta_3(p,q\,; z(p,q)) = 0$ for any $(p,q) \in D$. By Lemma 4.1 the point $z(p,q)$ is the unique eigenvalue of the operator $h_3(p,q)$ lying on the l.h.s. of m.

For any $(p,q) \in (\mathbf{T}^3)^2 \setminus D$ and $z < m$ we have $\Delta_3(p,q; z) > \Delta_3(p,q\,; m) \geq 0$.

Hence by Lemma 4.1 for each $(p,q) \in (\mathbf{T}^3)^2 \setminus D$ the operator $h_3(p,q)$ has no eigenvalues lying on the l.h.s. of m.

If $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) \geq 0$ resp. $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;m) < 0$, then $D = \emptyset$ resp. $D = (\mathbf{T}^3)^2$ and the above analysis leads again to the case 1) resp. 3). The lemma is completely proved. $\square$

The following two lemmas can be proved similarly to Lemma 4.8.

Lemma 4.9. 1) *Assume that* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;M) \leq 0$. *Then for any* $p,q \in \mathbf{T}^3$ *the operator* $h_3(p,q)$ *has no eigenvalues lying on the r.h.s. of* M.

2) *Assume that* $\max_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;M) > 0$ *and* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;M) \leq 0$. *Then there exists a non void open set* $D' \subset (\mathbf{T}^3)^2$ *such that* $D' \neq (\mathbf{T}^3)^2$ *and for any* $(p,q) \in D'$ *the operator* $h_3(p,q)$ *has a unique eigenvalue lying on the r.h.s. of* M *and for any* $(p,q) \in (\mathbf{T}^3)^2 \setminus D'$ *the operator* $h_3(p,q)$ *has no eigenvalues lying on the r.h.s. of* M.

3) *Assume that* $\min_{p,q\in\mathbf{T}^3} \Delta_3(p,q\,;M) > 0$. *Then for any* $p,q \in \mathbf{T}^3$ *the operator* $h_3(p,q)$ *has a unique eigenvalue lying on the r.h.s. of* M.

Lemma 4.10. *Let Assumption 2.1 be fulfilled.*

1) *Assume that* $\min_{p\in\mathbf{T}^3} \Delta_1(p\,;m) \geq 0$. *Then for any* $p \in \mathbf{T}^3$ *the operator* $h_1(p)$ *has no eigenvalues lying on the l.h.s. of* m.

2) *Assume that* $\min_{p\in\mathbf{T}^3} \Delta_1(p\,;m) < 0$ *and* $\max_{p\in\mathbf{T}^3} \Delta_1(p\,;m) \geq 0$. *Then there exists a non void open set* $D'' \subset \mathbf{T}^3$ *such that* $D'' \neq \mathbf{T}^3$ *and for any* $p \in D''$ *the operator* $h_1(p)$ *has a unique eigenvalue lying on the l.h.s. of* m *and for any* $p \in \mathbf{T}^3 \setminus D''$ *the operator* $h_1(p)$ *has no eigenvalues lying on the l.h.s. of* m.

3) *Assume that* $\max_{p\in\mathbf{T}^3} \Delta_1(p\,;m) < 0$. *Then for any* $p \in \mathbf{T}^3$ *the operator* $h_1(p)$ *has a unique eigenvalue lying on the l.h.s. of* m.

In the proof of Theorem 3.1 we also use the following lemmas.

Lemma 4.11. *For any* $p \in \mathbf{T}^3$ *the operator* $h_1(p)$ *has no eigenvalues lying on the r.h.s. of* M.

Proof. Since for any $p \in \mathbf{T}^3$ the function $\Delta_1(p\,;\cdot)$ is monotone decreasing on $(M;+\infty)$ and $\lim_{z\to+\infty} \Delta_1(p\,;z) = 1$, by Lemma 4.1 for any $p \in \mathbf{T}^3$ the operator $h_1(p)$ has no eigenvalues lying on the r.h.s. of M. $\square$

Lemma 4.12. *Let the functions* $v_3(\cdot)$, $w_2(\cdot,\cdot)$ *and* $w_3(\cdot,\cdot,\cdot)$ *be defined by*

$$v_3(t) \equiv \sqrt{\mu}, \quad w_2(p,q) = \varepsilon(p) + \varepsilon(q) + \lambda, \quad w_3(p,q,t) = \varepsilon(p) + \varepsilon(q) + \varepsilon(t),$$

where

$$\varepsilon(t) = 3 - \cos t^{(1)} - \cos t^{(2)} - \cos t^{(3)},\ t = (t^{(1)}, \quad t^{(2)}, t^{(3)}) \in \mathbf{T}^3$$

and

$$\mu = (b - a - 12)\Big(\int_{\mathbf{T}^3} \frac{ds}{\varepsilon(s)} - \int_{\mathbf{T}^3} \frac{ds}{\varepsilon(s) + 12} \Big)^{-1},\ \lambda = a + \mu \int_{\mathbf{T}^3} \frac{ds}{\varepsilon(s)}$$

for some real nonzero numbers a *and* b *such that* $b - a > 12$. *Then there exist the numbers* $C_1, C_2 > 0$ *and* $\delta_1,\ \delta_2 > 0$ *such that the functions* $\Delta_3(\cdot, \cdot\,; m)$ *and* $\Delta_3(\cdot, \cdot\,; M)$ *satisfy Assumption 2.1 with* $\alpha = \beta = 0$.

Proof. First we note that $p_0 = q_0 = \theta = (0, 0, 0) \in \mathbf{T}^3$, $p_1 = q_1 = \bar{\pi} = (\pi, \pi, \pi) \in \mathbf{T}^3$. By the definition of the numbers μ and λ we have

$$\Delta_3(p_0, q_0\,; m) = a \neq 0, \quad \Delta_3(p_1, q_1\,; M) = b \neq 0. \tag{4.9}$$

Since the functions $\Delta_3(\cdot, \cdot\,; m)$ and $\Delta_3(\cdot, \cdot\,; M)$ are continuous on $(\mathbf{T}^3)^2$ it follows in accordance with (4.9) that there exist the numbers $C_1, C_2 > 0$ and $\delta_1,\ \delta_2 > 0$ such that the inequalities stated in the Assumption 2.1 hold with $\alpha = \beta = 0$. □

5 Proof of the Main Results

Using the assertions were proved in section 4, we prove the main results of this paper.

Proof of Theorem 3.1. By Lemma 4.7 for any $p, q \in \mathbf{T}^3$ the operator $h_3(p, q)$ has no more than two simple eigenvalues lying outside of its essential spectrum. Then the theorem on the spectrum of decomposable operators and the equality (2.1) imply that the set $\sigma_{three}(H)$ consists of no more than two bounded closed intervals. Similarly, using Lemmas 4.5, 4.10, 4.11 and the equality (2.2) one can prove that the set $\sigma_{two}(H)$ consists of no more than four bounded closed intervals. Then Theorem 2.1 completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. First we prove part **III**.

Suppose that $\min_{p,q \in \mathbf{T}^3} \Delta_3(p, q\,; M) > 0$. By Lemma 4.9 for any $p, q \in \mathbf{T}^3$ the operator $h_3(p, q)$ has a unique eigenvalue $E_1(p, q) > M$. Since the functions $v_3(\cdot)$, $w_2(\cdot, \cdot)$ and $w_3(\cdot, \cdot, \cdot)$ are analytic functions on its domains, the function $E_1 : (p, q) \in (\mathbf{T}^3)^2 \to E_1(p, q)$ is continuous on the compact set $(\mathbf{T}^3)^2$. From here it follows that the range $Im E_1$ of $E_1(\cdot, \cdot)$ is a closed subset of $(M; +\infty)$, i.e., $Im E_1 = [a_3; b_3]$ with $a_3 > M$. Hence, equality (2.1) and Theorem 2.1 imply that $\sigma_{three}(H) \cap [m; +\infty) = [m; M] \cup [a_3; b_3]$.

3.1) Suppose that $\min_{p,q \in \mathbf{T}^3} \Delta_3(p, q\,; m) \geq 0$. For any $p, q \in \mathbf{T}^3$, Lemma 4.8 implies that the operator $h_3(p, q)$ has no eigenvalues lying on the l.h.s. of m. Then equality (2.1) and Theorem 2.1 complete the proof of assertion 3.1) of Theorem 3.2.

3.2) Suppose that $\min_{p,q\in \mathbf{T}^3} \Delta_3(p,q\,;m)<0$ and $\max_{p,q\in \mathbf{T}^3} \Delta_3(p,q\,;m)\geq 0$. Then by Lemma 4.8 there exists a non void open set $D\subset (\mathbf{T}^3)^2$ such that $D\neq (\mathbf{T}^3)^2$ and for any $(p,q)\in D$ the operator $h_3(p,q)$ has a unique eigenvalue $E_2(p,q)$ lying on the l.h.s. of m.

The functions $v_3(\cdot)$, $w_2(\cdot,\cdot)$ and $w_3(\cdot,\cdot,\cdot)$ are analytic functions on its domain, the function $E_2:(p,q)\in D\to E_2(p,q)$ is continuous on D.

Since for any $p,q\in \mathbf{T}^3$ the operator $h_3(p,q)$ is bounded and $(\mathbf{T}^3)^2$ is compact set, there exists a positive number C such that $\sup_{p,q\in \mathbf{T}^3}\|h_3(p,q)\|$ and for any $p,q\in \mathbf{T}^3$ we have

$$\sigma(h_3(p,q))\subset[-C;C]. \tag{5.1}$$

For any $(p,q)\in\partial D=\{(p,q)\in (\mathbf{T}^3)^2:\Delta_3(p,q\,;m)=0\}$ there exist $\{(p_n,q_n)\}\subset D$ such that $(p_n,q_n)\to(p,q)$ as $n\to\infty$. Set $E_2^{(n)}=E_2(p_n,q_n)$. Then by Lemma 4.8 for any $\{(p_n,q_n)\}\in D$ the inequality $E_2^{(n)}<m$ holds and from (5.1) we get $\{E_2^{(n)}\}\subset[-C;m]$. Without loss of generality (otherwise we would have to take a subsequence) we assume that $E_2^{(n)}\to E_2^{(0)}$ as $n\to\infty$ for some $E_2^{(0)}\in[-C;m]$.

From the continuity of the function $\Delta_3(\cdot,\cdot\,;\cdot)$ in $(\mathbf{T}^3)^2\times(-\infty;m]$ and $(p_n,q_n)\to(p,q)$ and $E_2^{(n)}\to E_2^{(0)}$ as $n\to\infty$ it follows that

$$0=\lim_{n\to\infty}\Delta_3(p_n,q_n\,;E_2^{(n)})=\Delta_3(p,q\,;E_2^{(0)}).$$

Since for any $p,q\in \mathbf{T}^3$ the function $\Delta_3(p,q;\cdot)$ is decreasing in $(-\infty;m]$ and $(p,q)\in\partial D$ we see that $\Delta_3(p,q\,;E_2^{(0)})=0$ if and only if $E_2^{(0)}=m$.

For any $(p,q)\in\partial D$ we define

$$E_2(p,q)=\lim_{(p',q')\to(p,q),\,(p',q')\in D}E_2(p',q')=m.$$

Since the function $E_2(\cdot,\cdot)$ is continuous on the compact set $D\cup\partial D$ and $E_2(p,q)=m$ for all $(p,q)\in\partial D$ we conclude that $ImE_2=[a_2;m]$, $a_2<m$.

Hence the set $\{z\in\sigma_{three}(H),\ z\leq m\}$ coincides with the set $ImE_2=[a_2;m]$. Then equality (2.1) and Theorem 2.1 complete the proof of assertion 3.2) of Theorem 3.2.

3.3) Let $\max_{p,q\in \mathbf{T}^3}\Delta_3(p,q\,;m)<0$. Then by Lemma 4.8 for all $p,q\in \mathbf{T}^3$ the operator $h_3(p,q)$ has a unique eigenvalue $E_2(p,q)$ lying on the l.h.s. of m.

The functions $v_3(\cdot)$, $w_2(\cdot,\cdot)$ and $w_3(\cdot,\cdot,\cdot)$ are analytic functions on its domain, the function E_2 is continuous on $(\mathbf{T}^3)^2$. Therefore the range ImE_2 of the function E_2 is a connected closed subset of $(-\infty;m)$, that is, $ImE_2=[a_2;b_2]$ with $b_2<m$. Then the equality (2.1) and Theorem 2.1 complete the proof of assertion 3.3) of Theorem 3.2. Other assertions of Theorem 3.2 are proved similarly. □

Theorem 3.3 can be proved similarly to Theorem 3.2. Therefore, to avoid repetition, it is not given here.

Acknowledgements

The author wishes to thank the Abdus Salam International Centre for Theoretical Physics, Trieste, Italy, for kind hospitality and support and the Commission on Development and Exchanges of the International Mathematical Union for the travel grant. He is grateful to Dr. Zahriddin I. Muminov for useful discussions and to the referees for valuable critical remarks.

References

[1] S. Albeverio, S. N. Lakaev and J. I. Abdullaev, On the finiteness of the discrete spectrum of a four-particle lattice Schrödinger operator, *Func. Anal. Appl.* **36** (2002), 212–216.

[2] S. Albeverio, S. N. Lakaev and Z. I. Muminov, Schrödinger operators on lattices. The Efimov effect and discrete spectrum asymptotics, *Ann. Inst. Henri Poincare, Phys. Theor.* **5** (2004), 1–30.

[3] S. Albeverio, S. N. Lakaev and Z. I. Muminov, On the structure of the essential spectrum for the three-particle Schrödinger operators on lattices, *Math. Nachr.* **280** (2007), 699–716.

[4] S. Albeverio, S. N. Lakaev and T. H. Rasulov, On the Spectrum of an Hamiltonian in Fock Space. Discrete Spectrum Asymptotics, *J. Stat. Phys.* **127** (2007), 191–220.

[5] S. Albeverio, S. N. Lakaev and T. H. Rasulov, The Efimov Effect for a Model Operator Associated with the Hamiltonian of a non Conserved Number of Particles, *Methods Func. Anal. Topol.* **13** (2007), 1–16.

[6] K. O. Friedrichs, *Perturbation of spectra in Hilbert space*, Amer. Math. Soc. Providence, Rhode Island, 1965.

[7] P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, Princeton, N. J. 1967.

[8] W. Hunziker, On the spectra of Schrödinger multiparticle Hamiltonians, *Helv. Phys. Acta* **39** (1966), 451–462.

[9] S. N. Lakaev and J. I. Abdullaev, Finiteness of the discrete spectrum of the three-particle Schrödinger operator on a lattice, *Theor. Math. Phys.* **111** (1997), 467–479.

[10] S. N. Lakaev and M. É. Muminov, Essential and discrete spectra of the three-particle Schrödinger operator on a lattices, *Theor. Math. Phys.* **135** (2003), 849–871.

[11] S. N. Lakaev and T. Kh. Rasulov, A Model in the Theory of Perturbations of the Essential Spectrum of Multiparticle Operators, *Math. Notes* **73** (2003), 521–528.

[12] S. N. Lakaev and T. Kh. Rasulov, Efimov's Effect in a Model of Perturbation Theory of the Essential Spectrum, *Func. Anal. Appl.* **37** (2003), 69–71.

[13] R. Minlos and H. Spohn, The Three-Body Problem in Radioactive Decay: The Case of One Atom and At Most Two Photons, *Amer. Math. Soc. Transl.* **177** (1996), 159–193.

[14] M. É. Muminov, A Hunziker-van Winter-Zhislin theorem for a four-particle lattice Schrödinger operator, *Theor. Math. Phys.* **148** (2006), 1236–1250.

[15] V. S. Rabinovich and S. Roch, The essential spectrum of Schrödinger operators on lattices, *J. Phys. A: Math. Gen.* **39** (2006), 8377–8394.

[16] T. Kh. Rasulov, Discrete Spectrum of a Model Operator in Fock Space, *Theor. Math. Phys.* **152** (2007), 1313–1321.

[17] T. Kh. Rasulov, On the Structure of the Essential Spectrum of a Model Many-Body Hamiltonian, *Math. Notes* **83** (2008), 80–87.

[18] T. Kh. Rasulov, The Faddeev Equation and the Location of the Essential Spectrum of a Model Multi-Particle Operator, *Russian Math.* **52** (2008), 50–59.

[19] T. H. Rasulov, M. I. Muminov and M. Hasanov, On the Spectrum of a Model Operator in Fock Space, *Methods Func. Anal. Topol.* **15** (2009), 369–383.

[20] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV: Analysis of Operators*, Academic Press, New York, 1979.

[21] I. M. Sigal, A. Soffer and L. Zielinski, On the spectral properties of Hamiltonians without conservation of the particle number, *J. Math. Phys.* **42** (2002), 1844–1855.

[22] G. R. Yodgorov and M. É. Muminov, Spectrum of a Model Operator in the Perturbation Theory of the Essential Spectrum, *Theor. Math. Phys.* **144** (2005), 1344–1352.

[23] G. M. Zhislin, Investigations of the spectrum of the Schrödinger operator for a many body system, *Trudy Moskov. Mat. Obshch.* **9** (1960), 81–128.

[24] Yu. Zhukov and R. Minlos, Spectrum and scattering in a “spin-boson” model with not more than three photons, *Theor. Math. Phys.* **103** (1995), 398–411.

T. H. Rasulov graduated from the Samarkand State University in 1998. His scientific interests are connected with the spectral and scattering theory of Hamiltonians without conservation of particle number, many-body Schroedinger operators, mathematical problems of quantum mechanics and quantum field theory. In the last years he has visited important scientific institutions and he has continued his scientific activity, working on the mathematical analysis of quantum systems where the number of particles is not fixed. Collaborating with Professor S. Albeverio, he has obtained some interesting mathematical results.