Applied Mathematics & Information Sciences Letters
*An International Journal*

http://dx.doi.org/10.12785/amisl/030301

# Analytic Solutions of Liouville Equation using Extended Trial Equation Method and The Functional Varible Method

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Received: 12 Dec. 2014, Revised: 18 May 2015, Accepted: 19 May 2015
Published online: 1 Sep. 2015

**Abstract:** In this paper, the functional variable method and extended trial method are applied to the Liouville equation to obtain its exact solution.

**Keywords:** Extended trial equation method, the functional varible method, Liouville equation, travelling wave solutions.
**AMS classification:** 35C07, 35A20, 35A25

## 1 Introduction

Nonlinear ordinary or mostly partial differential equations are generally used to model many physical phenomena in different fields of pysics and engineering. Especially nonlinear equations in the form

$$u_{tt} - ku_{xx} + f(u) = 0 \quad (1)$$

has an important application area such as solid state physics, nonlinear optics, plasma physics, fluid dynamics, mathematical biology, nonlinear optics, dislocations in crystals, kink dynamics, and chemical kinetics, and quantum field theory[1]. By taking

$$f(u) = e^{\pm u} \quad (2)$$

the equation becomes Liouville equation [1]. In this article we handle the equation of the form

$$u_{xt} + e^{u} = 0.$$

In applied mathematics, it has importance to obtain and search the exact solutions of these equations. Therefore, recently, a lot of efficient and accurate methods such as sine-cosine method[2], tanh function method[3], variational iteration method[4], homotopy perturbation method[5], homotopy analysis method[6], Exp-function method[7], F-expansion method[8], $(G'/G)$-expansion method[9], functional variable method[10], extended trial equation method[11] and others have been presented and successfully applied to have exact solutions of nonlinear partial differential equations. These methods differs greatly from each other with regards to the initial approximation, transformations, etc. For example, some of these methods involve using transformations to convert nonlinear equations into more easily handled simple equations, some others involve using trial functions in an iterative scheme to converge rapidly to the exact solution, and still others look for the solution of nonlinear evolution equations (NLEEs) viewed as polinomial in variable satisfying a supplementary nonlinear ordinary differential equation.

### 1.1 The Extended Trial Equation Method

We handle the general form of nonlinear partial differential equations

$$P(u, u_t, u_x, u_{xx}, \ldots) = 0 \quad (3)$$

and apply the wave transformation

$$u(x_1, x_2, \ldots, x_N, t) = u_\eta, \quad \eta = \lambda \left( \sum_{j=1}^{N} x_j - ct \right) \quad (4)$$

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to Eq. (3) where $\lambda \neq 0$ and $c \neq 0$ ,thus we have the following nonlinear ordinary differential equation

$$N(u,u',u'',...) = 0 \tag{5}$$

Lets accept that, the exact solutions of Eq. (5) can be stated as

$$u = \sum_{i=0}^{\delta} \tau_i \Gamma^i \tag{6}$$

where

$$(\Gamma')^2 = \Lambda(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + ... + \xi_1 \Gamma + \xi_0}{\zeta_\varepsilon \Gamma^\varepsilon + ... + \zeta_1 \Gamma + \zeta_0} \tag{7}$$

Now, by using the relations (6) and (7), we can calculate the terms $(u')^2$ and $u''$ as

$$(u')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left( \sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1} \right) \tag{8}$$

$$u'' = \frac{\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma)}{2\Psi^2(\Gamma)} \left( \sum_{i=0}^{\delta} i\tau_i \Gamma^{i-1} \right) + \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left( \sum_{i=0}^{\delta} i(i-1)\tau_i \Gamma^{i-2} \right) \tag{9}$$

where $\Phi(\Gamma)$ and $\Psi(\Gamma)$ are polynomials of $\Gamma$.If we substitute above calculated terms in Eq. (5), we have an algebraic equation of polynomial $\Omega(\Gamma)$ of $\Gamma$:

$$\Omega(\Gamma) = \rho_s \Gamma^s + ... + \rho_1 \Gamma^1 + \rho_0 = 0 \tag{10}$$

A relation of $\theta, \varepsilon$ and $\delta$ can be found out according to the balance principle. We can find the appropriate values of $\theta, \varepsilon$ and $\delta$.

Taking all the coefficients of $\Omega(\Gamma)$ zero creates an algebraic equations system:

$$\rho_i = 0, \;\; i = 0,...,s. \tag{11}$$

If we solve the equations system (11), we can find the values of $\xi_0,...,\xi_\theta, \zeta_0,...,\zeta_\varepsilon, \tau_0,...,\tau_\delta$.

Reduce Eq. (7) to elementary integral form,

$$\pm\eta - \eta_0 = \int \frac{d\Gamma}{\sqrt{\Lambda(\Gamma)}} = \int \sqrt{\frac{\Psi(\Gamma)}{\Phi(\Gamma)}} d\Gamma. \tag{12}$$

Using a complete discrimination system for polynomial to classify the roots of $\Phi(\Gamma)$,we solve the indefinite integral (12) and obtain the exact solutions of Eq. (5). Moreover we can write exact travelling wave solutions of (3).

### *1.2 The Functional Varible Method*

The general characteristics of the FVM can be explained as follows.We can write a nonlinear partial differential equation with several independent variables in the form of

$$P(u,u_t,u_x,u_y,u_z,u_{tt},u_{xt},u_{xx},...) = 0 \tag{13}$$

where $P$ is a function, the subscripts denote partial derivatives, and $u(t,x,y,z,...)$ is the unknown function to be determined. At the beginning, we can write the new wave variable as

$$\xi = \sum_{i=0}^{p} \alpha_i \chi_i + \gamma \tag{14}$$

where $\chi_i$'s are the independent variables, and $\gamma$ and $\alpha_i$'s are free parameters.

Now, we can state the following transformation to obtain a travelling wave solution of Eq. (13),

$$u(\chi_0,\chi_1,...) = U(\xi) \tag{15}$$

and the chain rule

$$\frac{\partial}{\partial \chi_i}(.) = \alpha_i \frac{\partial}{\partial \xi}(.), \qquad \frac{\partial}{\partial \chi_i \partial \chi_j}(.) = \alpha_i \alpha_j \frac{\partial^2}{\partial \xi^2}(.),.... \tag{16}$$

By using Eq. (15) and Eq. (16), the nonlinear partial differential equation (13) can be changed into an ordinary differential equation of the form

$$Q(U,U_\xi,U_{\xi\xi},U_{\xi\xi\xi},U_{\xi\xi\xi\xi},...) = 0. \tag{17}$$

Then,we use a transformation where the unknown function $U$ is handled as a functional variable of the form

$$U_\xi = F(U) \tag{18}$$

and the successive derivatives of $U$ of the forms

$$\begin{aligned}
U_{\xi\xi} &= \frac{1}{2}(F^2)', \\
U_{\xi\xi\xi} &= \frac{1}{2}(F^2)''\sqrt{F^2}, \\
U_{\xi\xi\xi\xi} &= \frac{1}{2}[(F^2)'''F^2 + (F^2)''(F^2)'], \\
&\vdots
\end{aligned} \tag{19}$$

where "$'$" means $\frac{d}{dU}$. When we substitute Eq. (19) in Eq. (17), the ordinary differential equation in Eq. (17) can be expressed in terms of $U$ and $F$ as follows

$$R(U,F,F',F'',F''',F^{(4)},...) = 0. \tag{20}$$

So Eq. (20) allows the analytical solutions of many nonlinear wave type equations. If we take the integral, the Eq. (20) becomes an expression of $F$, and the obtained result together with the Eq. (18) result in an appropriate solution of the original problem.

## 2 The Liouville Equation

We first consider the Liouville equation [12,13]

$$u_{xt} + e^u = 0 \tag{21}$$

To look for travelling wave solutions of (21), we use the wave transformation $\eta = x - wt$ and change Eq. (21) into the form of an ODE

$$-wu'' + e^u = 0 \tag{22}$$

Using the transformation

$$u = lnv \tag{23}$$

Eq. (22) reduces to nonlinear ODE in the form

$$-w(vv'' - v'^2) + v^3 = 0 \tag{24}$$

where the prime denotes differentiation with respect to $\eta$.

### *2.1 The Extended Trial Equation Method Solution*

If we substitutite Eqs. (8) and (9) into Eq. (24) and using the balance principle, we obtain

$$\theta = \varepsilon + \delta + 2. \tag{25}$$

Lets choose $\theta = 3$, $\varepsilon = 0$ and $\delta = 1$, then

$$(v'^2 = \frac{\tau_1^2(\xi_3\Gamma^3 + \xi_2\Gamma^2 + \xi_1\Gamma + \xi_0)}{\zeta_0}, \tag{26}$$

$$v'' = \frac{\tau_1(3\xi_3\Gamma^2 + 2\xi_2\Gamma + \xi_1)}{2\zeta_0} \tag{27}$$

where $\xi_3 \neq 0$ and $\zeta_0 \neq 0$. Respectively, solving the algebraic equation system (11) makes

$$\xi_0 = \frac{\tau_0\tau_1^2\xi_1 - \tau_0^3\xi_3}{2\tau_1^2}, \xi_1 = \xi_1, \xi_2 = \frac{\tau_1^2\xi_1 + 3\tau_0^2\xi_3}{2\tau_0\tau_1}, \xi_3 = \xi_3, \tau_0 = \tau_0, \tau_1 = \tau_1, w = \frac{2\tau_1\zeta_0}{\xi_3}. \tag{28}$$

Substituting these results into Eqs. (7) and (12), we get

$$\pm(\eta - \eta_0) = \sqrt{\frac{\zeta_0}{\xi_3}} \int \frac{d\Gamma}{\sqrt{\Gamma^3 + \frac{\tau_1^2\xi_1 + 3\tau_0^2\xi_3}{2\tau_0\tau_1\xi_3}\Gamma^2 + \frac{\xi_1}{\xi_3}\Gamma + \frac{\tau_0\tau_1^2\xi_1 - \tau_0^3\xi_3}{2\tau_1^2\xi_3}}}. \tag{29}$$

By integrating Eq. (29), we obtain the solutions to the Eq. (21) as follows:

$$\pm(\eta - \eta_0) = -2\sqrt{\frac{\zeta_0}{\xi_3}} \frac{1}{\sqrt{\Gamma - \alpha_1}}, \tag{30}$$

$$\pm(\eta - \eta_0) = 2\sqrt{\frac{\zeta_0}{\xi_3(\alpha_3 - \alpha_1)}} \arctan\sqrt{\frac{\Gamma - \alpha_3}{\alpha_3 - \alpha_1}}, \alpha_3 > \alpha_1, \tag{31}$$

$$\pm(\eta - \eta_0) = \sqrt{\frac{\zeta_0}{\xi_3(\alpha_1 - \alpha_3)}} \ln\left|\frac{\sqrt{\Gamma - \alpha_3} - \sqrt{\alpha_1 - \alpha_3}}{\sqrt{\Gamma - \alpha_3} + \sqrt{\alpha_1 - \alpha_3}}\right|, \alpha_1 > \alpha_3. \tag{32}$$

Also $\alpha_1 = \alpha_2$ and $\alpha_3$ are the roots of the polynomial equation

$$\Gamma^3 + \frac{\xi_2}{\xi_3}\Gamma^2 + \frac{\xi_1}{\xi_3}\Gamma + \frac{\xi_0}{\xi_3} = 0. \tag{33}$$

Substituting the solutions (30)-(33) into (6) and (23), we can obtain the following exact travelling wave solutions of Eq. (21), respectively:

$$u_1(x,t) = ln\left(\tau_0 + \tau_1\alpha_1 + \frac{4\tau_1\zeta_0}{\xi_3\left(x - \frac{2\tau_1\zeta_0}{\xi_3}t - \eta_0\right)^2}\right) \tag{34}$$

$$u_2(x,t) = ln\left(\tau_0 + \tau_1\alpha_3 + \tau_1(\alpha_1 - \alpha_3)\tanh\left(\frac{\sqrt{\xi_3(\alpha_1 - \alpha_3)}}{2\sqrt{\zeta_0}}\left(x - \frac{2\tau_1\zeta_0}{\xi_3}t - \eta_0\right)\right)\right)^2, \tag{35}$$

and

$$u_3(x,t) = ln\left(\tau_0 + \tau_1\alpha_1 + \tau_1(\alpha_1 - \alpha_3)mdcsch\left(\frac{\sqrt{\xi_3(\alpha_1 - \alpha_3)}}{2\sqrt{\zeta_0}}\left(x - \frac{2\tau_1\zeta_0}{\xi_3}t - \eta_0\right)\right)\right)^2. \tag{36}$$

### *2.2 The Functional Varible Method Solution*

If we apply the equalities $v_\eta = F$ and $v_{\eta\eta} = \frac{1}{2}(F^2)'$ to (24), we obtain the following expression for the function $F(v)$

$$(F^2)' - \frac{2}{v}F^2 = \frac{2v^2}{w}. \tag{37}$$

Hence we get first order differential equation. One can easily show that the solution of the Eq. (37) corresponds to

$$F = v\sqrt{\frac{2}{w}v}.$$

Using a transformation where the unknown function $v$ is considered as a functional variable of the form $v_\eta = F(v)$,we get $v$ as

$$v = \frac{2w}{\eta^2}.$$

Turning back to our unknown function $u(x,t)$ by using transformations $u = lnv$ and $\eta = x - wt$ , we have the exact solution of equation (21) as

$$u = \ln\left(\frac{2w}{(x - wt)^2}\right). \tag{38}$$

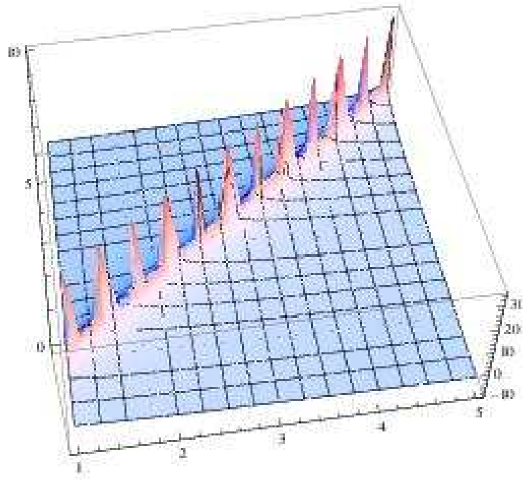

**Fig. 1:** The figure of the Liouville Equation's exact solution which is given in (38) for the values w=0.5, x=[1,5], t=[0,10].

## 3 Conclusions

In this paper, the functional variable method and extended trial method has been used to obtain some exact travelling wave solutions for Liouville equation. It is clearly seen from the obtained results that these two methods are obvious, brief and a feebleness tools for various general NLEEs arising in different fields of science and physics.

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